Context-Free languages (part IV)

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The main result of this section is a proof that the class $\mathcal{R}$ of regular languages coincides with $\mathcal{L}_3$.

Theorem

Let G be a type-3 grammar, and let L be the language generated by G. There is a transition system $\mathcal{T}$ such that $L = L(\mathcal{T})$.

Proof

Suppose that $G = (A_N, A_T, S, P)$ is a type-3 grammar. Define the transition system $\mathcal{T} = (A_T, A_N \cup \{Z\}, \theta, S, \{Z\})$, where Z is a new symbol, $Z \notin A_N \cup A_T$, and

$$\begin{aligned} \theta &= \{(X, u, Y) \mid X \to uY \in P\} \\ &\quad \cup \{(X, u, Z) \mid X \to u \in P\}. \end{aligned}$$

(Proof cont'd)

Let $w \in L(G)$. There exists a derivation

$$S \underset{G}{\Rightarrow} u_0 X_{i_0} \underset{G}{\Rightarrow} u_0 u_1 X_{i_1} \cdots \underset{G}{\Rightarrow}$$
$$u_0 u_1 \cdots u_{n-1} X_{i_{n-1}} \underset{G}{\Rightarrow} u_0 u_1 \cdots u_{n-1} u_n,$$

where $w = u_0 \cdots u_{n-1} u_n$. The productions used in this derivation are $S \to u_0 X_{i_0}$, $X_{i_{p-1}} \to u_p X_{i_p}$ for $1 \leq p \leq n-1$, and $X_{i_{n-1}} \to u_n$. Therefore, the triples

$$(S, u_0, X_{i_0}), (X_{i_0}, u_1, X_{i_1}), \ldots, (X_{i_{n-2}}, u_{n-1}, X_{i_{n-1}}), (X_{i_{n-1}}, u_n, Z)$$

must all be in θ, which implies that $(S, u_0 \cdots u_n, Z) \in \theta^*$. Since Z is a final state of $\mathcal{T}$, we have $u \in L(\mathcal{T})$, so $L(G) \subseteq L(\mathcal{T})$.

(Proof cont'd)

Conversely, if $u \in L(\mathcal{T})$, then $(S, u, Z) \in \theta^*$. Taking into account the definition of θ, there are n intermediate states in $\mathcal{T}$, $X_{i_0}, \ldots, X_{i_{n-1}}$ such that $u = u_0 \cdots u_n$ and the triples

$$(S, u_0, X_{i_0}), (X_{i_0}, u_1, X_{i_1}), \ldots, (X_{i_{n-2}}, u_{n-1}, X_{i_{n-1}}), (X_{i_{n-1}}, u_n, Z)$$

exist in θ. This implies the existence in P of the productions

$$S \to u_0 X_{i_0}, X_{i_0} \to u_1 X_{i_1}, \ldots, X_{i_{n-2}} \to u_{n-1} X_{i_{n-1}}, X_{i_{n-1}} \to u_n$$

Using these productions we obtain the derivation

$$\begin{aligned} S &\underset{G}{\Rightarrow} u_0 X_{i_0} \underset{G}{\Rightarrow} u_0 u_1 X_{i_1} \cdots \underset{G}{\Rightarrow} \\ & u_0 u_1 \cdots u_{n-1} X_{i_{n-1}} \underset{G}{\Rightarrow} u_0 u_1 \cdots u_{n-1} u_n, \end{aligned}$$

which implies that $x \in L(\mathcal{T})$. This proves the inclusion $L(\mathcal{T}) \subseteq L(G)$.

Theorem

For every regular language L there is a type-3 grammar G such that $L(G) = L$.

Proof.

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be a dfa such that $L = L(\mathcal{M})$. The type-3 grammar $G = (Q, A, q_0, P)$ whose productions are

$$\begin{array}{ll} q \to aq' & \text{for each } q, q', a \text{ with } q' = \delta(q, a) \\ q \to \lambda & \text{for each } q \in F. \end{array}$$

generates $L(\mathcal{M})$. □

Corollary

The class $\mathcal{L}_3$ coincides with the class $\mathcal{R}$ of regular languages.

