Boolean Algebra

Boolean algebra provides the operations and the rules for working with the set **{0, 1}.**

These are the rules that underlie **electronic circuits**, and the methods we will discuss are fundamental to **VLSI design**.

We are going to focus on three operations:

- Boolean complementation,
- Boolean sum, and
- Boolean product

Boolean Operations

The **complement** is denoted by a bar (on the slides, we will use a minus sign). It is defined by

-0 = 1 and -1 = 0.

The **Boolean sum**, denoted by + or by OR, has the following values:

1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0

The **Boolean product**, denoted by · or by AND, has the following values:

1 · 1 = 1, 1 · 0 = 0, 0 · 1 = 0, 0 · 0 = 0

Boolean Functions and Expressions

Definition: Let B = {0, 1}. The variable x is called a **Boolean variable** if it assumes values only from B.

A function from B^n, the set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in B, 1 \le i \le n\}$, to B is called a **Boolean function of degree n**.

Boolean functions can be represented using expressions made up from the variables and Boolean operations.

Boolean Functions and Expressions

The **Boolean expressions** in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as follows:

- $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions.
- If E_1 and E_2 are Boolean expressions, then $(-E_1)$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.

Boolean Functions and Expressions

For example, we can create Boolean expression in the variables x, y, and z using the “building blocks” 0, 1, x, y, and z, and the construction rules:

Since x and y are Boolean expressions, so is xy.

Since z is a Boolean expression, so is (-z).

Since xy and (-z) are expressions, so is xy + (-z).

… and so on…

Boolean Functions and Expressions

Example: Give a Boolean expression for the Boolean function F(x, y) as defined by the following table:

x	y	F(x, y)
0	0	0
0	1	1
1	0	0
1	1	0

Possible solution: F(x, y) = (-x)·y

Boolean Functions and Expressions

There is a simple method for deriving a Boolean expression for a function that is defined by a table. This method is based on **minterms**.

Definition: A **literal** is a Boolean variable or its complement. A **minterm** of the Boolean variables x_1, x_2, …, x_n is a Boolean product $y_1y_2…y_n$, where $y_i = x_i$ or $y_i = -x_i$.

Hence, a minterm is a product of n literals, with one literal for each variable.

Boolean Functions and Expressions

Definition: The Boolean functions F and G of n variables are **equal** if and only if $F(b_1, b_2, …, b_n) = G(b_1, b_2, …, b_n)$ whenever $b_1, b_2, …, b_n$ belong to B.

Two different Boolean expressions that represent the same function are called **equivalent**.

For example, the Boolean expressions xy, xy + 0, and xy·1 are equivalent.

Boolean Functions and Expressions

The **complement** of the Boolean function F is the function –F, where $–F(b_1, b_2, …, b_n) = -(F(b_1, b_2, …, b_n))$.

Let F and G be Boolean functions of degree n. The **Boolean sum F+G** and **Boolean product FG** are then defined by

$(F + G)(b_1, b_2, …, b_n) = F(b_1, b_2, …, b_n) + G(b_1, b_2, …, b_n)$

$(FG)(b_1, b_2, …, b_n) = F(b_1, b_2, …, b_n)\, G(b_1, b_2, …, b_n)$

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 2 are there?

Solution: There are 16 of them, F_1, F_2, …, F_{16}:

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree n are there?

Solution:

There are 2^n different n-tuples of 0s and 1s.

A Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples.

Therefore, there are 2^{2^n} different Boolean functions.

Boolean Identities

There are useful identities of Boolean expressions that can help us to transform an expression A into an equivalent expression B (see Table 5 on page 815 [6th edition: page 753] in the textbook).

--x = x, law of double complement
x+x = x, idempotent laws
x·x = x
x+0 = x, identity laws
x·1 = x
x+1 = 1, domination laws
x·0 = 0
x+y = y+x, commutative laws
x·y = y·x

x+(y+z) = (x+y)+z, associative laws
x·(y·z) = (x·y)·z
x+yz = (x+y)(x+z), distributive laws
x·(y+z) = (x·y)+(x·z)
-(xy) = -x + -y, De Morgan's laws
-(x+y) = (-x)(-y)
x+xy = x, Absorption laws
x(x+y) = x
x+-x = 1, unit property
x(-x) = 0, zero property

Duality

We can derive additional identities with the help of the **dual** of a Boolean expression.

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Duality

Examples:

The dual of x(y + z) is x + yz.
The dual of -x·1 + (-y + z) is (-x + 0)((-y)z).
The dual is essentially the complement, but with any variable x replaced by -x. (exercise 29, p. 881)

The **dual of a Boolean function F** represented by a Boolean expression is the function represented by the dual of this expression.

This dual function, denoted by F^d, **does not depend** on the particular Boolean expression used to represent F. (exercise 30, page 881 [6th ed. p.756])

Duality

Therefore, an identity between functions represented by Boolean expressions **remains valid** when the duals of both sides of the identity are taken.

