Finite Automata and Regular Languages (part IV)

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We discuss the behavior of the class of regular languages with respect to several operations previously defined.
We prove that this class is the smallest class of languages that

- contains all finite languages, and
- is closed with respect to union, product, and Kleene closure.

The class of regular languages is closed with respect to the complement operations as shown by the next theorem.

Theorem

Let A be an alphabet. If $L \subseteq A^*$ is a regular language, then its complement $\bar{L} = A^* - L$ is also a regular language.

Proof.

Suppose that $\mathcal{M} = (A, Q, \delta, q_0, F)$ is an automaton such that $L = L(\mathcal{M})$. Define the automaton $\mathcal{M}' = (A, Q, \delta, q_0, Q - F)$; thus, only the set of final states of $\mathcal{M}'$ differs from the corresponding set for $\mathcal{M}$. We have $x \in L(\mathcal{M}')$ if and only if $\delta^*(q_0, x) \in Q - F$, or, equivalently, if and only if $\delta^*(q_0, x) \notin F$. Therefore $L(\mathcal{M}') = A^* - L$, which proves that $A^* - L$ is regular. □

Transition systems are very convenient for proving closure properties of the class $\mathcal{R}$. To make use of these devices we use the following technical result.

Lemma

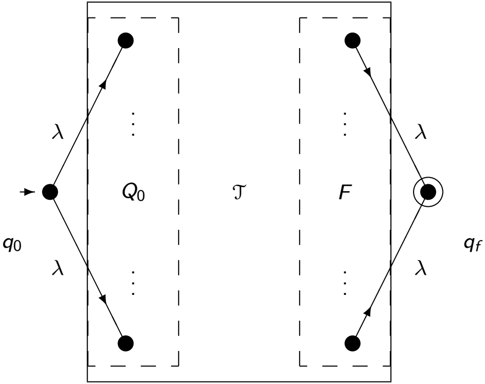
For every nonempty regular language $L \subseteq A^*$ there exists a transition system that accepts L and has a single initial state q_0 and a single final state q_f.

Proof

Since L is a nonempty regular language, there exists a transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ such that $L(\mathcal{T}) = L$ and $F \neq \emptyset$. Define the transition system $\mathcal{T}' = (A, Q \cup \{q_0, q_f\}, \theta', \{q_0\}, \{q_f\})$, where $q_0, q_f \notin Q$. The relation θ' is given by:

$$\theta' = \theta \cup \{(q_0, \lambda, q) \mid q \in Q_0\} \cup \{(q, \lambda, q_f) \mid q \in F\}.$$

The graph of $\mathcal{T}'$ is represented below.



Clearly, $(q, x, q') \in \theta^*$ for some $q \in Q_0$ and $q' \in F$ if and only if $(q_0, \lambda x \lambda, q_f) = (q_0, x, q_f) \in \theta'^*$. Therefore, $L(\mathcal{T}) = L(\mathcal{T}')$.

Theorem

If L is a regular language, then L^R is also regular.

Proof.

Define a new transition system $\mathcal{T}' = (A, Q, \theta', \{q_f\}, \{q_0\})$, where

$$\theta' = \{(q_2, w, q_1) \mid (q_1, w, q_2) \in \theta\}.$$

We have $(q, x^R, q') \in \theta'^*$ if and only if $(q', x, q) \in \theta^*$. Therefore, $x \in L(\mathcal{T})$ if and only if $x^R \in L(\mathcal{T}')$, so $L(\mathcal{T}') = (L(\mathcal{T}))^R$. $\square$

Next, we show that the class of regular languages is closed under union.

Theorem

If L_0, L_1 are regular languages, then $L_0 \cup L_1$ is regular.

Proof.

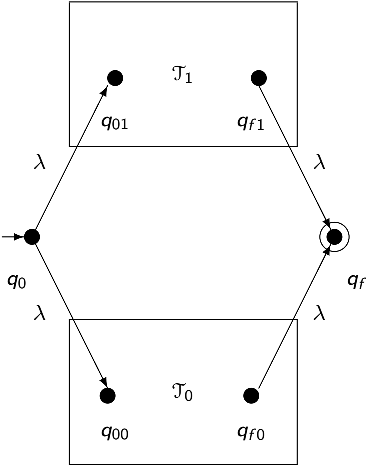
Without loss of generality, assume that both L_0 and L_1 are regular languages over the same alphabet A. Suppose that $L_i = L(\mathcal{T}_i)$, where $\mathcal{T}_i$, $i = 0, 1$, are two transition systems. We can assume that each $\mathcal{T}_i$ has a single initial and a single final state, $\mathcal{T}_i = (A, Q_i, \theta_i, \{q_{0i}\}, \{q_{fi}\})$ for $i = 0, 1$; also, assume that $Q_0 \cap Q_1 = \emptyset$.