Recall the Pumping Lemma for context-free languages:

Theorem

Let L be a context-free language. There exists a number $n_L \in \mathbb{N}$ such that if $w \in L$ and $|w| \geqslant n_L$, then we can write

$$w = xyzut$$

such that $|y| \geqslant 1$ or $|u| \geqslant 1$, $|yzu| \leqslant n_L$ and $xy^n zu^n t \in L$ for all $n \in \mathbb{N}$.

This is a necessary condition for the "context-freeness" of a language.

The Special Case of One-symbol Alphabets

Let $A = \{a\}$ be an one-symbol alphabet.

- Word concatenation in A^* is commutative.
- The formulation of the Pumping Lemma in this special case:
  Let L be a context-free language. There exists a number $n_L \in \mathbb{N}$ such that if $w \in L$ and $|w| \geqslant n_L$, then we can write

$$w = rs$$

  such that $1 \leqslant |s| \leq n_G$ and $rs^n \in L(G)$ for all $n \in \mathbb{N}$.

Note that $r \in L$ (since we can take $n = 0$).

If $|r| > n_L$ the same pumping lemma can be applied to r, and $r = r_1 w_1$ with $|w_1| \leqslant n_L$ such $r_1 w_1^{n_1} \in L$ for $n_1 \in \mathbb{N}$. Again $r_1 \in L$ (for $n = 0$), etc. This leads to a stronger form of the Pumping Lemma for languages over one-symbol alphabets.

If L is a context-free language on an one-symbol alphabet, there exists a number n_L such that every word $w \in L$ with $|x| \geqslant n_L$ can be written as

$$w = r s_1 s_2 \cdots s_k,$$

where $|r|, |s_1|, \ldots, |s_k| \leqslant n_L$ and

$$r s_1^{n_1} \cdots s_k^{n_k} \in L$$

for $n_1, \ldots, n_k \in \mathbb{N}$.

Note that the set $K_n(L)$ of words in L shorter than n_L is finite, so it is regular. Since $L = (L \cap K_n(L)) \cup (L - K_n(L))$. The set $L - K_n(L)$ has the form $\{w_1, w_2, \ldots, w_n\}^*$, where $w_1, \ldots, w_n$ are the words that can be "pumped". Thus, L is a regular language.

Theorem

Let $s : A^* \longrightarrow B^*$ be a substitution. If $s(a)$ is a context-free language for every $a \in A$ and $L \subseteq A^*$ is a context-free language, then $s(L)$ is a context-free language.

Proof

Suppose that $L = L(G)$, where $G = (A_N, A, S, P)$ is a context-free grammar and let $s(a)$ is generated by the context-free grammar $G_a = (A_N^a, B, S_a, P_a)$ for $a \in A$.
We may assume that the sets of nonterminal symbols A_N^a are pairwise disjoint.
Let P' be the set of productions obtained from P as follows. In each production of P replace every letter $a \in A$ by the nonterminal S_a. We claim that the language $s(L)$ is generated by the grammar $G' = (A_N \cup \bigcup_{a \in A} A_N^a, B, S, P' \cup \bigcup_{a \in A} P_a)$.

(Proof cont'd)

Let $y \in s(L)$. There exists a word $x = a_{i_0} \dots a_{i_{n-1}} \in L$ such that $y \in s(x)$. This means that $y = y_0 \dots y_{n-1}$, where $y_k \in s(a_{i_k}) = L(G_{a_{i_k}})$ for $0 \leq k \leq n-1$. Thus, we have the derivations $S_{a_{i_k}} \overset{*}{\underset{G_{a_{i_k}}}{\Rightarrow}} y_k$ for $0 \leq k \leq n-1$, and the same derivations can be done in G'. Consequently, we obtain the derivation

$$S \overset{*}{\underset{G'}{\Rightarrow}} S_{a_{i_0}} \dots S_{a_{i_{n-1}}} \overset{*}{\underset{G'}{\Rightarrow}} y_0 \dots y_{n-1} = y,$$

which implies $y \in L(G')$, so $s(L) \subseteq L(G')$.

