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Solving Recurrence Relations

•**Another Example:** Give an explicit formula for the Fibonacci numbers.

•**Solution:** The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

•The characteristic equation is $r^2 - r - 1 = 0$.

•Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

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•Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2.
We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

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•The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$