Now let us do some…

Counting

(Chapter 6)

Basic Counting Principles

Counting problems are of the following kind:

"How many different 8-letter passwords are there?"

"How many possible ways are there to pick 11 soccer players out of a 20-player team?"

Most importantly, counting is the basis for computing probabilities of discrete events.

("What is the probability of winning the lottery?")

Basic Counting Principles

The sum rule:
If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Example:
The department will award a free computer to either a CS student or a CS professor.
How many different choices are there, if there are 530 students and 15 professors?

There are 530 + 15 = 545 choices.

Basic Counting Principles

Generalized sum rule:

If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

Basic Counting Principles

The product rule:

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are n_1n_2 ways to do the procedure.

Generalized product rule:

If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

Basic Counting Principles

Example:
How many different license plates are there that contain exactly three English letters ?

Solution:
There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.

So there are $26 \cdot 26 \cdot 26$ = 17576 different license plates.

Basic Counting Principles

The sum and product rules can also be phrased in terms of set theory.

Sum rule: Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is $|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$.

Product rule: Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set independently in the order $A_1, A_2, \ldots, A_m$ is
$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$.

Inclusion-Exclusion

How many bit strings of length 8 start with a 1 or end with 00?

Task 1: Create a string of length 8 that starts with a 1.

There is one way to pick the first bit (1).
For each of these there are two ways to pick the second bit (0 or 1),
For each of these,two ways to pick the third bit (0 or 1),
:
:
For each, two ways to pick the eighth bit (0 or 1).

Product rule: Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Inclusion-Exclusion

Task 2: Construct a string of length 8 that ends with 00.

There are two ways to pick the first bit (0 or 1),
two ways to pick the second bit (0 or 1),
:
:
two ways to pick the sixth bit (0 or 1),
one way to pick the seventh bit (0), and
one way to pick the eighth bit (0).

Product rule: Task 2 can be done in $2^6 = 64$ ways.

Inclusion-Exclusion

Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?

No, because here Task 1 and Task 2 can be done at the same time.

When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

Therefore, we sometimes do Tasks 1 and 2 at the same time, so the sum rule does not apply.

Inclusion-Exclusion

If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.

How many cases are there, that is, how many strings start with 1 and end with 00?

There is one way to pick the first bit (1),
two ways for the second, …, sixth bit (0 or 1),
one way for the seventh, eighth bit (0).

Product rule: In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time.

Inclusion-Exclusion

Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are

$128 + 64 - 32 = 160$ ways to do either task.

In set theory, this corresponds to sets A_1 and A_2 that are not disjoint. Then we have:

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

This is called the principle of inclusion-exclusion.

Inclusion–Exclusion

When counting the number of points in a finite union of sets, we add the points in all the sets. If intersections are non empty we have overcounted, and subtract the number of points in the intersections to compensate. If there are more than three sets we have usually overcompensated and have to add back the number of points in the intersections of three of the sets. This process continues…

See Theorem 1, page 556. The notation is hard to put in powerpoint.

Inclusion–Exclusion

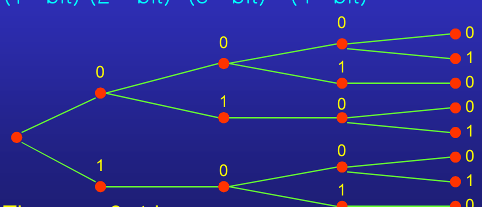
We'll draw a Venn diagram for the inclusion–exclusion theorem for $n = 3$ and $n = 4$.

The general case is trickier to prove but you can use the proof on p. 556 once you understand combinations and have done the Binomial Theorem.

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

Task 1 (1st bit) Task 2 (2nd bit) Task 3 (3rd bit) Task 4 (4th bit)



There are 8 strings.

The Pigeonhole Principle

The pigeonhole principle: If $(k + 1)$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1: If there are 11 players in a soccer team that wins 12–0, there must be at least one player in the team who scored at least twice.

Example 2: If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

Proof: If every box holds fewer than N/k objects then there are fewer than $k(N/k) = N$ objects overall. Thus some box must hold at least N/k objects, thus $\lceil N/k \rceil$, since each box holds an integral number of objects.

