CS/Math 320: F2009 Practice Exam 2 (Exam from May 6, 2009) With Solutions

NAME____

Open Book, Open Notes, Do any TEN of the following TWELVE problems: 10 pts per question. Show work for partial credit in questions below. You can use both sides of any sheet. NOTE 1: For this Practice Exam, I add three problems from Chapter 9, which was not covered in Spring 2009 bringing the number of questions to FIFTEEN. NOTE 2: I do NOT guarantee that I will have identical form questions on the coming Exam 2, although the same topics will be covered, of course. Do not freeze up if the Exam questions seem harder -- I grade on a Scale!

1. Let P(n) be the statement that a postage of n cents can be formed using 4-cent and 9-cent stamps. Find the value m such that P(n) is true for $n \geq m$.

Answer. We can represent 4, 8, 9, 12, 13, 16, 17, 18, 20, 21 22, 24, 25, 26, 27. The answer is 24. This sort of problem was on homework.

2. Prove that the value m chosen in Problem 1 guarantees P(n) is true for $n \geq m$.

Answer. Given 24, 25, 26, and 27 (or any 4 in a row) are represented, we can get 28 = 24 + 4, 29 = 25 + 4, etc.

3. Generalize the principle of inclusion-exclusion in terms of sets laid out in the third paragraph of Example 17, page 342. Let A, B, and C be three finite sets of integers. If we know cardinalities of all sets and intersections of sets ($|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$), show how to represent the total number of integers in $A \cup B \cup C$ counted only once! Note: it might help to look at the Venn diagram of three intersecting sets at the top of page 127.

Answer. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Argument (not required): $|A| + |B| + |C|$ leaves each of the intersections by pairs counted twice and the triple intersection counted three time. When we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ we correct the extra intersections by pairs, but subtract one too many of the triple intersection and must add it again.

4. How many functions are there from a set of 10 (distinct) elements to a set of 4 (distinct) elements?

Answer. The answer is 4^{10}.This was on homework.

5. How many strings of four decimal digits (from {0, 1,…, 9}) contain exactly two 7s? Explain.

Answer. C(4,2)*81, because there are C(4,2) ways to choose positions for the two 7s, then 9 ways to choose digits other than 7s in the two remaining distinct positions. This was on homework.

6. Use the pigeonhole principle to show that if seven integers are chosen from the first 10 integers, {1, 2, 3,…,10}, there will be at least two pairs of these integers that add up to 11.

Answer. Consider the set of pairs of integers {(1,10), (2,9), (3,8), (4,7), (5,6)}. Choosing 7 integers must choose 2 from two of these pairs, thus two pairs adding to 11. This was on homework.

7.* We generalize Example 13 of Section 5.2, that in a group of six people each pair of which are either friends or enemies, there are either 3 mutual friends or 3 mutual enemies. Assume instead that we have pairs of people that are friends, enemies, or strangers. Show that among seventeen people in a group there must be 3 mutual friends, 3 mutual enemies, or 3 mutual strangers. HINT: Start at a member M and note that there must be 6 members that are M's friends, enemies, or strangers. Then what can we say about those six?

Answer. Member M pairs with sixteen other people of three kinds, thus by the generalized pigeonhole principle there must be 6 that are friends or enemies or strangers. Choose strangers without loss of generality and note that of those 6, if any two are strangers then M and those two members will be mutual strangers. But if not then all the six will be either friends or enemies which we have already shown in Example 13, Section 5.2; this means three will be mutual friends or mutual enemies.

8. A grammar school has 6 boys and 8 girls, and we want to choose six children to go on a field trip, but wish to be sure the six include at least one boy and 1 girl. How many ways can we do this? (A formula is fine as solution -- no need for calculation.)

Answer. The number of ways to choose all boys is C(6,6) = 1. The number of ways to choose all girls is C(8,5). The number of ways to choose without restriction is C(14,6). Therefore the number of ways that include at least one boy and at least one girl is C(14,6) - C(8,6) - C(6,6) (the complement of the number of ways to choose 6 unisex). This was on homework.

NOTE: Betty O'Neil tried a solution where she first chose one boy and one girl in 6*8 ways, then chose the remaining 4 from the remaining 12 boys and girls in C(12,4) ways. This gives the wrong answer (twice as large as the right answer), but I am giving 1/2 CREDIT (5 pts) for this approach.

9. How many different combinations of pennies, nickels, dimes and quarters can a piggy bank contain if it has 12 coins in it and at least one of each type of coin?

Answer. This is like putting 12 balls (coins) into 4 slots (different coins), where we start originally with 1 ball in each slot, so the result is C(8 + 3, 3), the number of ways to place pickets separating slots. This was on homework.

