CS624 - Analysis of Algorithms

DFS and DAGs

November 10, 2025

Depth-First Search (DFS)

- Input: $G = (V, E)$, directed or undirected. No source vertex given!
- Output: 2 timestamps on each vertex. Integers between 1 and $2|V|$.
- $d[v]$ = discovery time (v turns from white to gray)
- $f[v]$ = finishing time (v turns from gray to black)
- $\pi[v]$ = predecessor of v. A vertex u such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS.

The DFS Algorithm

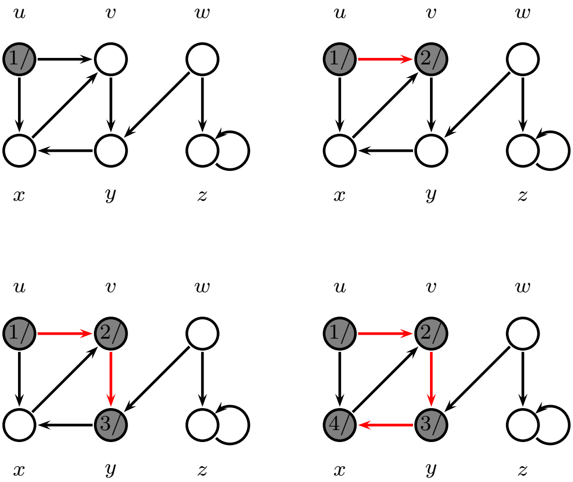
Algorithm 1 $DFS(G)$

```
for each u ∈ V[G] do
    color[u] ← white
    π[u] ← NIL
end for
time ← 0
for each u ∈ V[G] do
    if color[u] == white then
        DFS − Visit(u)
    end if
end for
```

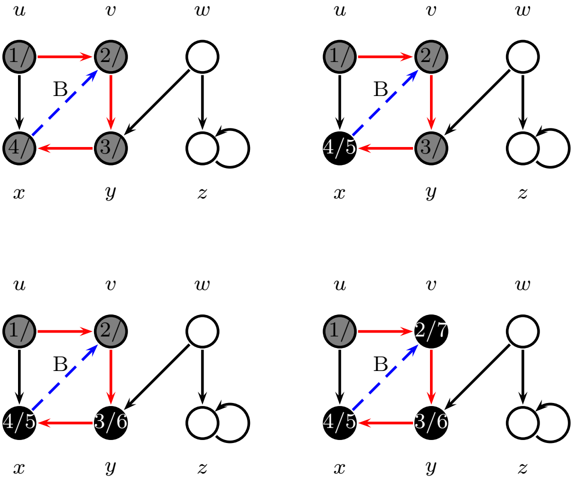
Algorithm 2 $DFS - Visit(u)$

```
color[u] ← GRAY
time ← time + 1
d[u] ← time
for each v ∈ Adj[u] do
    if color[v] == WHITE then
        π[v] ← u
        DFS − Visit(v)
    end if
end for
color[u] ← BLACK
f[u].time ← time + 1
```

