Matrices

A matrix is a rectangular array of numbers.
A matrix with m rows and n columns is called an m×n matrix.

Example: $A = \begin{bmatrix} -1 & 1 \\ 2.5 & -0.3 \\ 8 & 0 \end{bmatrix}$ is a 3×2 matrix.

A matrix with the same number of rows and columns is called **square**.

Two matrices are **equal** if they have the same number of rows and columns and the corresponding entries in every position are equal.

Matrices

A general description of an m×n matrix A = $[a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{mj} \end{bmatrix}$$

j-th column of A

$$[a_{i1}, \ a_{i2}, \ \dots, \ a_{in}]$$

i-th row of A

Matrix Addition

Let A = $[a_{ij}]$ and B = $[b_{ij}]$ be m×n matrices.
The sum of A and B, denoted by A+B, is the m×n matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element.
In other words, A+B = $[a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} -2 & 1 \\ 4 & 8 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ -3 & 6 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 1 & 14 \\ -7 & 1 \end{bmatrix}$$

Matrix Multiplication

Let A be an m×k matrix and B be a k×n matrix.
The product of A and B, denoted by AB, is the m×n
matrix with (i, j)th entry equal to the sum of the products of the corresponding elements from the i-th row of A and the j-th column of B.

In other words, if AB = $[c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj} = \sum_{t=1}^{k} a_{it}b_{tj}$$

Matrix Multiplication

A more intuitive description of calculating C = AB:

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \\ 0 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$$

- Take the first column of B
- Turn it counterclockwise by 90° and superimpose it on the first row of A
- Multiply corresponding entries in A and B and add the products: 3·2 + 0·0 + 1·3 = 9
- Enter the result in the upper-left corner of C

Matrix Multiplication

– Now superimpose the first column of B on the second, third, …, m–th row of A to obtain the entries in the first column of C (same order).

– Then repeat this procedure with the second, third, …, n–th column of B, to obtain to obtain the remaining columns in C (same order).

– After completing this algorithm, the new matrix C contains the product AB.

Matrix Multiplication

Let us calculate the complete matrix C:

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \\ 0 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 9 & 7 \\ 8 & 15 \\ 15 & 20 \\ -2 & -2 \end{bmatrix}$$

Identity Matrices

The identity matrix of order n is the n×n matrix $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$:

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Multiplying an m×n matrix A by an identity matrix of appropriate size does not change this matrix:

$AI_n = I_mA = A$

Powers and Transposes of Matrices

The power function can be defined for square matrices. If A is an n×n matrix, we have:

$A^0 = I_n$,
$A^r = AAA\dots A$ (r times the matrix A)

The transpose of an m×n matrix $A = [a_{ij}]$, denoted by A^t, is the n×m matrix obtained by interchanging the rows and columns of A.

In other words, if $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Powers and Transposes of Matrices

Example: $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix} \qquad A^t = \begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix}$

A square matrix A is called **symmetric** if $A = A^t$. Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & -9 \\ 3 & -9 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

A is symmetric, B is not.