10. How many 5-card poker hands are there that contain three-of-a-kind, but not four-of-a-kind or a full house (i.e., the two side cards should not form a pair). Provide a numerical solution.

Answer. We can choose 3-of-a-kind by choosing from 13 denominations and then a suit to miss, so 13*4. We can choose the other two cards by choosing the first card from 12 denominations and the second from 11, and choosing a suit for each of them in 4 ways. Thus the answer is 13*4*12*4*11*4.

11. In tossing a coin 8 times in a row, we wish to derive the numerical probability of event E, that the first two flips are Heads (H), and event F, that an equal number of Heads and Tails appear, and finally the probability that both E and F happen together. From this, determine if E and F are independent.

Answer. The probability that the first two flips are Tails is 1/4. The probability that an equal number of Heads and Tails appear is C(8,4)/256 = 70/256. The product of these is 70/1024. Finally, the probability that they both occur together is the probability that last six digits contain 4 Heads and 2 Tails divided by the total number, or C(6,2)/256 = 15/256 = 60/1024, so since 60/1024 ≠ 70/1024, the two cases are independent.

12. Urn 1 contains 4 red balls and 6 blue balls and Urn 2 contains 2 red balls and 8 white balls.
You flip two coins and if you get two heads, you draw a ball from Urn 1, otherwise you draw from Urn 2. The result is that a red ball is drawn. Calculate the probability that the red ball came from Urn 1!

Answer. F = Draw from Urn 1, F^c from Urn 2; E = Red ball is drawn, E^c = non-Red ball is drawn.

$$p(F \mid E) = \frac{p(E \mid F)\, p(F)}{p(E \mid F)\, p(F) + p(E \mid F^c)\, p(F^c)} = \frac{0.4 * 0.25}{0.4 * 0.25 + 0.2 * 0.75} = 0.1 / (0.1 + 0.15) = 0.4$$

13.Look at the graph listed as 33 at the top right of pg. 645, but with the vertex g and the edge from b to g left out. Does the resulting graph have a Hamiltonian path? If so, find such a path, if not argue why such a path is impossible.

Answer. Yes. The path is: e, c, a, b, d, f, or instead: e, c, b, a, d, f.

14. Provide an adjacency list (defined on pg. 612, example there in TABLE 1) for the graph listed as 33 at the top right of pg. 645, but with the vertex g and the edge from b to g left out. (i.e., the same graph as in Problem 13 above). Let the order of vertices on the left and in each list on the right be a, b, c, d, e, f.

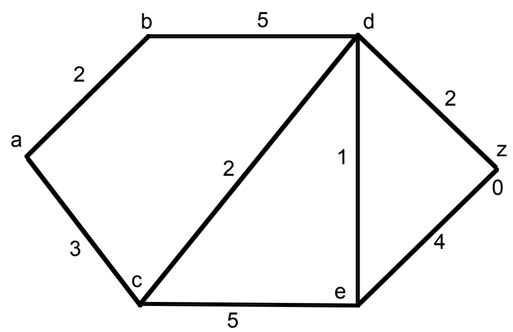
Answer.

Vertex	Adjacent Vertices
a	b, c, d
b	a, c, d
c	a, b, d, e
d	a, b, c, f
e	c
f	d

15. Name all pairs of values (m, n) for which $K_{m,n}$ has an Euler circuit and explain why.

Answer. Consider the bipartite graphs at the top of pg. 604, and assume that the m vertices of $K_{m,n}$ are at the top and the n vertices are at the bottom. If n is odd then every vertex at the top will have an odd number of edges leaving them, so there will be no Euler circuit; therefore n must be even. By the same argument, m must be even. If both m and n are even then a Hamiltonian circuit will exist.

16. Draw the weighted graph G given below, then find the path distance starting from z and going to all other vertices in the graph. Follow the approach of Figure 4 on pg. 652 except of course that you start with distance 0 at z and you only need to draw a single graph as long as you take the following steps. **(1)** You should cross out old distances at various vertices as you find newer, smaller distances; e.g., in Figure 4, starting from a, you would start with ∞ at z, then cross that out and put below it the smaller distance 14 once you look at edges out of node d with distance 8, then later cross out 14 and put below it 13 when you look at edges out of node e with distance 10. **(2)** You don't have to write path lists (such as in Figure 4(g) the list at z: (a, c, b, d, e)) at each vertex; you will know the order of vertices marked as minimum by the increasing order of their distances and can check the final distance values and determine paths in that way. Here is the graph G.



Answer. Note below: (1) Final distance to d is 2, direct from z. (2) Final distance to e is 3, path z, d, e. (3) Final distance to c is 4, path z, d, c. (4) Final distance to b is 7, path z, d, b. (5) Final distance to a is 7, path z, d, c, a.