CS/MA320 HW5 Solution, Fall 2009

Reading Assignments for the following Sections are in the Notes.

4.2 Exercises With Solutions: pg. 291: 5.a-c, 15.

Exercises For You To Solve: pg. 291: 6.a-c. 14. (HINT: in 6.a, generate successive integers that can be represented until successive integers are continuous. Be careful.)

6.a 3=3, 6=3+3, 9=3+3+3, 10=10, 12 = 3+3+3+3, 13=10+3, 15=3+3+3+3+3, 16=10+3+3, 18 = 3+3+3+3+3+3, 19=10+3+3+3, 20=10+10. We claim we can form all amounts of postage >= 18 cents. See proof in 6.b.

6.b Let P(n) be the statement that we can form n cents of postage with just 3-cent and 10-cent stamps. We want to prove that P(n) is true for all n >=18 using regular mathematical induction. The basis step, n = 18, is handled above. Assume P(k), and we will show P(k+1), i.e., that we can form k+1 cents of postage. If the k cents included two 10-cent stamps, replace them by seven 3-cent stamps. Otherwise, k cents was formed either form just 3-cent stamps, or from one 10-cent stamp and the remainder, k-10 cents, in 3-cent stamps. Because k >=18, there must be at least three 3-cent stamps involved in either case (one 10-cent stamp and the remainder in three 3-cent stamps occurs only for 19). Replace the three 3-cent stamps by one 10-cent stamp, and we have formed k+1 cents in postage.

6.c P(n) is the same as in part b. To prove that P(n) is true for all n>= 18, we note for the basis step that from part a, P(n) is true for n=18, 19, 20. Assume the strong inductive hypothesis, that P(j) is true for all j with 18 <= j <= k, where k is a fixed integer greater to or equal to 20. We want to show that P(k+1) is true. Because k>= 18, we know that P(k-2) is true, that is, that we can form k-2 cents of postage. Put one more 3-cent stamp on the envelope, and we have formed k+1 cents of postage, as desired. In this proof our strong inductive hypothesis included all values between 18 and k inclusive, and that enabled us to jump back three steps to a value for which we knew how to form the desired postage.

14. We prove this by strong induction. It is clearly true for n=1, because no splits are performed, so the sum computed is 0, which equals n(n-1)/2 when n=1. Assume the strong inductive hypothesis, and suppose that our first splitting is into piles of i stones and n-i stones, where i is a positive integer less than n. This gives a product i(n-i). The rest of the products will be obtained from splitting the piles thus formed, and so by the inductive hypothesis, the sum of the products will be i(i-1)/2 + (n-i)(n-i-1)/2. So we must show that

$$i(n-i) + i(i-1)/2 + (n-i)(n-i-1)/2 = n(n-1)/2$$

no matter what i is. This follows from elementary algebra, and our proof is complete.

4.3 Exercises With Solutions: pg. 309: 5.a-b, 35.

Exercises For You To Solve: pg. 309/10: 6.a&c. 38.

6.a: This is valid, since we are provided with the value at n = 0, and each subsequent value is determined by the previous one. Since all that changes from one value to the next is the sign, we conjecture that $f(n) = (-1)^n$. This is true for n=0, since $(-1)^0 = 1$. If it is true for n=k, then we have $f(k+1) = -f(k+1-1) = f(k) = -(-1)^k$ by the inductive hypothesis, whence $f(k+1) = (-1)^{k-1}$.

6.c. This is invalid. We are told that f(2) is defined in terms of f(3) or perhaps that f(3) is determined in terms of f(2), but neither f(2) nor f(3) has not yet been defined.

38. There are two types of palindromes, so we need two base cases, namely λ is a palindrome, and x is a palindrome for every symbol x. The recursive step is that if α is a palindrome and x is a symbol, the xα x is a palindrome.

4.4 Exercises With Solutions: pg. 321: 5, 9.

Exercises For You To Solve: pg. 321: 8, 16.

