Homework 1
Due October 5, 2011

1. Let x, y, z be three words from A^* such that $xy = yz$ and $x \neq \lambda$. Prove that there exist $u, v \in A^*$ such that $x = uv$, $y = (uv)^n u$ and $z = vu$ for some $n \in \mathbb{N}$.

2. Define the mapping $f : A^* \longrightarrow A^*$ by

$$\begin{aligned} f(\lambda) &= \lambda \\ f(ax) &= xa, \end{aligned}$$

   for every $x \in A^*$ and $a \in A$. Prove that f is a bijection, that is, it is one-to-one and onto.

3. Let $A = \{a, b\}$. Prove that there are no words x, y in A^* such that $xay = ybx$.

   Prove that there is no word $x \in \{a, b\}^*$ such that $ax = xb$.

4. Two words $u, v \in A^*$ are *conjugate* if we can write $u = xy$ and $v = yx$ for some words $x, y \in A^*$.

   Define the binary relation κ, where $(x, y) \in \kappa$ if they are conjugate. Prove that κ is an equivalence relation on A^*.

5. A word on an alphabet A is *square-free* if it contains no infix of the form xx, where $x \in A^+$.

   (a) List all square-free words of length three over the alphabet $\{a, b\}$.

   (b) Show that for the alphabet $\{a, b\}$ there are no square-free words of length at least equal to 4.

   (c) Let $f : A \longrightarrow A$ be a one-to-one mapping. Prove that if x is square-free then so is $f(x)$.

6. Let A be an alphabet and let $a \in A$.

   (a) Prove that $a^{-1}A^* = a^{-1}A^+ = A^*$.

   (b) Prove that $a^{-1}A^n = A^{n-1}$.

Optional Exercise – Please hand it in separately from the regular homework.

Let $A = \{0, 1\}$ be an alphabet and let $L = A^*1A^n$. Prove that the collection of languages $\{x^{-1}L \mid x \in A^*\}$ consists of 2^{n+1} distinct languages.