(Proof cont'd)

Conversely, if $y \in L(G')$, then any derivation $S \overset{*}{\underset{G'}{\Rightarrow}} y$ is of the previous form.

The word y can be written as $y = y_0 \dots y_{n-1}$, where $S_{a_{i_k}} \overset{*}{\underset{G'}{\Rightarrow}} y_k$ for $0 \leq k \leq n-1$, so $y_k \in L(G_{a_{i_k}}) = s(a_{i_k})$ for $0 \leq k \leq n-1$. This implies $y = y_0 \cdots y_{n-1} \in s(a_{i_0} \cdots s(a_{i_{n-1}}) = s(x) \in s(L)$, so $L(G') \subseteq s(L)$.
Since $s(L) = L(G')$, it follows that $s(L)$ is a context-free language.

Corollary

If $h : A^* \longrightarrow B^*$ is a morphism and $L \subseteq A^*$ is a context-free language, then $h(L)$ is a context-free language.

The class $\mathcal{L}_2$ is closed with respect to inverse morphic images. In other words, if $h : B^* \longrightarrow A^*$ is a morphism, and $L \subseteq A^*$ is a context-free language, then $h^{-1}(L)$ is a context-free language.

Proof

Suppose that $B = \{b_0, \ldots, b_{m-1}\}$ and that $h(b_i) = x_i$ for $0 \leq i \leq m-1$. Let $B' = \{b'_0, \ldots, b'_{m-1}\}$, and let s be the substitution given by $s(a) = B'^* a B'^*$ for $a \in A$.

