CS624: Analysis of Algorithms

Assignment 4

Due: Saturday, Nov. 8, 2025

1. Determine an LCS of $< 1, 0, 0, 1, 0, 1, 0, 1 >$ and $< 0, 1, 0, 1, 1, 0, 1, 1, 0 >$. Of course you could probably solve this problem easily enough just by looking at it. What I want you to do is to explicitly use the algorithm presented in class. Write out your derivation neatly, including the table.

2. Suppose you are managing the construction of billboards on the highway, a heavily-traveled stretch of road that runs west-east for M miles. The possible sites for billboards are given by numbers $x_1, x_2, \ldots, x_n$, each in the interval $[0, M]$ (specifying their position along the highway, measured in miles from its western end). If you place a billboard at location x_i , you receive a revenue of $r_i > 0$. You want to place billboards at a subset of the sites in $x_1, \ldots, x_n$ so as to maximize your total revenue, subject to the following restrictions:

   - You cannot build two billboards within 5 miles or less of one another on the highway.
   - You cannot build a billboard within 5 miles or less of the western or eastern ends of the highway.

   A subset of sites satisfying these two restrictions will be called valid. For example, Suppose M $= 20$, n $= 4$, $x_1, x_2, x_3, x_4 = 6, 7, 12, 14$, and $r_1, r_2, r_3, r_4 = 5, 6, 5, 1$. Then the optimal solution would be to place billboards at x_1 and x_3 with a revenue of 10.

   (a) Describe a recursive algorithm that solves the problem (inefficiently). Hint: You either select x_n and collect the revenue or not... What is the solution in each case?

   (b) Show that the problem has optimal substructure and overlapping subproblems, and formulate the dynamic programming approach.

   (c) Describe the dynamic programming solution, including the array(s) for the example above.

3. Exercise 3.3 in the Lecture 8 handout (on page 12).

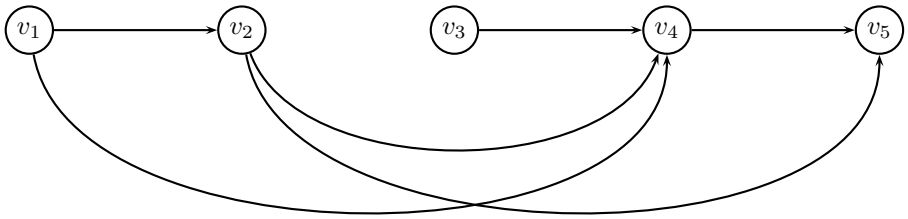
4. A palindrome is a nonempty string over some alphabet that reads the same forward and backward. Examples of palindromes are all strings of length 1, civic, racecar, and aibohphobia (fear of palindromes). Give an efficient algorithm to find the longest palindrome that is a subsequence of a given input string. For example, given the input character, your algorithm should return carac. What is the running time of your algorithm?

5. Let $G = (V, E)$ be a directed graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$. The set E of edges has the following property:

   - Each edge goes from a vertex of lower index to a vertex of higher index. For instance, there might be an edge (v_1, v_7) from v_1 to v_7. But there definitely will *not* be an edge (v_7, v_1) from v_7 to v_1.

- Each vertex except v_n has at least one edge leaving it. That is, for every vertex v_i ($1 \leq i \leq n-1$), there is at least one edge of the form (v_i, v_j) (of course with $i < j$).

Here is an example of a graph with this property:



It is called a DAG (directed acyclic graph). We will discuss them later on in the course. By a *path* we mean as usual a sequence of vertices $\{v_{i_1}, v_{i_2}, \ldots v_{i_k}\}$,

such that for each successive pair in the sequence there is an edge from the first to the second. (That is, there is an edge from v_{i_1} to v_{i_2}, another edge from v_{i_2} to v_{i_3}, and so on.)

The *length* of a path is just the number of edges in it. Equivalently, it is 1 less than the number of vertices in it.

We want to find an algorithm to solve the following problem, given such a graph:

> Find the length of the longest path that begins at v_1 and ends at v_n.

