CS724: Topics in Algorithms
Problem Set 4

Prof. Dan A. Simovici

Problem 1:

Let ν be a norm on $\mathbb{C}^n$. Prove that there exists a number $k \in \mathbb{R}$ such that for any vector $\boldsymbol{x} \in \mathbb{C}^n$ we have $\nu(\boldsymbol{x}) \leqslant k \sum_{i=1}^{n} |x_i|$.

Solution 1:

Starting from the equality $\boldsymbol{x} = \sum_{i=1}^n x_i \boldsymbol{e}_i$ we have $\nu(\boldsymbol{x}) \leqslant \sum_{i=1}^n \nu(x_i \boldsymbol{e}_i) = \sum_{i=1}^n |x_i| \nu(\boldsymbol{e}_i) \leqslant k \sum_{i=1}^n |x_i|$, where $k = \max\{\nu(\boldsymbol{e}_i) \mid 1 \leqslant i \leqslant n\}$.

Problem 2:

If $T \subseteq V$, then the set $T^{\perp}$ is defined by:

$$T^{\perp} = \{\boldsymbol{v} \in V \mid \boldsymbol{v} \perp \boldsymbol{t} \text{ for every } \boldsymbol{t} \in T\}$$

Note that $T \subseteq U$ implies $U^{\perp} \subseteq T^{\perp}$.
If S, T are two subspaces of an inner product space, then S and T are *orthogonal* if $\boldsymbol{s} \perp \boldsymbol{t}$ for every $\boldsymbol{s} \in S$ and every $\boldsymbol{t} \in T$. This is denoted as $S \perp T$.
Let V be an inner product space and let $T \subseteq V$. The set $T^{\perp}$ is a subspace of V. Furthermore, $\langle T \rangle^{\perp} = T^{\perp}$.

Solution 2:

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two members of T. We have $(\boldsymbol{x}, \boldsymbol{t}) = (\boldsymbol{y}, \boldsymbol{t}) = 0$ for every $\boldsymbol{t} \in T$. Therefore, for every $a, b \in F$, by the linearity of the inner product we have

$$(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{t}) = a(\boldsymbol{x}, \boldsymbol{t}) + b(\boldsymbol{y}, \boldsymbol{t}) = 0,$$

for $\boldsymbol{t} \in T$, so $a\boldsymbol{x} + b\boldsymbol{t} \in T^{\perp}$. Thus, $T^{\perp}$ is a subspace of V.
By a previous observation, since $T \subseteq \langle T \rangle$, we have $\langle T \rangle^{\perp} \subseteq T^{\perp}$. To prove the converse inclusion, let $\boldsymbol{z} \in T^{\perp}$.
If $\boldsymbol{y} \in \langle T \rangle$, $\boldsymbol{y}$ is a linear combination of vectors of T,
$\boldsymbol{y} = a_1\boldsymbol{t}_1 + \cdots + a_m\boldsymbol{t}_m$, so $(\boldsymbol{y}, \boldsymbol{z}) = a_1(\boldsymbol{t}_1, \boldsymbol{z}) + \cdots + a_m(\boldsymbol{t}_m, \boldsymbol{z}) = 0$.
Therefore, $\boldsymbol{z} \perp \boldsymbol{y}$, which implies $\boldsymbol{z} \in \langle T \rangle^{\perp}$. This allows us to conclude that $\langle T \rangle^{\perp} = T^{\perp}$.

Problem 3:

Let $\mathbf{x} \in \mathbb{R}^n$. Prove that for every $\epsilon > 0$ there exists $\mathbf{y} \in \mathbb{R}^n$ such that the components of the vector $\mathbf{x} + \mathbf{y}$ are distinct and $\parallel \mathbf{y} \parallel_2 < \epsilon$.

Solution 3:

Partition the set $\{1, \ldots, n\}$ into the blocks $B_1, \ldots, B_k$ such that all components of $\boldsymbol{x}$ that have an index in B_j have a common value c_j. Suppose that $|B_j| = p_j$. Then, $\sum_{j=1}^{k} p_j = n$ and the numbers $\{c_1, c_2, \ldots, c_k\}$ are pairwise distinct. Let $d = \min_{i,j} |c_i - c_j|$. The vector $\boldsymbol{y}$ can be defined as follows. If $B_j = \{i_1, \ldots, i_{p_j}\}$, then

$$y_{i_1} = \eta \cdot 2^{-1}, y_{i_2} = \eta \cdot 2^{-2}, \ldots, y_{i_{p_j}} = \eta \cdot 2^{-p},$$

where $\eta > 0$, which makes the numbers $c_j + y_{i_1}, c_j + y_{i_2}, \ldots, c_j + y_{i_{p_j}}$ pairwise distinct. It suffices to take $\eta < d$ to ensure that the components of $\boldsymbol{x} + \boldsymbol{y}$ are pairwise distinct. Also, note that $\parallel \boldsymbol{y} \parallel_2^2 \leqslant \sum_{j=1}^{k} p_j \frac{\eta^2}{4} = \frac{n\eta^2}{4}$. It suffices to choose η such that $\eta < \min\{d, \frac{2\epsilon}{n}\}$ to ensure that $\parallel \boldsymbol{y} \parallel_2 < \epsilon$.