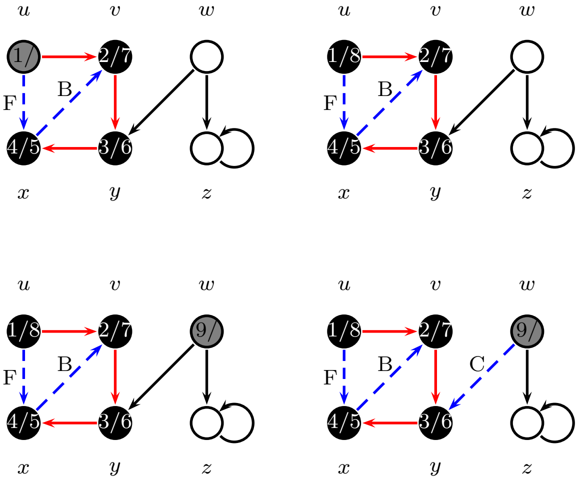
The DFS Algorithm – Example



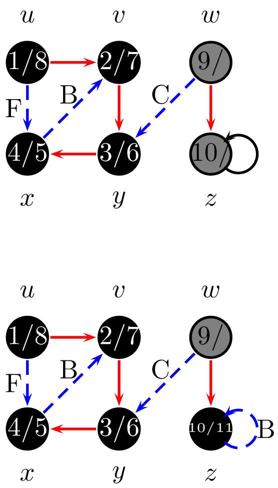
The DFS Algorithm – Example



The DFS Algorithm – Example



The DFS Algorithm – Example



The DFS Algorithm – Runtime and Properties

- The loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in$ V when it's painted gray the first time.
- Lines 3-6 of DFS-Visit is executed —Adj[v]— times. The total cost of executing DFS-Visit is $\sum_{v \in V|Adj[v]|} = \Theta(E)$.
- Total running time of DFS is $\Theta(V + E)$.

The Parenthesis Theorem

Theorem

For all u, v, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ *or* $d[v] < f[v] < d[u] < f[u]$ *and neither u nor v is a descendant of the other.*
2. $d[u] < d[v] < f[v] < f[u]$ *and v is a descendant of u.*
3. $d[v] < d[u] < f[u] < f[v]$ *and u is a descendant of v.*

- So $d[u] < d[v] < f[u] < f[v]$ cannot happen, just like parentheses.
- OK: ( ) [ ] ([ ]) [( )]
- Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u iff $d[u] < d[v] < f[v] < f[u]$.

The Parenthesis Theorem

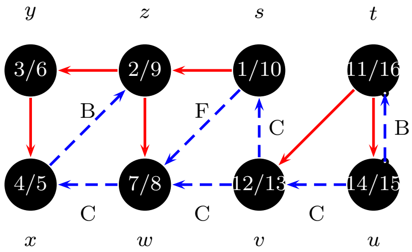
Proof.

- If *start*$[x] <$ *start*$[y] <$ *finish*$[x]$ then x is on the stack when y is first reached.
- Therefore the processing of y starts while x is on the stack, and so it also must finish while x is on the stack:
- we have *start*$[x] <$ *start*$[y] <$ *finish*$[y] <$ *finish*$[x]$.
- The case when *start*$[y] <$ *start*$[x] <$ *finish*$[y]$ is handled in the same way.

$\square$

- Another way to state the parenthesis nesting property is that given any two nodes x and y, the intervals [start[x], finish[x]] and [start[y], finish[y]] must be either nested or disjoint.

The Parenthesis Theorem – Example



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

Depth First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq NIL\}$.
- How does it differ from that of BFS?
- The predecessor subgraph G_π forms a depth-first forest composed of several depth-first trees.
- The edges in E_π are called tree edges.

Definition (Forest)

An acyclic graph G that may be disconnected.

White Path Theorem

Theorem

v is a tree descendant of u if and only if at time d[u], there is a path $u \rightsquigarrow v$ consisting of only white vertices (Except for u, which was just colored gray.)

Proof.

One direction: (if v is a tree descendant of u then there is a white path $u \rightsquigarrow v$ at time d[u]) is obvious from the definition of a tree descendant (see the parenthesis theorem). $\square$

White Path Theorem

Cont. – Reverse Direction.

- Is it possible that v is not a descendant of u in the DFS forest?
- By induction on all the vertices along the path: Of course u is a descendant of itself.
- Let us pick any vertex p on the path other than the first vertex u, and let q be the previous vertex on the path [so it can be that q is u].
- We assume that all vertices along the path from u to q inclusive are descendants of u (inductive hypothesis).
- We will argue that p is also a descendant of u.

White Path Theorem

Cont. – Reverse Direction.

- At time d[u] vertex p is white [by assumption about the white path], So $d[u] < d[p]$.
- But there is an edge from q to p, so q must explore this edge before finishing.
- At the time when the edge is explored, p can be:
- **WHITE**, then p becomes a descendant of q, and so of u.
- **BLACK**, then $f[p] < f[q]$ [because f[p] must have already been assigned by that time, while f[q] will get assigned later].
- But since q is a descendant of u [not necessarily proper], $f[q] \leq f[u]$, we have $d[u] < d[p] < f[p] < f[q] \leq f[u]$, and we can use the Parenthesis theorem to conclude that p is a descendant of u.