$$B = \{b_0, \ldots, b_{m-1}\}$$

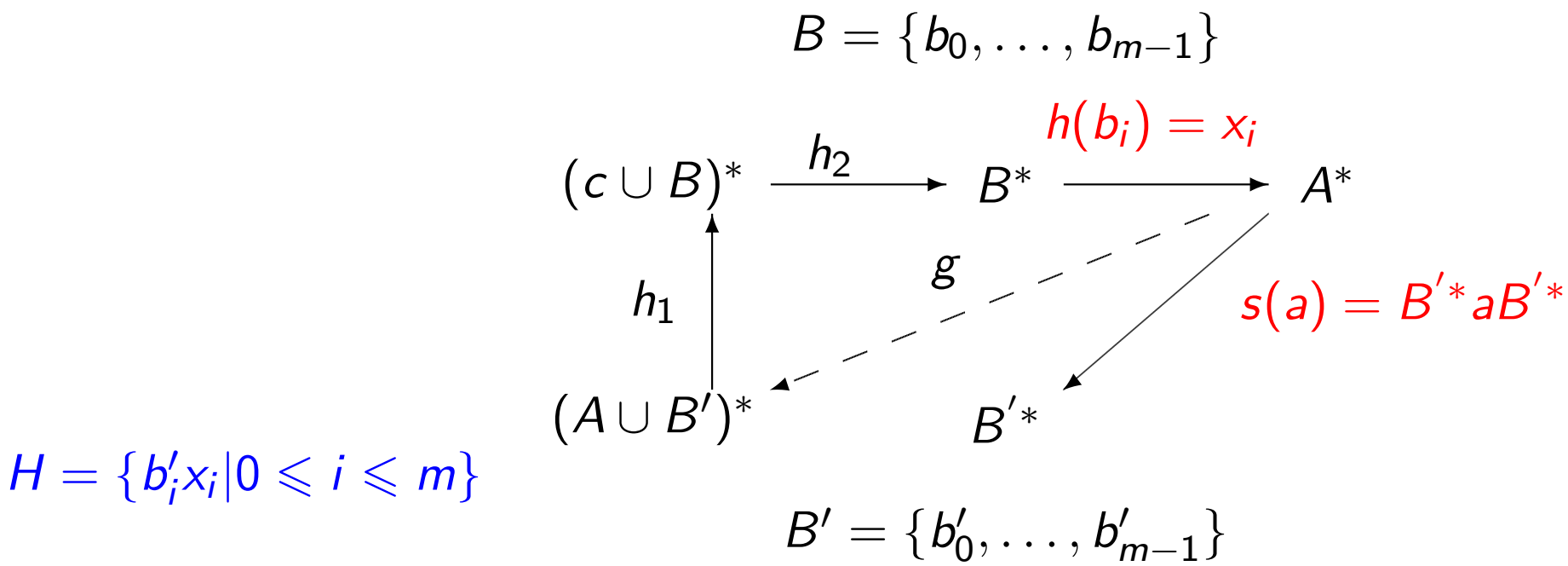
$$h(b_i) = x_i$$

$$B^* \longrightarrow A^*$$

$$s(a) = B'^* a B'^*$$

$$B'^*$$

$$B' = \{b'_0, \ldots, b'_{m-1}\}$$



Consider the finite language $H = \{b'_i x_i \mid 0 \leq i \leq m-1\}$ in $(B' \cup A)^*$ and the mapping $g : \mathcal{P}(A^*) \longrightarrow \mathcal{P}((A \cup B')^*)$ given by $g(L) = s(L) \cap H^*$.
Define $h_1 : (A \cup B')^* \longrightarrow (\{c\} \cup B)^*$ and $h_2 : (\{c\} \cup B)^* \longrightarrow B^*$ by
$h_1(a) = c$ for $a \in A$, $h_1(b') = b$ for all $b' \in B'$,
and $h_2(c) = \lambda$, $h_2(b) = b$ for $b \in B$.

We claim that for every language $L \in \mathcal{P}(A)$ such that $\lambda \notin L$, $h^{-1}(L) = h_2(h_1(g(L)))$ and hence, $h^{-1}(L)$ is context-free. This follows from the following equivalent statements:

1. $u = b_{i_0} \cdots b_{i_{k-1}} \in h^{-1}(L)$;
2. $h(u) = x_{i_0} \cdots x_{i_{k-1}} \in L$;
3. $b'_{i_0} x_{i_0} \cdots b'_{i_{k-1}} x_{i_{k-1}} \in g(L)$;
4. $h_1(b'_{i_0} x_{i_0} \cdots b'_{i_{k-1}} x_{i_{k-1}}) = b_{i_0} c \cdots c \cdots b_{i_{k-1}} c \cdots c \in h_1(g(L))$;
5. $h_2(b_{i_0} c \cdots c \cdots b_{i_{k-1}} c \cdots c) = b_{i_0} \cdots b_{i_{k-1}} = u \in h_2(h_1(g(L)))$.

(Proof cont'd)

If $\lambda \in L$, the language $L - \{\lambda\}$ is context-free, so $h^{-1}(L - \{\lambda\})$ is also context-free. Note that $h^{-1}(L) = h^{-1}(L - \{\lambda\}) \cup h^{-1}(\{\lambda\})$ and that $h^{-1}(\{\lambda\}) = \{a \in A \mid h(a) = \lambda\}^*$. Since $h^{-1}(\{\lambda\})$ is regular it follows that $h^{-1}(L)$ is context-free.

Reminder

We defined the shuffle of languages

Definition

Let A be an alphabet and let G, K be two languages over A. The *shuffle* of G and K is the language

$$\text{shuffle}(G,K) = \{x_0y_0x_1y_1\cdots x_{n-1}y_{n-1} \mid x_0x_1\cdots x_{n-1} \in G \text{ and } y_0y_1\cdots y_{n-1} \in K\}.$$

We proved

Theorem

There is an alphabet B and there exist three morphisms g, k, h from B^* to A^* such that h is a very fine morphism, g, k are fine morphisms and $\text{shuffle}(G, K) = h(g^{-1}(G) \cap k^{-1}(K))$.

Corollary

Let $L \subseteq A^*$ be a context-free language and let $R \subseteq A^*$ be a regular language. Then, $\mathrm{shuffle}(L, R)$ *is a context-free language.*