8. The sum of the first n positive integers is the sum of the first n-1 positive integers plus n. This trivial observation leads to the recursive algorithm shown here:

> **procedure** *sum of first*(n: positive integer)
>
> **if** n=1 **then** *sum of first*(n) := 1
>
> **else** *sum of first*(n) := *sum of first*(n-1) + n;

16. The sum of the first one positive integer is 1, and that is the answer to recursive algorithm gives when n=1, so the basis step is correct. Now assume that the algorithm works correctly for n=k. If n= k+1, then the else clause of the algorithm is executed, and k+1 is added to the (assumed correct) sum of the first k positive integers. Thus the algorithm correctly finds the sum of the first k+1 positive integers. More intuitively, we know that the sum of the first n positive numbers is (n(n+1))/2. For n = 1, (n(n+1))/2 = 1. Assume known for (k(k+1))/2 + (k + 1) = (k(k+1) +2k + 2)/2 = (k + k +2k + 2)/2 = ((k+1)(k+2))/2, which is the formula for k+1.

Skip Section 4.5.

5.1. Exercises With Solutions: pg. 345: 19.

Exercises For You To Solve: pg. 345: 20, 30.a-f, 32.a&b

20.a $\lfloor 999/7 \rfloor$ = FLOOR(999/7) = 142. Note we use the FLOOR function because the kth multiple of 7 does not occur until the number 7k has been reached, so fractions after dividing by 7 must be dropped.

Use Inclusion-exclusion for parts b-f.

20.b. There are $\lfloor 999/11 \rfloor$, = 90 numbers divisible by 11, and there are $\lfloor 999/7 \rfloor$, = 142 numbers divisible by 7. There are $\lfloor 999/77 \rfloor$, = 12 numbers divisible by both. Thus 141 – 12 = 130 numbers divisible by 7 but not by 11.

20.c. As explained in b, there are 12.

20.d. 142 + 90 – 12 = 220. (Divisible by one or the other or both.)

20.e. 142 -12 + 90 -12 = 208. (Divisible by one or the other but not by both.)

20.f. 999 – 220 (from d.) = 779.

20.g. Consider numbers written without leading zeroes and use 3 cases: one-digit, two-digit, and three-digit numbers. There are 9 one-digit numbers, all with distinct digits. There are 90 two-digit numbers, 10 thru 99, and all but 9 of them have distinct digits, so 81 with distinct digits. Alternatively, count 9 ways for first, then 9 ways for second (not matching first), 81 in all. Similarly for 3-digit numbers, 9*9*8 ways, 648 ways. Total is 9 + 81 + 648 = 738.

20.h. It is easier to count the odd numbers with distinct digits and subtract from result of g. There are 5 odd one-digit numbers. Two digits: 5 ways for 1s digit, then 8 ways for the 10s digit, which must avoid both the 1s digit and 0. So 5*8 = 40 ways for two digits. Three digits: 5 for 1s digit, 8 for 100s digit, then 8 for 10s digit that avoids both other digits, so 320 in all. Total: 5+40+320 = 365 odd numbers with distinct digits. Thus answer is 738 (from g.) – 365 = 373.

30. a By the product rule, the answer is 26^8 = 208,827,064,576.

30.b By the product rule, 26*25*24*...*19 = 62 billion plus.

30.c This is the same as a., except that there are only seven slots to fill, so 26^7 = 8 billion plus.

30.d This is similar to b., except that there is only one choice in the first slot, rather than 26, so the answer is 1*25*24*...*19 = 2.4 billion.

30.e Same as c, except that there are only six slots to fill, so 26^6 = 308 million.

30.f Same as e.; again there are six slots to fill, so 26^6 = 308 million.

32. In each case the answer is n^{10}, where n is the number of elements in the codomain, since there are n choices for a function value for each of the 10 elements in the domain.

32.a 2^{10} = 1024 b. 3^{10} = 59,049.

5.2 Exercises With Solutions: pg. 353: 9, 15, 21, 25.

Exercises For You To Solve: pg. 353/4: 8, 16, 22, 26, 40

8. This is just a restatement of the pigeonhole principle, with k = |T|.

16. We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 16, namely {1,15}, {3, 13}, {5, 11}, and {7,9}. If we select five numbers from the set { 1, 3, 5,..., 15} then at least two of them must fall within the same subset, since there are only four subsets. Two numbers in the same subset are the desired pair that adds up to 16. Four numbers in not enough, since we could choose {1,3,5,7}, and no pair adds up to more than 12.

22. This follows immediately from Theorem 3, with n=10.

26. Let A be one of the people. She must have either 10 friends or 10 enemies, since if there were 9 or fewer of each, then that would account for at most 18 of the 19 other people. Suppose A has 10 friends. By Exercise 25 there are either 4 mutual enemies among these 10 people, or 3 mutual friends. In the former case we have our desired set of 4 mutual enemies; in the latter case, these 3 people together with A form the desired set of 4 mutual friends. We can use a similar argument if A has 10 enemies.