□

White Path Theorem

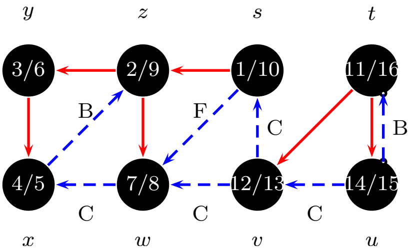
Cont. – Reverse Direction.

- **GRAY**, then p is already discovered, while q is not yet finished, so $d[p] < f[q]$.
- Since q is a descendant of u [not necessarily proper], by the Parenthesis theorem, $f[q] \leq f[u]$.
- Hence $d[u] < d[p] < f[q] \leq f[u]$. So d[p] belongs to the set $\{d[u], \ldots, f[u]\}$, and so we can use the the Parenthesis theorem again to conclude that p must be a descendant of u.
- The conclusion thus far is that p is a descendant of u. Now, as long as there is a vertex on the remainder of the path from p to v, we can repeatedly apply the inductive argument, and finally conclude that the vertex v is a descendant of u, too.

$\square$

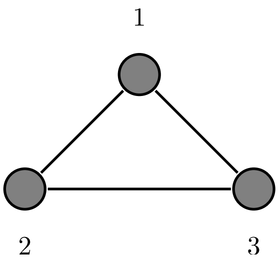
Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.
- Edge type for edge (u, v) can be identified when it is first explored by DFS based on the color of v.
- White – tree edge. Gray – back edge. Black – forward or cross edge.



The edge $x \rightarrow z$ will be discovered when exploring x, hence it's a back edge.

Classification of Edges



Theorem

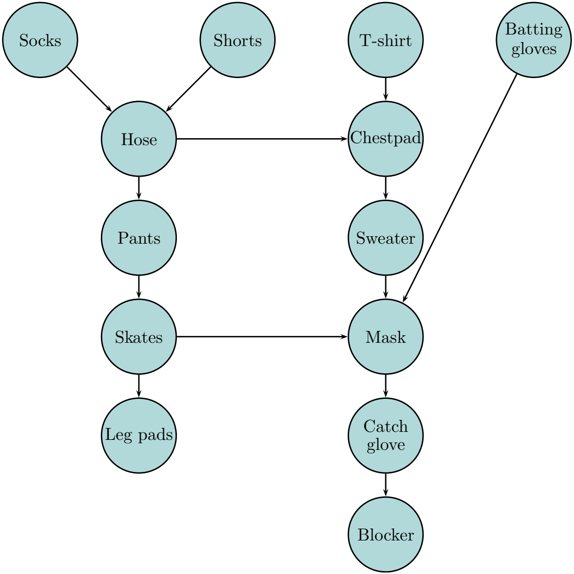
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Starting from 1, either 2 discovers 3 or vice versa, therefore one of them is the other's descendant, Hence no cross edges.

Directed Acyclic Graph (DAG)

- DAG – Directed graph with no cycles.
- Good for modeling processes and structures that have a partial order:
- $a > b$ and $b > c \Rightarrow a > c$.
- But may have a and b such that neither $a > b$ nor $b > a$.
- Can always make a total order (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order.

Directed Acyclic Graph (DAG) – Example



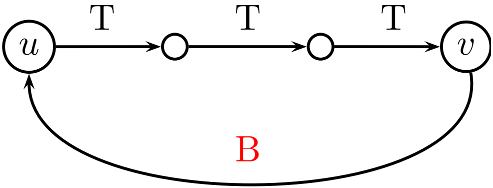
Characterizing a DAG

Lemma

A directed graph G is acyclic iff a DFS of G yields no back edges.

Proof.

$\Rightarrow$ Show that back edge $\rightarrow$ cycle:
Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest. Therefore, there is a path $v \rightsquigarrow u$, so $v \rightsquigarrow u \rightsquigarrow v$ is a cycle. □



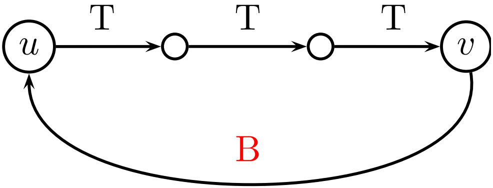
Characterizing a DAG

Proof.

$\Rightarrow$: Show that a cycle implies a back edge.