We can use this fact, called the **duality principle**, to derive new identities.

For example, consider the absorption law x(x + y) = x.

By taking the duals of both sides of this identity, we obtain the equation x + xy = x, which is also an identity (and also called an absorption law).

Definition of a Boolean Algebra

All the properties of Boolean functions and expressions that we have discovered also apply to **other mathematical structures** such as propositions and sets and the operations defined on them.

If we can show that a particular structure is a Boolean algebra, then we know that all results established about Boolean algebras apply to this structure.

For this purpose, we need an **abstract definition** of a Boolean algebra.

Definition of a Boolean Algebra

Definition: A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation – such that the following properties hold for all x, y, and z in B:

$x \vee 0 = x$ and $x \wedge 1 = x$ (identity laws)

$x \vee (-x) = 1$ and $x \wedge (-x) = 0$ (domination laws)

$(x \vee y) \vee z = x \vee (y \vee z)$ and
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and (associative laws)

$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutative laws)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributive laws)

Boolean Algebras

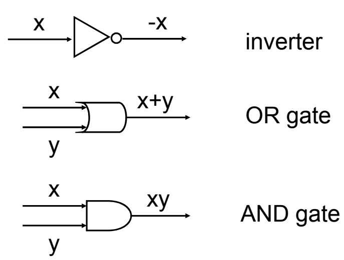
Examples of Boolean Algebras are:

1. The algebra of all subsets of a set U, with $+ = \cup$, $\cdot = \cap$, $- =$ complement, $0 = \varnothing$, $1 = U$.
2. The algebra of propositions with symbols $p_1, p_2, \ldots, p_n$, with $+ = \vee$, $\cdot = \wedge$, $- = \neg$, $0 = F$, $1 = T$.
3. If $B_1, \ldots, B_n$ are Boolean Algebras, so is $B_1 \times .. \times B_n$, with operations defined coordinate-wise.

Logic Gates

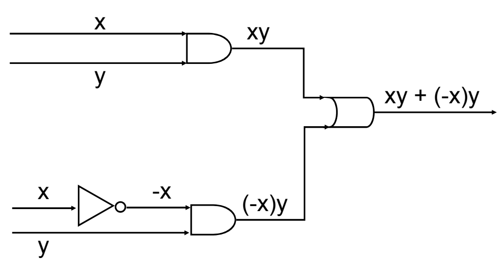
Electronic circuits consist of so-called gates. There are three basic types of gates.

In each case the input is a Boolean expression and the output is another Boolean expression.



Logic Gates

Example: How can we build a circuit that computes the function xy + (-x)y ?



Multi switch light circuit

Suppose we want a circuit for a light controlled by two switches, where changing the state of either switch changes the state of the light (on or off).

If we let x and y be the states of the switches (0 or 1) then the boolean expression xy + (-x)(-y) (- = complement) will do the job.

Multi switch light circuit

This is because if both x and y are "on" (1) or "off" (0) xy + (-x)(-y) will be 1, and otherwise will be 0.

We can generalize this method. For three switches the Boolean expression xyz + x(-y)(-z) + (-x)y(-z) + (-x)(-y)z will work.

Can you draw circuits implementing these expressions? (see pp. 825, 826 [6^{th} ed. pp. 763, 764])

Adding binary integers

If we add two one bit integers x and y we get a sum for that bit position plus a carry bit.

If we don't consider a carry bit from a lower bit addition we get what's called a half adder.

If we do get consider an input carry bit we have a full adder. (see p. 827, 6th ed.765)

Half Adder

Given input bits x and y, the result bit will be x+y unless both x and y are 1, in which case the result is 0.

This means that we can express the result bit as $(x+y)(-(xy))$, or $(x+y)(-x + -y)$.

The carry bit will be xy (we carry if both x and y are 1)

Full Adder

If we add a carry bit c_0 from the previous order bit sum our result for this bit would be 1 if one or three of c_0, x, y are 1, and 0 otherwise.

This means $xyc_0+x(-y)(-c_0)+(-x)y(-c_0)+(-x)(-y)c_0$ would work, with carry bit $xyc_0+xy(-c_0)+x(-y)c_0+(-x)yc_0$

See p. 827 to check your implementation.

Minimizing Circuits

A Boolean function can be implemented by many different Boolean expressions.

Disjunctive normal form, the sum-of-products expansion we got from the table of values of the expression, is often not the most efficient.

Minimizing Circuits

For example, the Boolean expression $x_1(-x_2)x_3 + x_1x_2x_3 + (-x_1)x_3$

$= x_1((-x_2)+x_2)x_3 + (-x_1)x_3$

$= x_1x_3 + (-x_1)x_3 = (x_1 + (-x_1))x_3 = x_3$

This last expression is a lot easier to compute. No gates required. Much simpler circuit.

Minimizing Circuits

Karnaugh Maps and the Quine-McCluskey Method are used for simplifying Boolean expressions.

See section 12.4.

We'll do some examples on the board.