Here is a "greedy" algorithm that it is natural to think of at first:

Algorithm 1 FindPath

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1: w ← v_1  w is the current node we are considering.
2: L ← 0  L will hold the greatest length of any path so far.
3: while there is an edge out of w do
4:    Choose the edge (w, v_j) for which j is as small as possible.
5:    w ← v_j
6:    L ← L + 1
7: end while
8: return L
```

(a) Show that this algorithm gives the correct answer for the graph drawn above.

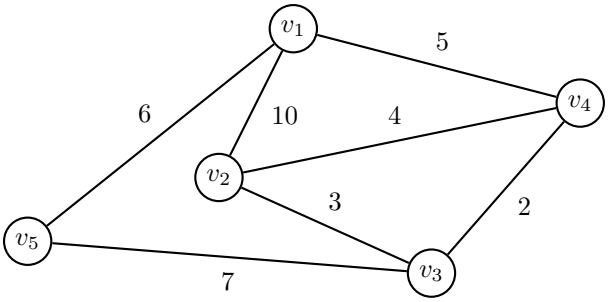
(b) Show that this algorithm does not give the correct answer in general.

(c) One could of course solve the problem in general by considering all possible paths from v_1 to v_n. Why is this not an efficient algorithm?

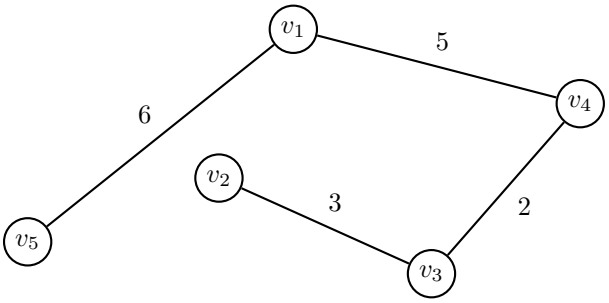
(d) Construct an efficient algorithm that does give the correct answer in general. (Hint: use dynamic programming. You have to prove that the algorithm is correct.)

(e) How efficient is your algorithm?

6. The minimum spanning tree problem is defined on undirected, weighted graphs as follows: a spanning tree of that graph is a subgraph that is a tree and connects all the vertices together. The minimum spanning tree is a spanning tree of minimum weight. For example, given the following graph:

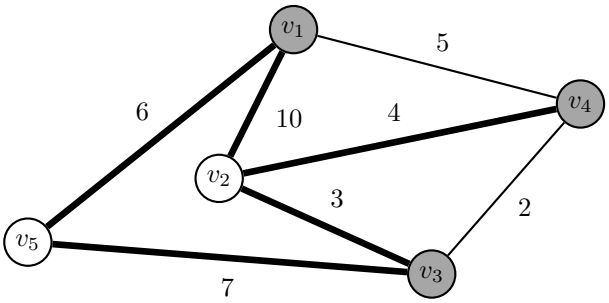


Its minimum spanning tree is:



(a) Show that the minimum spanning tree has an optimal substructure by showing that given an MST for a graph $G = (V, E)$, any sub-tree of the MST is a minimum spanning tree with respect to the subset of vertices and their edges in G.

(b) A cut in a graph $G = (V, E)$ is a division of the vertices into two disjoint non-empty subsets, S and T. An edge that crosses the cut is an edge that has one vertex in S and one vertex in T. An example here shows a cut in the graph above, where $S = \{v_1, v_3, v_4\}$ and $T = \{v_2, v_5\}$. Edges crossing the cut are shown in bold. The greedy choice property says that for any cut in the graph, the lightest edge crossing the cut is a part of a minimum spanning tree. Prove it is true for any cut (not only this example!). You can show by a simple exchange argument.



(c) Show that the lightest edge in the graph is always a part of an MST (there may be more than one MST if the weights are not unique).

7. Given a tree $T = (V, E)$ (not necessarily binary) an *independent set* is subset of nodes in the tree such that no two nodes are adjacent (in a tree it means, no parent and child can be in the same independent set). The problem of finding the maximum independent set in a tree involves finding an independent set of maximum size (obviously).

(a) Show that the problem has an optimal substructure by showing that given a maximum independent set on a tree, every subset of it represents a maximum independent set with respect to its subtree.

(b) Every tree has at least one leaf. Show that any leaf node v in a tree must be a part of a maximum size independent set. **Hint:** Assume we have a maximum independent set S on the tree. It either contains v of not, take it from there.

(c) Give a linear time algorithm to obtain a maximum size independent set in a tree.

8. Prove that a binary tree that is not full (i.e., that at least one of its nodes has only one child) cannot correspond to an optimal prefix code.

9. Suppose that a data file contains a sequence of 8-bit characters such that all 256 characters are about equally common: the maximum character frequency is less than twice the minimum character frequency. Prove that Huffman coding in this case is no more efficient than using an ordinary 8-bit fixed-length code.