Let us finally solve Example III by finding the transitive closure of the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$.

$R$ can be represented by the following matrix $M_R$:

$$
M_R = \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Solution: The transitive closure of the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$ is given by the relation

$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$

Proof: We'll use induction.

Base case: $k=0$. $W_0 = W_R$ because there can be no interior vertices, so just a single edge.

Induction step: If true for $k-1$, show $w^{[k]}_{ij} = w^{[k-1]}_{ij} \lor (w^{[k-1]}_{ik} \land w^{[k-1]}_{kj})$ because there is a path from $v_i$ to $v_j$ using interior vertices from $\{v_1, v_2, \ldots v_k\}$ if:

- There is a path without $v_k$ as an interior vertex (so $w^{[k-1]}_{ik} = 1$) or
- There is path with $v_k$ as an interior vertex, in which case both $w^{[k-1]}_i$ and $w^{[k-1]}_k$ are 1. (there must be a k-1 path from $v_i$ to $v_k$ and from $v_k$ to $v_j$)

A more efficient way of computing the transitive closure of a relation with digraph on vertices $\{v_1, v_2, \ldots, v_n\}$:

Theorem (p. 606). Let $W_k = (w^{[k]}_{ij})$ be the 0,1 matrix $w^{[k]}_{ij} = 1$ iff there is a path from $v_i$ to $v_j$ with any interior vertices in the set $\{v_1, v_2, \ldots v_k\}$. Then

$w^{[k]}_{ij} = w^{[k-1]}_{ij} \lor (w^{[k-1]}_{ik} \land w^{[k-1]}_{kj})$

$W_0 = W_R$, $W_n = W_{R^*}$.

Using Warshall’s Algorithm

As shown in the book, the formula giving Warshall’s Algorithm easily translates to computer code.

If you do it by hand, just note that in $w^{[k]}_{ij} = w^{[k-1]}_{ij} \lor (w^{[k-1]}_{ik} \land w^{[k-1]}_{kj})$ you go from $W_{k-1}$ to $W_k$ by looking at the matrix for $W_{k-1}$. If you can go from $v_i$ to $v_j$ in $W_{k-1}$ then in $W_k$ you can add an entry $ij$ if $v_k$ goes to $v_j$ in $W_{k-1}$. (this is easier than it sounds)
Transitive Closure
via Warshall's Algorithm

\[ W_0 = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ W_1 = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ W_2 = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ W_3 = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ M_f = W_f = W_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]