# Random Variables

•In some experiments, we would like to assign a numerical value to each possible outcome in order to facilitate a mathematical analysis of the experiment.

### •For this purpose, we introduce random variables.

•Definition: A random variable is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

•Note: Random variables are functions, not variables, and they are not random, but map random results from experiments onto real numbers in a well-defined manner.

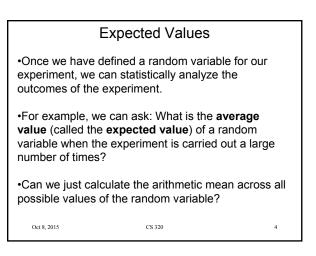
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## Random Variables •Example: •Let X be the result of a rock-paper-scissors game. •If player A chooses symbol a and player B chooses symbol b, then •X(a, b) = 1, if player A wins, = 0, if A and B choose the same symbol, = -1, if player B wins. Oct 8, 2015 CS 320

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•A(rock, rock) =	•0	
•A(rock, paper) =	•-1	
•A(rock, scissors) =	•1	
•A(paper, rock) =	•1	
•A(paper, paper) =	•0	
•A(paper, scissors) =	•-1	
•A(scissors, rock) =	•-1	
•A(scissors, paper) =	•1	
•A(scissors, scissors) =	•0	
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### **Expected Values**

•No, we cannot, since it is possible that some outcomes are more likely than others.

•For example, assume the possible outcomes of an experiment are 1 and 2 with probabilities of 0.1 and 0.9, respectively.

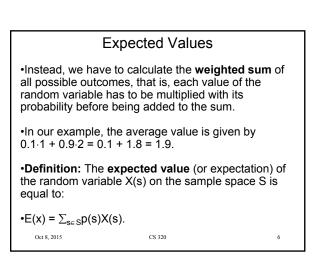
•Is the average value 1.5?

•No, since 2 is much more likely to occur than 1, the average must be larger than 1.5.

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### Expected Values

•Example: Let X be the random variable equal to the sum of the numbers that appear when a pair of dice is rolled.

•There are **36 outcomes** (= pairs of numbers from 1 to 6).

•The range of X is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

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•Are the 36 outcomes equally likely?

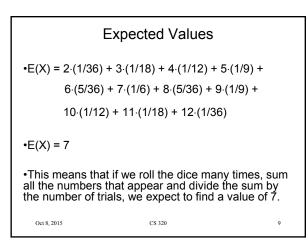
•Yes, if the dice are not biased.

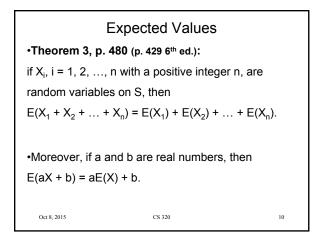
•Are the 11 values of X equally likely to occur?

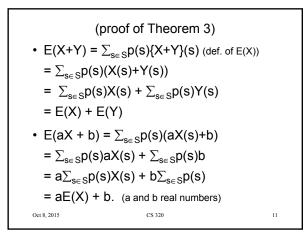
•No, the probabilities vary across values.

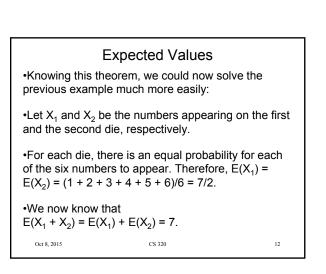
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	Expected Values	
•P(X = 2) = •P(X = 3) =		
•P(X = 4) = •P(X = 5) =		
•P(X = 6) =	5/36	
•P(X = 7) = •P(X = 8) =		
•P(X = 9) = •P(X = 10) =		
•P(X = 11) =	2/36 = 1/18	
•P(X = 12) =		
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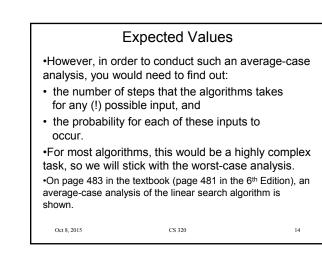


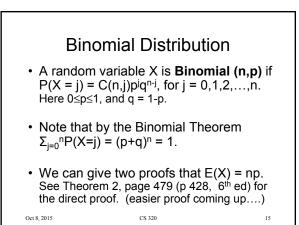


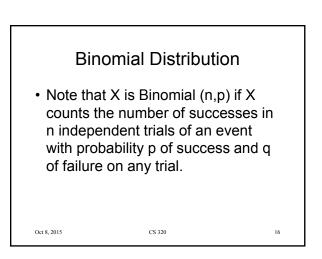




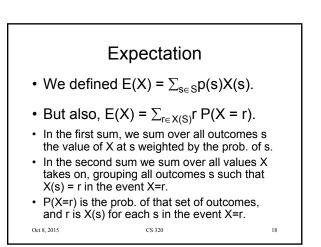
### **Expected Values** •We can use our knowledge about expected values to compute the average-case complexity of an algorithm. •Let the sample space be the set of all possible inputs $a_1, a_2, ..., a_n$ , and the random variable X assign to each $a_j$ the number of operations that the algorithm executes for that input. •For each input $a_j$ , the probability that this input occurs is given by $p(a_j)$ . •The algorithm's average-case complexity then is: •E(X) = $\sum_{j=1,...,n} p(a_j) X(a_j)$

