CS724: Topics in Algorithms
Problem Set 5

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Problem 1:

Consider the linear system:

$$
\begin{aligned}
x_1 + 2x_2 &= 4, \\
2x_1 + 3.999x_2 &= 7
\end{aligned}
$$

Is this system well-conditioned?

Solution 1:

The answer is negative. In matrix form this is $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3.999 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ 7.999 \end{pmatrix}$$

Solving this in MATLAB with yields:

```
>> x=inv(A)*b
x =
     2
     1
```

Solution 1 cont'd:

A small change to the $\boldsymbol{b}$ as in

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3.999 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 4.001 \\ 7.998 \end{pmatrix}$$

yields the solution

```
x =
   -3.9990
    4.0000
```

Thus, we get a huge variation!

Solution 1 cont'd:

Similarly, a small change in the components of A causes a big variation in the solution. Suppose that

$$A = \begin{pmatrix} 1.001 & 2.001 \\ 2.001 & 3.998 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 4 \\ 7.999 \end{pmatrix}$$

yields

```
x =
    6.9890
   -1.4973
```

Solution 1 cont'd:

Thus, the system is ill-conditioned. This can be seen in the condition number of A:

```
>> A = [1 2; 2 3.999]
A =
    1.0000    2.0000
    2.0000    3.9990

>> cond(A)
ans =
   2.4992e+04
```

which is quite large!

Note that the columns of A are almost proportional! This is bad for conditioning.

Problem 2:

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ be two matrices. Prove that if AB is invertible, then both A and B are invertible.

Solution 2:

Since AB is invertible we have $AB(AB)^{-1} = I_n$. Thus, A is invertible and $A^{-1} = B(AB)^{-1}$. Similarly, since $(AB)^{-1}AB = I_n$ it follows that B is invertible and $B^{-1} = (AB)^{-1}A$.

Problem 3:

Let A and B be two matrices in $\mathbb{C}^{p \times q}$. Prove that
$\text{rank}(A + B) \leqslant \text{rank}(A\ B) \leqslant \text{rank}(A) + \text{rank}(B)$.

Solution 3:

We have $\text{rank}(A\ B) = \text{rank}(A\ A+B) \geqslant \text{rank}(A+B)$ because adding the first q columns of the matrix $(A\ B)$ to the last q columns does not change the rank of a matrix, and the rank of a submatrix is not larger than the rank of the matrix.
On another hand, $\text{rank}(A\ B) = \text{rank}((A\ O_{p,q}) + (O_{p,q}\ B)) \leqslant$
$\text{rank}(A\ O_{p,q}) + \text{rank}(O_{p,q}\ B) = \text{rank}(A) + \text{rank}(B)$.

Problem 4:

Let $A \in \mathbb{C}^{n \times n}$. Prove that A is Hermitian if and only if $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Solution 4:

We have $(A\boldsymbol{x}, \boldsymbol{y}) = (A\boldsymbol{x})^{\mathsf{H}}\boldsymbol{y} = \boldsymbol{x}^{\mathsf{H}}A^{\mathsf{H}}\boldsymbol{y}$ and $(\boldsymbol{x}, A\boldsymbol{y}) = \boldsymbol{x}^{\mathsf{H}}A\boldsymbol{y}$.
If A is Hermitian, $(A\boldsymbol{x}, \boldsymbol{y}) = (A\boldsymbol{x})^{\mathsf{H}}\boldsymbol{y} = \boldsymbol{x}^{\mathsf{H}}A^{\mathsf{H}}\boldsymbol{y} = \boldsymbol{x}^{\mathsf{H}}A\boldsymbol{y}$, hence $(A\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, A\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$.
Conversely, suppose that $(A\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, A\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$. Choosing $\boldsymbol{x} = \boldsymbol{e}_i$ and $\boldsymbol{y} = \boldsymbol{e}_j$ we have

$$\begin{aligned} (A\boldsymbol{e}_i, \boldsymbol{e}_j) &= (A\boldsymbol{e}_i)^{\mathsf{H}}\boldsymbol{e}_j = \overline{a_{ji}} \\ (\boldsymbol{e}_i, A\boldsymbol{e}_j) &= \boldsymbol{e}_i^{\mathsf{H}}A\boldsymbol{e}_j = a_{ij}, \end{aligned}$$

hence $a_{ij} = \overline{a_{ji}}$. Thus, A is Hermitian.

Problem 5:

Let $U \in \mathbb{C}^{n\times n}$ be a matrix whose set of columns is orthonormal and let $V \in \mathbb{C}^{n\times n}$ be a matrix whose set of rows is orthonormal. Prove that $\|UA\|_2 = \|AV\|_2 = \|A\|_2$ and, therefore, $\|UAV\|_2 = \|A\|_2$.

Solution 5:

By hypothesis, we have $U^{\mathsf{H}}U = I_n$ and $VV^{\mathsf{H}} = 1$. Therefore, $|\!|\!| UA |\!|\!|_2^2 = \max\{\| UA\boldsymbol{x} \|_2^2 \mid \| \boldsymbol{x} \|_2 = 1\} = \max\{\boldsymbol{x}^{\mathsf{H}}A^{\mathsf{H}}U^{\mathsf{H}}UA\boldsymbol{x} \mid \| \boldsymbol{x} \|_2 = 1\} = \max\{\boldsymbol{x}^{\mathsf{H}}A^{\mathsf{H}}A\boldsymbol{x} \mid \| \boldsymbol{x} \|_2 = 1\} = \max\{\| A\boldsymbol{x} \|_2^2 \mid \| \boldsymbol{x} \|_2 = 1\} = |\!|\!| A |\!|\!|_2^2$. This allows us to conclude that $|\!|\!| UA |\!|\!|_2 = |\!|\!| A |\!|\!|_2$.

The second equality follows immediately from the first.