40. Look at the pigeonholes {1000,1001}, {1002,1003}, {1004,1005}, ... {1098, 1099}. There are clearly 50 sets in this list. By the pigeonhole principle, if we have 51 numbers in the range 1000 to 1099 inclusive, then at least two of them must come from the same set. These are the desired two consecutive house numbers. If we had only 50 houses, the addresses could be chosen to be the max of each of the 2-integer pigeonholes, 1001, 1003, 1005,…, 1099, with no two successive.

5.3 Exercises With Solutions: pg. 361: 9, 11.a-b.

Exercises For You To Solve: pg. 361/2: 12.a&d, 30, 36.

12a. To specify a bit string of length 12 that contains exactly three 1s, we simply need to choose the three positions for the 1s. There are C(12,3) = 220 ways to do that.

d. To have an equal number of 0s and 1s in this case means to have six 1s. Therefore the asnwer is C(12,6) = 924.

30.a There are C(16,5) ways to select a committee if there are no restrictions. There are C(9,5) ways to select a committee of 5 men. Therefore there are C(16,5) – C(9,5) = 4368 – 126 = 4242 committees with at least one woman.

30.b There are C(16,5) ways to select a committee if there are no restrictions. There are C(9,5) ways to select a committee of 9 men. There are C(7,5) ways to select a committee from just the 7 women. These two possibilities do not overlap. Therefore there are C(16,5) –C(9,5) – C(7,5) = 4368 – 126 – 21 = 4221 committees with at least one woman and at least one man.

5.4 Exercises With Solutions: pg. 369/70: 3, 11, 13, 33.

Exercises For You To Solve: pg. 369/70: 4, 10 (HARD but see 11), 12, 34.

4. C(13, 8) = 1287.

10. By the Binomial Theorem, the typical term in this expansion is C(100,j)x^{100-j} $(1/x)^j$, which can be rewritten as C(100, j) x^{100-2j}. As j runs from 0 to100, the exponent runs from 100 down to -100 in decrements of 2. If we let k denote the exponent, then solving k= 100 - 2j for j we obtain j=(100-k)/2. Thus the values of k for which x^k appears in this expansion are -100, -98, ... -2, 0, 2, 4, ..., 100, and for such values of k the coefficient is C(100, (100-k)/2)).

12. We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course)

1 11 55 165 330 462 462 330 165 55 11 1

34. By exercise 33 there are C(n-k+k, k) = C(n,k) paths from (0,0) to (n-k,k) and C(k+n-k,n-k) = C(n,n-k) paths from (0,0 to (k,n-k). By symmetry, these two quantities must be the same (flip the picture around the 45 degree line).

5.5 Exercises With Solutions: pg. 379/80: 5, 15.a&c, 31, 37.

Exercises For You To Solve: pg. 379/80: 4, 16.a&c, 30, 32, 38.

4. There are 6 choices each of 7 times, so 6^7 = 279,936.

16.a We require each x_i >= 2. This uses up 12 of the 29 total required to form 6 bins containing at least 2 balls each; now we have to distribute the 17 remaining balls in the 6 bins, or in other words counting 5 separator pickets, permute 17+5 objects where 17 are alike of one kind and 5 are alike of the other. The number of solutions is therefore C(22,5) = C(22,17) = 26,334.

16.c The number of solutions with no restriction at all is C(29+5, 29) = C(34, 29) = 278256. The number of solutions violating the restriction by having x_1 >= 6 is calculated by starting with 6 of the 29 balls in one bin, then permuting remaining balls and pickets, so C(23+5,23) = C(28, 23) = 98280. Therefore, answer is 278256 – 98280 = 179,976.

30. By Theorem 3 the answer is 11!/(4!4!2!) = 34, 650.

32. We can treat the 3 consecutive As as one letter. Thus we have 6 letters, of which 2 are the same (the two Rs), so by Theorem 3 the answer is 6!/2! = 360.

38. We assume that the forty issues are distinguishable.

38.a Theorem 4 says that the answer is $40!/(10!^4) = 4.7 \times 10^{21}$.

38.b Each distribution into identical boxes gives rise to 4! = 24 distributions into labeled boxes, since once we have made the distribution into unlabeled boxes we can arbitrarily label the boxes. Therefore the answer is the same as the answer in a. divided by 24, about 2.0×10^{20}.