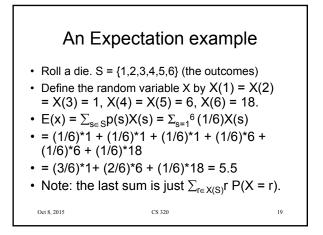


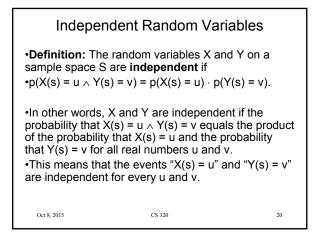


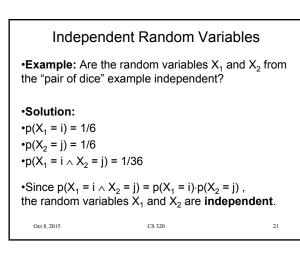


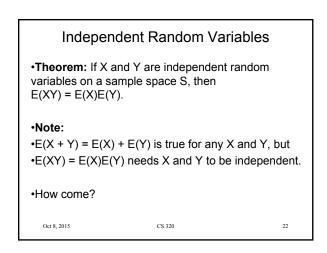
# Expectation of a Binomial R.V. If X is binomial(n,p), let X<sub>i</sub> be 1 if the i<sup>th</sup> trial gives a success, 0 otherwise. Then E(X<sub>i</sub>) = 1p + 0q = p and X = Σ<sub>i=1</sub><sup>n</sup> X<sub>i</sub> (X is the number of successes in the n trials) so E(X) = Σ<sub>i=1</sub><sup>n</sup> E(X<sub>i</sub>) = np. For a direct proof see Theorem 2, p 479











- Proof: the proof is subtle.
   E(XY) = Σ<sub>s∈S</sub>X(s)Y(s)p(s), sum over outcomes.
- Σ<sub>u∈X(S),v∈Y(S)</sub> uvP(X=u and Y=v), sum over values X & Y take on, grouping outcomes.
- = Σ<sub>u∈X(S),v∈Y(S)</sub>uvP(X=u)P(Y=v), since X,Y are independent.
- =  $(\Sigma_{u \in X(S)} u P(X=u))(\Sigma_{v \in Y(S)} v P(Y=v))$ • =E(X)E(Y)

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Independent Random Variables •Example: Let X and Y be random variables on the sample space, and each of them assumes the values 1 and 3 with equal probability. •Then E(X) = E(Y) = 2•If X and Y are **independent**, we have: • $E(X + Y) = 1/4 \cdot (1 + 1) + 1/4 \cdot (1 + 3) + 1/4 \cdot (3 + 1) + 1/4 \cdot (3 + 3) = 4 = E(X) + E(Y)$ • $E(XY) = 1/4 \cdot (1 \cdot 1) + 1/4 \cdot (1 \cdot 3) + 1/4 \cdot (3 \cdot 1) + 1/4 \cdot (3 \cdot 3) = 4 = E(X) \cdot E(Y)$ 

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### Independent Random Variables

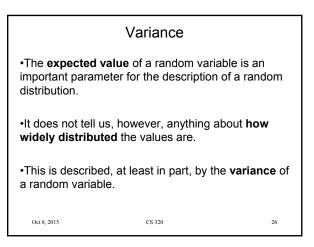
•Let us now assume that X and Y are **correlated** in such a way that Y = 1 whenever X = 1, and Y = 3 whenever X = 3. •E(X + Y) =  $1/2 \cdot (1 + 1) + 1/2 \cdot (3 + 3)$ = 4 = E(X) + E(Y) = 2 + 2 •E(XY) =  $1/2 \cdot (1 \cdot 1) + 1/2 \cdot (3 \cdot 3)$ = 5  $\neq$  E(X) ·E(Y) = 2 · 2

•So, we can guarantee the average value of XY to be the average value of X \* the average value of Y  $\underline{if}$  X and Y are independent

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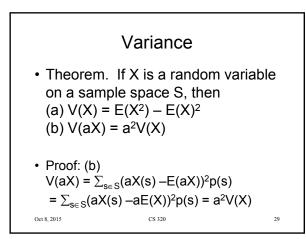
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	Variance		
• <b>Definition:</b> Let X be a random variable on a sample space S. The <b>variance</b> of X, denoted by V(X), is			
•V(X) = $\sum_{s \in S} (X(s) -$	- E(X))²p(s).		
•The standard deviation of X, denoted by $\sigma(X)$ , is defined to be the square root of V(X).			
<ul> <li>A large variance means the distribution is spread out, a small variance means it is more localized.</li> </ul>			
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Variance			
•If X is a random variable on a sample space S, then $V(X) = E(X^2) - E(X)^2$ . $V(aX) = a^2V(X)$			
•If X and Y are two independent random variables on a sample space S, then $V(X + Y) = V(X) + V(Y)$ . •Furthermore, if X <sub>i</sub> , i = 1, 2,, n, with a positive integer n, are pairwise independent random variables on S, then $V(X_1 + X_2 + + X_n) = V(X_1) + V(X_2) + + V(X_n)$ .			
•Proofs coming up, and in the textbook on page 489 (6 <sup>th</sup> edition 436, 437).			
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• Proof (a): 
$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$
  
 $= \sum_{s \in S} (X(s)^2 - 2E(X)X(s) + E(X)^2)p(s)$   
 $= \sum_{s \in S} X(s)^2 p(s) - 2E(X)\sum_{s \in S} X(s)p(s)$   
 $+ E(X)^2 \sum_{s \in S} p(s)$   
 $= E(X^2) - 2E(X)E(X) + E(X)^2$   
 $= E(X^2) - E(X)^2$   
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