Permutations with repetitions

Theorem (p.423)(371 in 6th ed.): The number of r-permutations from a set of n objects with repetition allowed is n^r.

Proof:
Since we are allowed to repeat, we have n choices for each of r positions. The set we get is just the Cartesian product r times of the set.

Combinations with repetition

Theorem (p.425) (373 in 6th ed.)
There are C(n+r-1,r) ways to choose r objects from n if repetition of objects is allowed.

Proof:
An example of this is: in how many ways can we choose 6 drinks, if we choose from water, juice, milk?

Combinations with repetition

We can think of the n objects as described by n bins. We choose an object by putting a marker, *, in a bin.

We think of the bins as marked by n+1 vertical bars on a line, and we put r stars, *, on the line in the n bins.

Combinations with repetition

The two bars on the ends are fixed, but the n-1 internal bars can move as we add stars.

So the number of choices is the number of ways we can arrange r stars and n-1 bars in a line.

That is C(n-1+r, r), the number of ways to choose r of the n-1+r positions for stars.

Combinations with repetition

Example:

In how many ways can we choose 3 drinks, if we can choose water, juice, milk, or beer?

Answer: C(4+3-1, 3) = C(6,3) = 6!/(3!3!) = 20.

Permutations with indistinguishable objects

Example: In how many ways can we arrange 7 red beads, 4 blue beads and 5 yellow beads in a row?

Theorem (p.375). If we have n objects, n_i indistinguishable of type i, i = 1 to k, they can be permuted in $n!/(n_1!\, n_2! \dots n_k!)$ ways.

Permutations with indistinguishable objects

Proof:
If the n objects are all distinguishable there are n! permutations.

If we now identify n_1 objects of type 1 then we can permute these n_1 objects among themselves in n_1! ways, giving distinct permutations of the distinguishable objects, but the same permutation if the n_1 objects are indistinguishable.

Permutations with indistinguishable objects

Thus, dividing n! by n_1! gives the number of permutations of n objects with n_1 of them being identical.

Repeating, to identify n_2 objects of type 2, …, n_k objects of type k, gives $n!/(n_1!\ n_2!\ \ldots\ n_k!)$ as the result.

Putting objects into boxes

Theorem (p.429) (377 in 6th ed.).

There are $n!/(n_1!\ n_2!\ \ldots\ n_k!)$ ways to put n distinguishable objects into k boxes, so that the i^{th} box contains n_i objects.

Putting objects into boxes

Proof: Think of distributing the objects into n positions on a line, with fixed bars separating locations which will be the boxes. There are n! ways of permuting the objects.

But any permutation of the objects in a given box corresponds to a single method of putting the objects in the boxes. Thus we divide by $n_1!\ n_2!\ \ldots\ n_k!$ To make up for counting the arrangement multiple times.

Now it's time to look at…

Discrete Probability

Section 7.1

Discrete Probability

Everything you have learned about counting constitutes the basis for computing the **probability** of events to happen.

In the following, we will use the notion **experiment** for a procedure that yields one of a given set of possible outcomes.

This set of possible outcomes is called the **sample space** of the experiment.

An **event** is a subset of the sample space.

Discrete Probability

If all outcomes in the sample space are equally likely, the following definition of probability applies:

The probability of an event E, which is a subset of a finite sample space S of equally likely outcomes, is given by p(E) = |E|/|S|.

Probability values range from **0** (for an event that will **never** happen) to **1** (for an event that will **always** happen whenever the experiment is carried out).

Discrete Probability

Example I:

An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

Solution:

There are nine possible outcomes, and the event "blue ball is chosen" comprises four of these outcomes. Therefore, the probability of this event is 4/9 or approximately 44.44%.

Discrete Probability

Example II:

What is the probability of winning the lottery 6/49, that is, picking the correct set of six numbers out of 49?

Solution:

There are C(49, 6) possible outcomes. Only one of these outcomes will actually make us win the lottery.

p(E) = 1/C(49, 6) = 1/13,983,816

Complementary Events

Let E be an event in a sample space S. The probability of an event –E, the **complementary event** of E, is given by

p(-E) = 1 – p(E).

We see this when all outcomes are equally likely:

p(-E) = (|S| - |E|)/|S| = 1 - |E|/|S| = 1 – p(E).

This rule is useful if it is easier to determine the probability of the complementary event than the probability of the event itself.

Complementary Events

Example I: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is zero?

Solution: There are 2^{10} = 1024 possible outcomes of generating such a sequence. The event –E, **"none of the bits is zero"**, includes only one of these outcomes, namely the sequence 1111111111.

Therefore, p(-E) = 1/1024.

Now p(E) can easily be computed as
p(E) = 1 – p(-E) = 1 – 1/1024 = 1023/1024.

Complementary Events

Example II: What is the probability that at least two out of 36 people have the same birthday?

Solution: The sample space S encompasses all possibilities for the birthdays of the 36 people, so |S| = 365^{36}.

Let us consider the event –E ("no two people out of 36 have the same birthday"). –E includes P(365, 36) outcomes (365 possibilities for the first person's birthday, 364 for the second, and so on).

