The Other Rice Theorem

Let \( T(x_1, x_2, z) \) be the predicate

\[
T(x_1, x_2, z) = \begin{cases} 
1 & \text{if program } x_2 \text{ halts on } x_1 \text{ in exactly } z \text{ steps} \\
0 & \text{otherwise.} 
\end{cases}
\]

Prove that \( T \) is a primitive recursive predicate (which is easy; you may use STP).

Next, we give a method to show that certain sets are not r.e.

For two partial functions \( f, g \) we write \( f \subseteq g \) if \( x \in \text{Dom}(f) \) implies \( x \in \text{Dom}(g) \) and \( f(x) = g(x) \) for \( x \in \text{Dom}(f) \).

**The Other Rice’s Theorem**

If \( \Gamma \) is a set of computable functions and there exist \( m, m' \in R_\Gamma \) such that \( \Phi_m \in \Gamma, \Phi_{m'} \not\in \Gamma \) and \( \Phi_m \subseteq \Phi_{m'} \), then \( R_\Gamma \) is not r.e.

**Proof:** Let \( \mathcal{P} \) be the program defined in flowchart shown in Figure 1.

![Flowchart of \( \mathcal{P} \)](image)

Suppose that \( \#(\mathcal{P}) = p \), so the program computes \( \Phi(x_1, x_2, p) \).
If $x_2 \not\in K$, then $\Phi(x_1, x_2, p) = \Phi_m(x_1)$. Otherwise, $\Phi(x_1, x_2, p) = \Phi_{m'}(x_1)$.

Thus,

$$\Phi(x_1, x_2, p) = \begin{cases} 
\Phi_m(x_1) & \text{if } x_2 \not\in K, \\
\Phi_{m'}(x_1) & \text{if } x_2 \in K.
\end{cases}$$

By the s-m-n theorem, we have

$$\Phi_{S^1_2(x_2,p)}(x_1) = \begin{cases} 
\Phi_m(x_1) & \text{if } x_2 \not\in K, \text{ (that is } x_2 \not\in K) \\
\Phi_{m'}(x_1) & \text{if } x_2 \in K.
\end{cases}$$

If we define the computable function $f$ as $f(x_2) = S^1_2(x_2, p)$ we may conclude that

- $x \in \bar{K}$ if and only if $\Phi_f(x_2) = \Phi_m$, that is, if and only if $f(x_2) \in R^\Gamma$ and
- $x \in K$ if and only if $\Phi_f(x_2) = \Phi_{m'}$, that is, if and only if $f(x_2) \not\in R^\Gamma$.

Thus, $x \in \bar{K}$ if and only if $f(x) \in R^\Gamma$ so $\bar{K} \leq_m R^\Gamma$, which implies that $R^\Gamma$ is not r.e.

Using the “Other Rice’s Theorem” prove that the following sets are not r.e.:

- $\text{EMPTY} = \{ x \mid \text{Dom}(\Phi_x) = \emptyset \}$
- $\{ x \mid \text{Ran}(\Phi_x) = \emptyset \}$
- $\text{FIN} = \{ x \mid \text{Dom}(\Phi_x) \text{ is finite } \}$
- $\text{NOT-TOT} = \{ x \mid \Phi_x \text{ is not total} \}$
- $\{ x \mid x_0 \not\in W_x \}$
- $\{ x \mid x_0 \not\in \text{Ran}(\Phi_x) \}$
- $\{ x \mid W_x \text{ is not recursive} \}$
- $\{ x \mid W_x \text{ is recursive} \}$