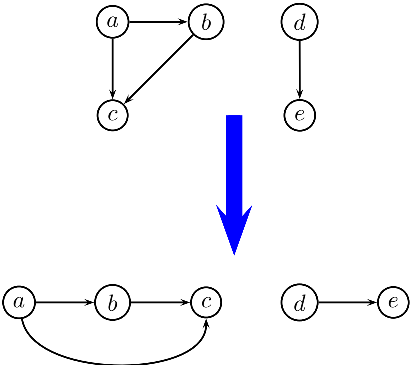
- c : cycle in G, u : first vertex discovered in c, (v, u) : preceding edge in c.
- At time d[v], vertices of c form a white path $u \rightsquigarrow v$. Why?
- By white-path theorem, v is a descendant of u in depth-first forest.
- Therefore, (v, u) is a back edge.

$\square$



Topological Sorting

- We want to "sort" a DAG.
- Think of original DAG as a partial order.
- We want a total order that extends this partial order.



Topological Sorting

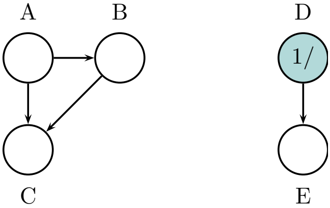
- Performed on a DAG.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

TopologicalSort(G)

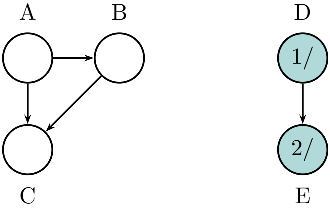
1. call DFS(G) to compute finishing times f[v] for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices

Runtime – $\Theta(V + E)$

Topological Sorting – Example

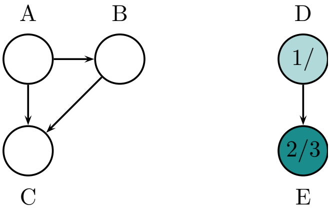


Linked list:



Linked list:

Topological Sorting – Example



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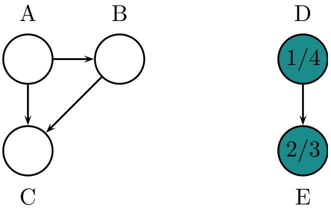

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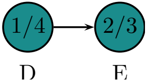
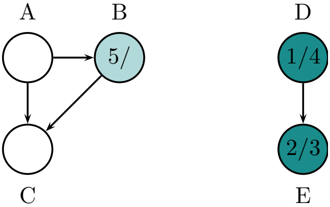
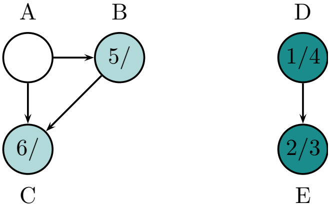


Topological Sorting – Example





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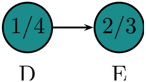

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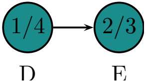
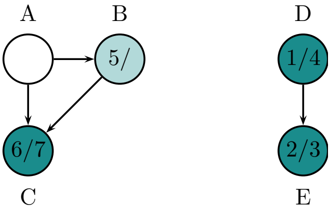
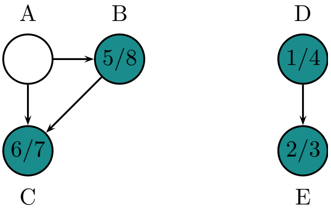


Topological Sorting – Example





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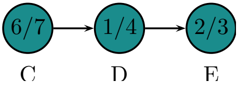

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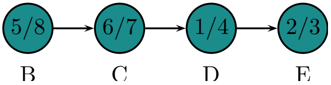

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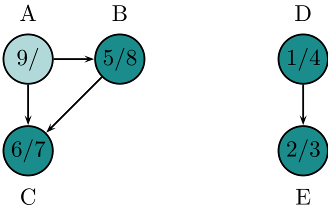

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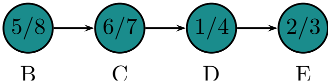