Then p(-E) = P(365, 36)/365^{36} = 0.168,
so p(E) = 0.832 or 83.2%

Discrete Probability

Let E_1 and E_2 be events in the sample space S. Then we have:

$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Does this remind you of something?

Of course, the principle of **inclusion-exclusion**.

Discrete Probability

Example: What is the probability of a positive integer selected at random from the set of positive integers not exceeding 100 to be divisible by 2 or 5?

Solution:

E_2: "integer is divisible by 2"
E_5: "integer is divisible by 5"

$E_2 = \{2, 4, 6, \ldots, 100\}$

$|E_2| = 50$

$p(E_2) = 0.5$

Discrete Probability

$E_5 = \{5, 10, 15, \ldots, 100\}$

$|E_5| = 20$

$p(E_5) = 0.2$

$E_2 \cap E_5 = \{10, 20, 30, \ldots, 100\}$

$|E_2 \cap E_5| = 10$

$p(E_2 \cap E_5) = 0.1$

$p(E_2 \cup E_5) = p(E_2) + p(E_5) - p(E_2 \cap E_5)$

$p(E_2 \cup E_5) = 0.5 + 0.2 - 0.1 = 0.6$

Discrete Probability

What happens if the outcomes of an experiment are **not** equally likely?

In that case, we assign a probability p(s) to each outcome $s \in S$, where S is the sample space.

Two conditions have to be met:

(1): $0 \leq p(s) \leq 1$ for each $s \in S$, and

(2): $\sum_{s \in S} p(s) = 1$

This means, as we already know, that (1) each probability must be a value between 0 and 1, and (2) the probabilities must add up to 1, because one of the outcomes is **guaranteed** to occur.

Discrete Probability

How can we obtain these probabilities p(s) ?

Sometimes we know from the structure of the problem.

We can sometimes estimate it experimentally because the probability p(**s**) assigned to an outcome **s** equals the limit of the number of times **s** occurs divided by the number of times the experiment is performed.

Once we know the probabilities p(s), we can compute the **probability of an event E** as follows:

$p(E) = \sum_{s \in E} p(s)$

A non-discrete probabilty example

It is sometimes useful to think of the problem of picking a point at random from the unit square. It's a good concrete example for thinking about probability theorems.

In this case the P(A), the probability that the point chosen is in the set A is the area of A.

Discrete Probability

Example I: A die is biased so that the number 3 appears twice as often as each other number.

What are the probabilities of all possible outcomes?

Solution: There are 6 possible outcomes $s_1, \ldots, s_6$.

$p(s_1) = p(s_2) = p(s_4) = p(s_5) = p(s_6)$, $p(s_3) = 2p(s_1)$

Since the probabilities must add up to 1, we have:

$5p(s_1) + 2p(s_1) = 1$

$7p(s_1) = 1$

$p(s_1) = p(s_2) = p(s_4) = p(s_5) = p(s_6) = 1/7$, $p(s_3) = 2/7$

Discrete Probability

Example II: For the biased die from Example I, what is the probability that an odd number appears when we roll the die?

Solution:

$A_{odd} = \{s_1, s_3, s_5\}$

Remember the formula $p(A) = \sum_{s \in A} p(s)$.

$p(A_{odd}) = \sum_{s \in A_{odd}} p(s) = p(s_1) + p(s_3) + p(s_5)$

$p(A_{odd}) = 1/7 + 2/7 + 1/7 = 4/7 = 57.14\%$

Conditional Probability

If we toss a coin three times, what is the probability that an odd number of tails appears **(event E)**, if the first toss is a tail **(event F)** ?

If the first toss is a tail, the possible sequences are TTT, TTH, THT, and THH.

In two out of these four cases, there is an odd number of tails.

Therefore, the probability of E, under the condition that F occurs, is 0.5.

We call this **conditional probability**.

Conditional Probability

If we want to compute the conditional probability of E given F, we use F as the sample space.

For any outcome of E to occur under the condition that F also occurs, this outcome must also be in $E \cap F$.

Definition: Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $p(E \mid F)$, is defined as

$p(E \mid F) = p(E \cap F)/p(F)$

Conditional Probability

Example: What is the probability of a random bit string of length four to contain at least two consecutive 0s, given that its first bit is a 0 ?

Solution:

E: "bit string contains at least two consecutive 0s"

F: "first bit of the string is a 0"

We know the formula $p(E \mid F) = p(E \cap F)/p(F)$.

$E \cap F = \{0000, 0001, 0010, 0011, 0100\}$

$p(E \cap F) = 5/16$

$p(F) = 8/16 = 1/2$

$p(E \mid F) = (5/16)/(1/2) = 10/16 = 5/8 = 0.625$

Independence

Let us return to the example of tossing a coin three times.

Does the probability of event E (odd number of tails) **depend** on the occurrence of event F (first toss is a tail) ?

In other words, is it the case that $p(E \mid F) \neq p(E)$?

We actually find that $p(E \mid F) = 0.5$ and $p(E) = 0.5$, so we say that E and F are **independent events**.

