CS724: Topics in Algorithms
Problem Set 1

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Problem 1:

The set of polynomials over a field $\mathbb{F}$, $\mathbb{F}[x]$, consists of all functions $f : \mathbb{F} \longrightarrow \mathbb{F}$ of the form $f(x) = c_0 + c_2 x + \cdots + c_n x^n$, where $c - 0, c_1, \ldots, c_n$ are fixed scalars from $\mathbb{F}$.
Prove that the set $\{x, x^2\}$ is linearly independent in $\mathbb{R}[x]$.

Solution 1:

Suppose that $ax + bx^2 = \mathbf{0}$ for $x \in \mathbb{R}$, where $\mathbf{0}(x) = 0$ for $x \in rr$. This holds for every a and b, so we can write

$$\begin{aligned} a + b &= 0 (\text{by taking } x = 1) \\ -a + b &= 0 (\text{by taking } x = -1), \end{aligned}$$

which implies $a = b = 0$.

Problem 2:

Prove that the set of complex numbers $\mathbb{C}$ can be regarded as a linear space over the field $\mathbb{R}$ of real numbers.

Solution 2: Let $u = a + ib$ and $v = c + id$ be two complex numbers. Their sum is $u + v = (a + c) + i(b + d)$; for $\alpha \in \mathbb{R}$, the product αu is $\alpha u = \alpha a + i\alpha b$. The addition and multiplication of complex numbers satisfy the definition of a complex space over $\mathbb{R}$.

Problem 3:

Let W_1, W_2 be subspaces of a real linear space V such that the set union $W_1 \cup W_2$ is also a subspace. Prove that one of the subspaces W_i is included in the other.

Solution 3:

Suppose that $W_1 \cup W_2$ is a subspace but neither subspace is contained in the other. Then, there exist $x \in W_1 - W_2$ and $y \in W_2 - W_1$. We claim that $x + y$ cannot be in either subspace, hence it cannot be in their union $W_1 \cup W_2$, which is a contradiction.
If $x + y \in W_1$, then $(x + y) - x \in W_1$, but this is y and we have a contradiction. Similarly, $x + y$ cannot belong to W_2.

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Problem 4

Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$; let V_{even} be the subset of all even functions, $f(x) = f(-x)$; let V_{odd} be the subset of all odd functions, $f(-x) = -f(x)$.

Prove that:

- V_{even} and V_{odd} are subspaces of V;
- $V_{even} + V_{odd} = V$;
- $V_{even} \cap V_{odd} = \{0\}$.

Solution 4

Suppose that $f, ginV_{even}$, that is $f(x) = f(-x)$ and $g(x) = g(-x)$. Then

$$(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$$

so $f + g$ is even. Also, $(af)(x) = af(x) = af(-x)$, so V_{even} is a subspace. A similar argument works for V_{odd}.

Solution 4 cont'd

If $h \in V$ we can write h as

$$h(x) = \frac{h(x) + h(-x)}{2} + \frac{h(x) - h(-x)}{2}$$

The first function is even, and the second is odd, so $V_{even} + V_{odd} = V$.
If $f \in V_{even} \cap V_{odd}$ we have both $f(x) = f(-x)$ and $f(-x) = -f(x)$.
Therefore, $f(x) = 0$, and f is 0.

Problem 5

Let W_1 and W_2 be subspaces of a vector space V such that

$$W_1 + W_2 = \{u + v \mid u \in W_1, v \in W_2\} = V,$$

and $W_1 \cap W_2 = \{\mathbf{0}\}$. Prove that each vector v in V can be *uniquely* written as a sum $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

Solution 5

By the definition of $W_1 + W_2$ it is clear that each vector v in V can be written as a sum $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. What needs to be shown is the uniqueness part.
Suppose that v can be written as:

$$v = w_1 + w_2 = \tilde{w}_1 + \tilde{w}_2,$$

where $w_1, \tilde{w}_1 \in W_1$ and $w_2, \tilde{w}_2 \in W_2$.
Since $w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2$, $w_1 - \tilde{w}_1 \in W_1$, $\tilde{w}_2 - w_2 \in W_2$ it follows that these vector differences belong to $W_1 \cap W_2 = \{\mathbf{0}\}$, which means that $w_1 - \tilde{w}_1 = \mathbf{0}$ and $\tilde{w}_2 - w_2 = \mathbf{0}$. Thus, $w_1 = \tilde{w}_1$ and $\tilde{w}_2 = w_2$, which proves the uniqueness.

Things to remember when you do homework:

- name your file as "hw1-John.Doe.pdf"; this would allow me to recognize your file in the mail;
- write neatly, using latex;
- use clear and correct English;
- do not use the expression "it is easy to see"; fully justify your statements.