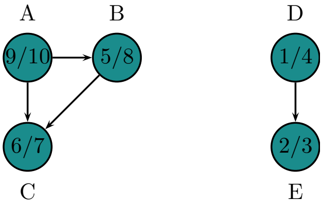

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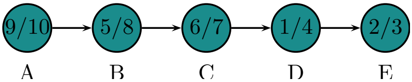
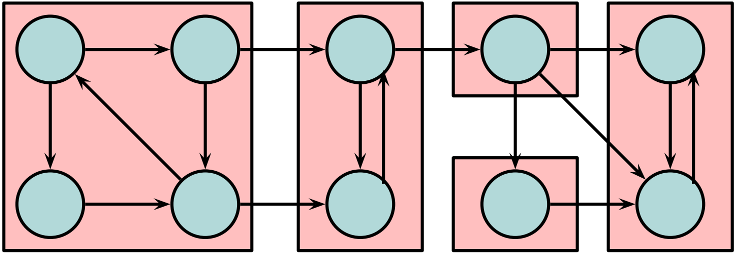


Topological Sorting – Proof of Correctness

- Just need to show if $(u, v) \in E$, then $f[v] < f[u]$.
- When we explore (u,v) then u is gray. What is the color of v?
- Is v **gray**?
- No, because then v would be ancestor of u. $\Rightarrow (u, v)$ is a back edge, which contradicts the fact that A DAG has no back edges.
- Is v **white**?
- Then becomes descendant of u.
- By parenthesis theorem, $d[u] < d[v] < f[v] < f[u]$.
- Is v **black**?
- Then v is already finished.
- Since we're exploring (u,v), we have not yet finished u.
- Therefore, $f[v] < f[u]$.

Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.
- A strongly connected component (SCC) of G is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, there is a path from u to v and from v to u.



$G^{SCC} isaDAG$

Theorem

Let C and C' be distinct SCC's in G, let u, $v \in C, u', v' \in C'$, and suppose there is a path $u \leadsto u'$ in G. Then there cannot also be a path $v' \leadsto v$ in G.

Proof.

- Suppose there is a path from v' to v in G.
- Then there are paths from u to u' to v' and from v' to v to u in G.
- Therefore, u and v' are reachable from each other, so they are not in separate SCC's.

□

Transpose of a Directed Graph

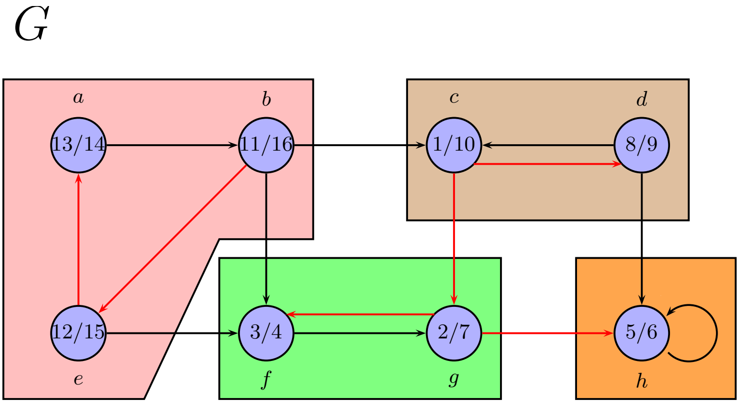
- G^T = transpose of directed G.
- $G^T = (V, E^T), E^T = (u, v) : (v, u) \in E$.
- G^T is G with all edges reversed.
- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.
- G and G^T have the same SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^T).

Algorithm to Determine SCC

1. Call $DFS(G)$ to compute finishing times $f[u]$ for all u
2. Compute G^T
3. Call $DFS(G^T)$, but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. Output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

