**Matrices**

A matrix is a rectangular array of numbers. A matrix with \( m \) rows and \( n \) columns is called an \( m \times n \) matrix.

Example: \[
A = \begin{bmatrix}
-1 & 1 \\
2.5 & -0.3 \\
8 & 0
\end{bmatrix}
\]

is a 3×2 matrix.

A matrix with the same number of rows and columns is called square.

Two matrices are equal if they have the same number of rows and columns and the corresponding entries in every position are equal.

**Matrices**

A general description of an \( m \times n \) matrix \( A = [a_{ij}] \):

\[
\begin{bmatrix}
\cdots \\
\vdots \\
\cdots
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
\cdots \\
\vdots \\
\cdots
\end{bmatrix}
\]

In other words, \( A = [a_{ij}] \).

**Matrix Addition**

Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be \( m \times n \) matrices. The sum of \( A \) and \( B \), denoted by \( A + B \), is the \( m \times n \) matrix that has \( a_{ij} + b_{ij} \) as its \((i, j)\)th element.

In other words, \( A + B = [a_{ij} + b_{ij}] \).

Example:

\[
\begin{bmatrix}
-2 & 1 \\
4 & 8 \\
-3 & 0
\end{bmatrix} + \begin{bmatrix}
5 & 9 \\
-3 & 6 \\
-4 & 1
\end{bmatrix} = \begin{bmatrix}
3 & 10 \\
1 & 14 \\
-7 & 1
\end{bmatrix}
\]

**Matrix Multiplication**

Let \( A \) be an \( m \times k \) matrix and \( B \) be a \( k \times n \) matrix. The product of \( A \) and \( B \), denoted by \( AB \), is the \( m \times n \) matrix with \((i, j)\)th entry equal to the sum of the products of the corresponding elements from the \(i\)-th row of \( A \) and the \(j\)-th column of \( B \).

In other words, if \( AB = [c_{ij}] \), then

\[
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj} = \sum_{k=1}^{k} a_{ik}b_{kj}
\]

**Matrix Multiplication**

A more intuitive description of calculating \( C = AB \):

- Take the first column of \( B \)
- Turn it counterclockwise by 90° and superimpose it on the first row of \( A \)
- Multiply corresponding entries in \( A \) and \( B \) and add the products: \( 3 \cdot 2 + 0 \cdot 0 + 1 \cdot 3 = 9 \)
- Enter the result in the upper-left corner of \( C \)

- Now superimpose the first column of \( B \) on the second, third, ..., \( m\)-th row of \( A \) to obtain the entries in the first column of \( C \) (same order).
- Then repeat this procedure with the second, third, ..., \( n\)-th column of \( B \), to obtain to obtain the remaining columns in \( C \) (same order).
- After completing this algorithm, the new matrix \( C \) contains the product \( AB \).
Matrix Multiplication

Let us calculate the complete matrix $C$:

$$
A = \begin{bmatrix}
3 & 0 & 1 \\
-2 & -1 & 4 \\
0 & 0 & 5 \\
\end{bmatrix}, \quad
B = \begin{bmatrix}
2 & 1 \\
0 & -1 \\
3 & 4 \\
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
9 & 7 \\
8 & 15 \\
15 & 20 \\
-2 & -2 \\
\end{bmatrix}
$$

Identity Matrices

The identity matrix of order $n$ is the $n \times n$ matrix

$$
I_n = [\delta_{ij}], \text{ where } \delta_{ij} = 1 \text{ if } i = j \text{ and } \delta_{ij} = 0 \text{ if } i \neq j:
$$

$$
A = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
$$

Multiplying an $m \times n$ matrix $A$ by an identity matrix of appropriate size does not change this matrix:

$$
A I_n = I_m A = A
$$

Powers and Transposes of Matrices

The power function can be defined for square matrices. If $A$ is an $n \times n$ matrix, we have:

$$
A^0 = I_n, \\
A^r = A A A \ldots A \ (r \text{ times the matrix } A)
$$

The transpose of an $m \times n$ matrix $A = [a_{ij}]$, denoted by $A^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $A$.

In other words, if $A^t = [b_{ji}]$, then $b_{ji} = a_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

A square matrix $A$ is called symmetric if $A = A^t$.

Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Example:

$$
A = \begin{bmatrix}
2 & 0 & -1 \\
1 & 3 & 4
\end{bmatrix}, \quad
A' = \begin{bmatrix}
2 & 0 & 3 \\
1 & -1 & 4
\end{bmatrix}
$$

A square matrix $A$ is called symmetric if $A = A'$.

Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$$
A = \begin{bmatrix}
5 & 1 & 3 \\
1 & 2 & -9 \\
3 & -9 & 4
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1
\end{bmatrix}
$$

A is symmetric, $B$ is not.