Independence

Because we have $p(E \mid F) = p(E \cap F)/p(F)$,
$p(E \mid F) = p(E)$ if and only if $p(E \cap F) = p(E)p(F)$.

Definition: The events E and F are said to be independent if and only if $p(E \cap F) = p(E)p(F)$.

Obviously, this definition is **symmetrical** for E and F. If we have $p(E \cap F) = p(E)p(F)$, then it is also true that $p(F \mid E) = p(F)$. This last condition would be an equivalent definition for independence.

Independence

Intuitively, events A and B are independent if you won't change your bet on B if you know A happened.

Independence is a property of the numbers P(A), P(B), $P(A \cap B)$, though often we argue that events are independent from physical considerations, e.g. arguing that flips of two coins will be independent.

Independence

Example: Suppose E is the event that a randomly generated bit string of length four begins with a 1, and F is the event that a randomly generated bit string contains an even number of 0s. We have 16 outcomes. Are E and F independent?

Solution: Obviously, $p(E) = p(F) = 0.5$.

F = {0000, 0011, 0110, 0101, 1100, 1001, 1010, 1111}

$E \cap F$ = {1111, 1001, 1010, 1100}
$p(E \cap F) = 0.25$
$p(E \cap F) = p(E)p(F)$

Conclusion: E And F are **independent**. If a bit string is generated and you know E happened you won't change a bet on whether F also happened.

Bernoulli Trials

Consider an experiment with **two possible outcomes**, such as tossing a coin.

Each performance of such an experiment is called a **Bernoulli trial**.

We will call the two possible outcomes a **success** or a **failure**, respectively.

If p is the probability of a success and q is the probability of a failure, it is obvious that
$p + q = 1$.

Bernoulli Trials

Often we are interested in the probability of **exactly k successes** when an experiment consists of **n independent Bernoulli trials**.

Example:
A coin is biased so that the probability of head is 2/3. What is the probability of exactly four heads to come up when the coin is tossed seven times?

Bernoulli Trials

Solution:

There are $2^7 = 128$ possible outcomes.

The number of possibilities for four heads among the seven trials is C(7, 4).

The seven trials are independent, so the probability of each of these outcomes is
$(2/3)^4(1/3)^3$.

Consequently, the probability of exactly four heads to appear is

$C(7, 4)(2/3)^4(1/3)^3 = 560/2187 = 25.61\%$

Bernoulli Trials

Theorem: The probability of k successes in n independent Bernoulli trials, with probability of success p and probability of failure q = 1 – p, is $C(n, k)p^kq^{n-k}$ **.**

We denote by b(k; n, p) the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure q = 1 – p.

Considered as function of k, we call b the **binomial distribution**.

Bernoulli Trials

Illustration: Let us denote a success by 'S' and a failure by 'F'. As before, we have a probability of success p and probability of failure q = 1 – p.

What is the probability of **two** successes in **five** independent Bernoulli trials?

Let us look at a possible sequence:

SSFFF

What is the probability that we will generate exactly this sequence?

Bernoulli Trials

Sequence: S S F F F

Probability: p p q q q = p^2q^3

Another possible sequence:

Sequence: F S F S F

Probability: q p q p q = p^2q^3

Each sequence with two successes in five trials occurs with probability p^2q^3.

Bernoulli Trials

And how many possible sequences are there?

In other words, how many ways are there to pick two items from a list of five?

We know that there are C(5, 2) = 10 ways to do this, so there are 10 possible sequences, each of which occurs with a probability of p^2q^3.

Therefore, the probability of **any** such sequence to occur when performing five Bernoulli trials is $C(5, 2)\ p^2q^3$.

In general, for k successes in n Bernoulli trials we have a probability of $\mathbf{C(n,k)p^kq^{n-k}}$.

Bayes' Theorem

The idea of Bayes' Theorem:

Urn 1 has 5 red balls and 5 blue balls.

Urn 2 has 2 red balls and 8 blue.

We flip two coins. If we get 2 heads we draw a ball at random from urn 1, otherwise from urn 2.

If we do this and get a red ball, what's the probability it came from urn 1? (assuming we didn't observe the drawing)

Bayes' Theorem Example

Let U_1 = "get 2 H, draw from urn1"

U_2 = "We draw from urn2"

R = "get red ball", B = "get blue ball"

We want $P(U_1 \mid R) = P(U_1 \cap R) / P(R)$

$= P(R|U_1)P(U_1)\ /\ [\ P(R \cap U_1) + P(R \cap U_2)\ ]$

$= P(R|U_1)P(U_1)\ /\ [\ P(R|U_1)P(U_1) + P(R|U_2)P(U_2)\ ]$

= 0.5*0.25 / [0.5*0.25 + 0.2*0.75]

= 0.125 / 0.275 = 0.454545…

Bayes' Theorem

Suppose A and B_i are events with P(A), $P(B_i)$ not 0 for all i and $S = U_{i=1}^{n} B_i$ and the B_i pairwise disjoint. Then

$P(B_k \mid A) = P(B_k \cap A) / P(A) =$

$= P(A \mid B_k)\, P(B_k) / \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$