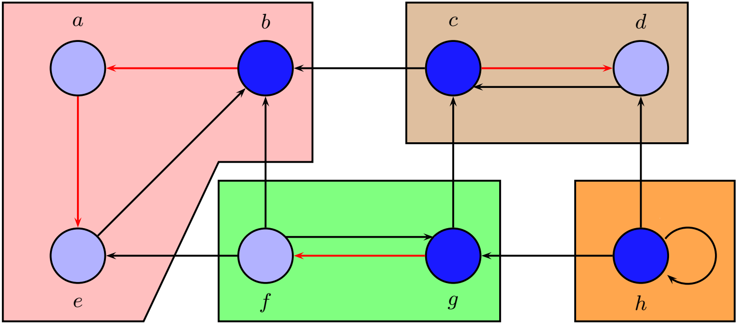
Runtime – $\Theta(V + E)$

Example

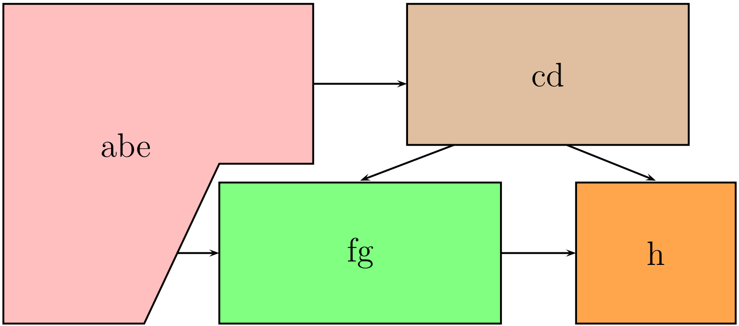


Example

G^T



Example



How Does it Work?

- Idea:
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - Because we are running DFS on G^T, we will not be visiting any v from a u, where v and u are in different components.
- Notation:
  - d[u] and f[u] always refer to first DFS.
  - Extend notation for d and f to sets of vertices $U \subseteq V$:
  - $d(U) = \min_{u \in U}\{d[u]\}$ (earliest discovery time)
  - $f(U) = \max_{u \in U}\{f[u]\}$ (latest finishing time)

SCCs and DFS finishing times

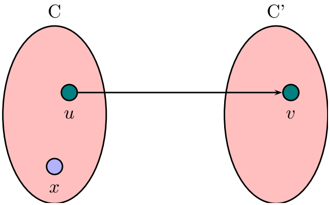
Lemma

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Case 1: $d(C) < d(C')$.

- Let x be the first vertex discovered in C.
- At time d[x], all vertices in C and C' are unvisited. Thus, there exist paths of unvisited vertices from x to all vertices in C and C'.
- All vertices in C and C' are descendants of x in depth-first tree.
- Therefore, $f[x] = f(C) > f(C')$.

□



SCCs and DFS finishing times

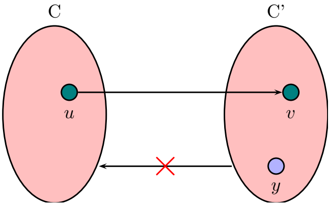
Lemma

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Case 2: $d(C) > d(C')$.

- Let y be the first vertex discovered in C'.
- At time d[y], all vertices in C' are unvisited and there is an unvisited path from y to each vertex in C'.
- All vertices in C' become descendants of y. Again, f [y] = f (C').
- At time d[y], all vertices in C are also unvisited.

□



SCCs and DFS finishing times

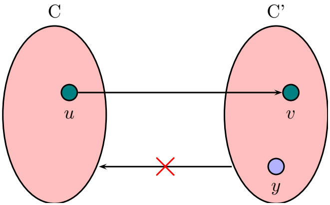
Lemma

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Case 2: $d(C) > d(C')$.

- By earlier lemma, since there is an edge (u, v), we cannot have a path from C' to C.
- So no vertex in C is reachable from y.
- Therefore, at time f[y], all vertices in C are still white.
- Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.

□



SCCs and DFS finishing times

Corollary

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof.

$(u, v) \in E^T \Rightarrow (v, u) \in E$. Since SCC's of G and G^T are the same, $f(C') > f(C)$, by former Lemma. □

Correctness of SCC

- When we do the second DFS, on G^T, start with SCC C such that f(C) is maximum.
- The second DFS starts from some $x \in C$, and it visits all vertices in C.
- Corollary above says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
- Therefore, DFS will visit only vertices in C.
- Which means that the depth-first tree rooted at x contains exactly the vertices of C.

Correctness of SCC

- The next root chosen in the second DFS is in SCC C' such that f(C') is maximum over all SCC's other than C.
- DFS visits all vertices in C', but the only edges out of C' go to C, which we've already visited.
- Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only vertices in its SCC–get tree edges to these,
- Vertices in SCC's already visited in second DFS–get no tree edges to these.