Homework 2

Posted: February 21, 2024
Due: March 6, 2024

1. Let $S^{n\times n}$ be the set of $n \times n$ symmetric matrices in $\mathbb{R}^{n\times n}$. Prove that $S^{n\times n}$ is a subspace of $\mathbb{R}^{n\times n}$ and $\dim(S^{n\times n}) = \frac{n(n+1)}{2}$.
2. Let $A, B \in \mathbb{C}^{n\times n}$ be two matrices. Prove that if $AB = BA$, then we have the following equality known as *Newton's binomial*:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k.$$

   Give an example of matrices $A, B \in \mathbb{C}^{2\times 2}$ such that $AB \neq BA$ for which the above formula does not hold.
3. Let $M \in \mathbb{R}^{n\times n}$ be a partitioned matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

   where $A \in \mathbb{R}^{m\times m}$ and $m < n$. Prove that, if all inverse matrices mentioned next exist and $Q = (A - BD^{-1}C)^{-1}$, then

$$M^{-1} = \begin{pmatrix} Q & -QBD^{-1} \\ -D^{-1}CQ & D^{-1} + D^{-1}CQBD^{-1} \end{pmatrix}.$$

4. Prove or disprove:
   (a) If $\mathbf{0} \in W$, where $W = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$, then W is linearly independent.
   (b) If $W = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ is linearly independent and $\boldsymbol{w}$ is not a linear combination of the vectors of W, then $W \cup \{\boldsymbol{w}\}$ is linearly independent.

(c) If $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ is linearly dependent, then any of $\boldsymbol{w}_i$ is a linear combination of the others.

(d) If $\boldsymbol{y}$ is not a linear combination of $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$, then $\{\boldsymbol{y}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ is linearly independent.

(e) If any $n-1$ vectors of the set $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ are linearly dependent, then W is linearly independent.

5. An *involutive matrix* is a matrix $A \in \mathbb{C}^{n\times n}$ such that $A^2 = I_n$. An *idempotent matrix* is a matrix $B \in \mathbb{C}^{n\times n}$ such that $B^2 = B$.
   Prove that if $B \in \mathbb{C}^{n\times n}$ is an idempotent matrix, then $A = 2B - I_n$ is an involutive matrix.