Example 1: In a 60-student class, at least 12 students will get the same letter grade (A, B, C, D, or F).

Example 2: In a 61-student class, at least 13 students will get the same letter grade.

The Pigeonhole Principle

Example 3: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

Permutations and Combinations

How many different sets of 3 people can we pick from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so are there $6 \cdot 5 \cdot 4 = 120$ ways to do this?

This is not the correct result!

For example, picking person C, then person A, and then person E leads to the same group as first picking E, then C, and then A.

However, these cases are counted separately in the above equation.

Permutations and Combinations

So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking) ?

To find out about this, we need to look at permutations.

A permutation of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an r-permutation.

Permutations and Combinations

Example: Let S = {1, 2, 3}.
The arrangement 3, 1, 2 is a permutation of S.
The arrangement 3, 2 is a 2-permutation of S.

The number of r-permutations of a set with n distinct elements is denoted by P(n, r).

We can calculate P(n, r) with the product rule:

$P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1)$.

(n choices for the first element, (n – 1) for the second one, (n – 2) for the third one…)

Permutations and Combinations

Example:

$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
$= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

General formula:

$P(n, r) = n!/(n - r)!$

Knowing this, we can return to our initial question:

How many different sets of 3 people can we pick from a group of 6?

Permutations and Combinations

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Example: Let S = {1, 2, 3, 4}.
Then {1, 3, 4} is a 3-combination from S.

The number of r-combinations of a set with n distinct elements is denoted by C(n, r).

Example: C(4, 2) = 6, since, for example, the 2-combinations of {1, 2, 3, 4} are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}.

Permutations and Combinations

How can we calculate C(n, r)?

Consider that we can obtain the r-permutation of a set in the following way:

First, we form all the r-combinations of the set (there are C(n, r) such r-combinations).

Then, we generate all possible orderings in each of these r-combinations (there are P(r, r) such orderings in each case).

Therefore, we have:

$P(n, r) = C(n, r) \cdot P(r, r)$

Combinations

C(n, r) = P(n, r)/P(r, r)
= n!/(n – r)!/(r!/(r – r)!)
= n!/(r!(n – r)!)

Now we can answer our initial question:

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

C(6, 3) = 6!/(3!⋅3!) = 720/(6⋅6) = 720/36 = 20

There are 20 different ways, that is, 20 different groups that may be picked.

Combinations

Corollary:
Let n and r be nonnegative integers with r ≤ n.
Then C(n, r) = C(n, n – r).

Note that “picking a group of r people from a group of n people” is the same as “splitting a group of n people into a group of r people and another group of (n – r) people”.

Please also look at the proof on page 411 (page 359 of the 6^{th} edition).

Combinations

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n,r)$$

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

Picking two elements and **leaving four** is essentially the same as **picking four** elements and **leaving two**.

In either case, our number of choices is the number of ways to **divide** the set into one set containing two elements and another set containing four elements.

Combinations

Example:
A soccer club has 8 female and 7 male members. For today’s match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

C(8, 6) ⋅ C(7, 5) = 8!/(6!⋅2!) ⋅ 7!/(5!⋅2!)
= 28⋅21
= 588

Combinations

Pascal’s Identity (page 418) [366 in 6^{th} ed.]:

Let n and k be positive integers with n ≥ k.
Then C(n + 1, k) = C(n, k – 1) + C(n, k).

How can this be explained?

What is it good for?

Combinations

Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element a.

Calculating C(n + 1, k) is equivalent to answering the question: How many subsets of T containing k items are there?

Case I: The subset contains (k – 1) elements of S plus the element a: C(n, k – 1) choices.

Case II: The subset contains k elements of S and does not contain a: C(n, k) choices.

Sum Rule: C(n + 1, k) = C(n, k – 1) + C(n, k).