UMASS BOSTON

Problem 4:

Let $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be a sequence of vectors, where $n \geqslant 2$. Prove that the volume $\mathcal{V}_n$ of the parallelepiped constructed on these vectors equals to the square root of the Gramian of the sequence $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$.

Solution 4:

For the base case $n = 2$ the area A of the parallelogram is given by $A = \parallel \boldsymbol{u} \parallel_2 \parallel \boldsymbol{v} \parallel_2 \sin\alpha$, where $\alpha = \angle(\boldsymbol{u}, \boldsymbol{v})$. In another words,

$$\begin{aligned}
\det(G_{\boldsymbol{u},\boldsymbol{v}}) &= \det\begin{pmatrix} (\boldsymbol{u},\boldsymbol{u}) & (\boldsymbol{u},\boldsymbol{v}) \\ (\boldsymbol{u},\boldsymbol{v}) & (\boldsymbol{v},\boldsymbol{v}) \end{pmatrix} \\
&= \parallel \boldsymbol{u} \parallel_2^2 \parallel \boldsymbol{v} \parallel_2^2 - \parallel \boldsymbol{u} \parallel_2^2 \parallel \boldsymbol{v} \parallel_2^2 \cos^2\alpha \\
&= \parallel \boldsymbol{u} \parallel_2^2 \parallel \boldsymbol{v} \parallel_2^2 \sin^2\alpha = \mathcal{V}_2^2.
\end{aligned}$$

Sol.cont'd

Suppose that the statement holds for sequences of n vectors and let $L = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1})$ be a sequence of $n+1$ vectors. Let $\boldsymbol{v}_{n+1} = \boldsymbol{x} + \boldsymbol{y}$ be the orthogonal decomposition of $\boldsymbol{v}_{n+1}$ on the subspace $U_n = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle$, where $\boldsymbol{x} \in U_n$ and $\boldsymbol{y} \perp U_n$. Since $\boldsymbol{x} \in U_n$ there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $\boldsymbol{x} = a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n$. Let

$$\det(G_L) = \begin{vmatrix} (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_1, \boldsymbol{v}_n) & (\boldsymbol{v}_1, \boldsymbol{v}_{n+1}) \\ (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_2, \boldsymbol{v}_n) & (\boldsymbol{v}_2, \boldsymbol{v}_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\boldsymbol{v}_n, \boldsymbol{v}_1) & (\boldsymbol{v}_n, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_n, \boldsymbol{v}_n) & (\boldsymbol{v}_n, \boldsymbol{v}_{n+1}) \\ (\boldsymbol{v}_{n+1}, \boldsymbol{v}_1) & (\boldsymbol{v}_{n+1}, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_{n+1}, \boldsymbol{v}_n) & (\boldsymbol{v}_{n+1}, \boldsymbol{v}_{n+1}) \end{vmatrix}.$$

Sol. cont'd

By subtracting from the last row the first row multiplied by a_1, the second row multiplied by a_2, etc., the value of the determinant remains the same and we obtain

$$\det(G_L) = \begin{vmatrix} (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_1, \boldsymbol{v}_n) & (\boldsymbol{v}_1, \boldsymbol{v}_{n+1}) \\ (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_2, \boldsymbol{v}_n) & (\boldsymbol{v}_2, \boldsymbol{v}_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\boldsymbol{v}_n, \boldsymbol{v}_1) & (\boldsymbol{v}_n, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_n, \boldsymbol{v}_n) & (\boldsymbol{v}_n, \boldsymbol{v}_{n+1}) \\ (\boldsymbol{y}, \boldsymbol{v}_1) & (\boldsymbol{y}, \boldsymbol{v}_2) & \cdots & (\boldsymbol{y}, \boldsymbol{v}_n) & (\boldsymbol{y}, \boldsymbol{v}_{n+1}) \end{vmatrix}.$$

Sol. cont'd

Note that $(\boldsymbol{y}, \boldsymbol{v}_1) = (\boldsymbol{y}, \boldsymbol{v}_2) = \cdots = (\boldsymbol{y}, \boldsymbol{v}_n) = 0$ because $\boldsymbol{y} \perp U_n$ and $(\boldsymbol{y}, \boldsymbol{v}_{n+1}) = (\boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y}) = \parallel \boldsymbol{y} \parallel_2^2$, which allows us to further write

$$\begin{aligned} \det(G_L) &= \begin{vmatrix} (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_1, \boldsymbol{v}_n) & (\boldsymbol{v}_1, \boldsymbol{v}_{n+1}) \\ (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_2, \boldsymbol{v}_n) & (\boldsymbol{v}_2, \boldsymbol{v}_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\boldsymbol{v}_n, \boldsymbol{v}_1) & (\boldsymbol{v}_n, \boldsymbol{v}_2) & \cdots & (\boldsymbol{v}_n, \boldsymbol{v}_n) & (\boldsymbol{v}_n, \boldsymbol{v}_{n+1}) \\ 0 & 0 & \cdots & 0 & \parallel \boldsymbol{y} \parallel_2^2 \end{vmatrix} \\ &= \mathcal{V}_n^2 \parallel \boldsymbol{y} \parallel_2^2 = \mathcal{V}_{n+1}^2. \end{aligned}$$