Pascal's Triangle

In Pascal's triangle, each number is the sum of the numbers to its upper left and upper right:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

… … … … … …

Pascal's Triangle

Since we have C(n + 1, k) = C(n, k – 1) + C(n, k) and
C(0, 0) = 1, we can use Pascal's triangle to simplify the computation of C(n, k):

k →

n ↓

C(0, 0) = 1

C(1, 0) = 1 C(1, 1) = 1

C(2, 0) = 1 C(2, 1) = 2 C(2, 2) = 1

C(3, 0) = 1 C(3, 1) = 3 C(3, 2) = 3 C(3, 3) = 1

C(4, 0) = 1 C(4, 1) = 4 C(4, 2) = 6 C(4, 3) = 4 C(4, 4) = 1

Binomial Coefficients

Expressions of the form C(n, k) are also called **binomial coefficients**.

How come?

A **binomial expression** is the sum of two terms, such as (a + b).

Now consider $(a + b)^2 = (a + b)(a + b)$.

When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:

$(a + b)^2 = a \cdot a + a \cdot b + b \cdot a + b \cdot b$

Then we can sum identical terms:

$(a + b)^2 = a^2 + 2ab + b^2$

Binomial Coefficients

For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have

$(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

There is only one term a^3, because there is only one possibility to form it: Choose **a** from all three factors: C(3, 3) = 1.

There is three times the term a^2b, because there are three possibilities to choose **a** from a **subset** of two out of the three factors: C(3, 2) = 3.

Similarly, there is three times the term ab^2 (C(3, 1) = 3) and once the term b^3 (C(3, 0) = 1).

Binomial Coefficients

This leads us to the following formula:

$$(a+b)^n = \sum_{j=0}^{n} C(n,j) \cdot a^{n-j} b^j \quad \text{(Binomial Theorem)}$$

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle (1 – 4 – 6 – 4 – 1) helps us to compute $(a + b)^4$:

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Binomial Theorem

We can prove the Binomial Theorem in two ways.

First, an argument about how the term a^jb^{n-j} appears when we expand $(a+b)^n = (a+b)\ldots(a+b)$, by choosing a from j of the factors (a+b) and b from n−j of them. This choice can be made in C(n,j) ways, so the coefficient of a^jb^{n-j} will be C(n,j)

Binomial Theorem

Binomial Theorem: $(a+b)^n = \sum_{j=0}^{n} C(n,j)\cdot a^{n-j}b^j, n \geq 0$

Second proof (induction on n)

Base case: n = 0. LHS = $(a+b)^0 = 1$

RHS = $C(0,0)a^0b^0 = 1$.

Induction step: Suppose the formula is true for n.

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{j=0}^{n} C(n,j)\cdot a^{n-j}b^j$$

$$= \sum_{j=0}^{n} C(n,j)a^{n-j+1}b^j + \sum_{j=0}^{n} C(n,j)\cdot a^{n-j}b^{j+1}$$

$$= a^{n+1} + \sum_{j=1}^{n} C(n,j)a^{n+1-j}b^j + \sum_{k=0}^{n-1} C(n,k)a^{(n+1)-(k+1)}b^{k+1} + b^{n+1}$$

$$= a^{n+1} + \sum_{j=1}^{n} C(n,j)a^{n+1-j}b^j + \sum_{j=1}^{n} C(n,j-1)a^{(n+1)-j}b^j + b^{n+1}$$

$$= a^{n+1} + \sum_{j=1}^{n} (C(n,j)+C(n,j-1))a^{n+1-j}b^j + b^{n+1}$$

$$= a^{n+1} + \sum_{j=1}^{n} C(n+1,j)a^{n+1-j}b^j + b^{n+1} = \sum_{j=0}^{n+1} C(n+1,j)a^{n+1-j}b^j$$

Proof of Binomial Theorem

In slide 2 of the proof using induction:

From line 1 to line 2 we split off the term j=0, a^n,from the first sum and k=n, b^n, from the second sum (having replaced j by k). Note that C(n,0) = C(n,n) = 1.

From line 2 to line 3 we let k+1=j, so k=j−1, and fixed the sum limits.

Then we combined the sums and applied Pascal's theorem.

Then we put the a^{n+1}, j=0, and b^{n+1},j=n+1 terms back in the sum, to get the result.