□

Proof (cont'd)

Define a new transition system $\mathcal{T}' = (A, Q', \theta', \{q_0\}, \{q_f\})$ given by $Q' = Q_0 \cup Q_1 \cup \{q_0, q_f\}$, and

$$\theta = \theta_0 \cup \theta_1 \cup \{(q_0, \lambda, q_{00}), (q_0, \lambda, q_{01}), (q_{f0}, \lambda, q_f), (q_{f1}, \lambda, q_f)\}.$$



Proof (cont'd)

Since $Q_0 \cap Q_1 = \emptyset$, a path ϖ in $\mathcal{T}$ that joins q_0 to q_f exists in $\mathcal{T}$ if and only if that path passes through q_{00} and q_{f0}, or through q_{01} and q_{f1}. If x is the label of the path ϖ, then x belongs to $L(\mathcal{T}_0)$ or $L(\mathcal{T}_1)$, respectively. This amounts to $L(\mathcal{T}) = L(\mathcal{T}_0) \cup L(\mathcal{T}_1) = L_0 \cup L_1$, which implies that $L_0 \cup L_1$ is regular.

Corollary

The class of regular languages is closed under intersection. In other words, if L_0, L_1 are regular languages, then $L_0 \cap L_1$ is regular.

Proof.

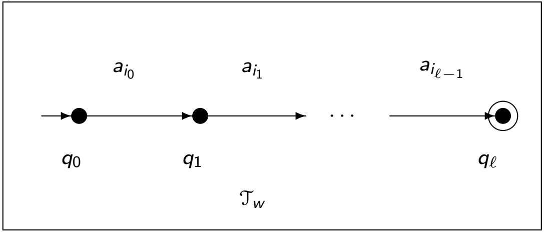
This statement follows immediately by previous theorems and by De Morgan's law. Specifically, $L_0 \cap L_1 = \overline{\overline{L_0} \cup \overline{L_1}}$, and each subexpression of the right hand side is regular if L_0 and L_1 are. □

Corollary

Every finite language over an alphabet A is regular.

Proof.

The empty language is clearly regular. Thus it suffices to show that one-word languages are regular. It is easy to see that if $L = \{w\}$, where $w = a_{i_0} \dots a_{i_{\ell-1}}$, then L is accepted by the transition system $\mathcal{T}_w$ given below



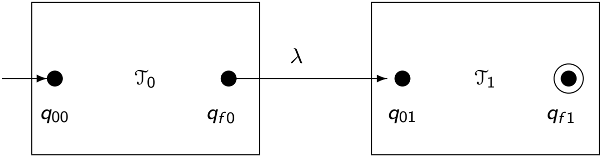
which implies the regularity of L. □

Theorem

If L_0, L_1 are regular languages, then $L_0 L_1$ is regular.

Proof.

Assume that both L_0 and L_1 are regular languages over the same alphabet A such that $L_i = L(\mathcal{T}_i)$, where $\mathcal{T}_i$, $i = 0, 1$, are two transition systems. We assume that each $\mathcal{T}_i$ has a single initial and a single final state, $\mathcal{T}_i = (A, Q_i, \theta_i, \{q_{0i}\}, \{q_{fi}\})$ for $i = 0, 1$; also, assume that $Q_0 \cap Q_1 = \emptyset$. Define the transition system $\mathcal{T} = (A, Q_0 \cup Q_1, \theta, \{q_{00}\}, \{q_{f1}\})$, where $\theta = \theta_0 \cup \theta_1 \cup \{(q_{f0}, \lambda, q_{01})\}$ □



Since $Q_0 \cap Q_1 = \emptyset$, to reach the state q_{f1} from the initial state q_{00} reading the symbols of the word x, the transition system $\mathcal{T}$ must pass through the states q_{f0} and q_{01} (via the null transition $(q_{f0}, \lambda, q_{01})$). This happens if and only if $x = uv$, where $(q_{00}, u, q_{f0}) \in \theta_0^*$ and $(q_{01}, v, q_{f1}) \in \theta_1^*$, so $L(\mathcal{T}) = L(\mathcal{T}_0)L(\mathcal{T}_1) = L_0L_1$. Hence, L_0L_1 is a regular language.

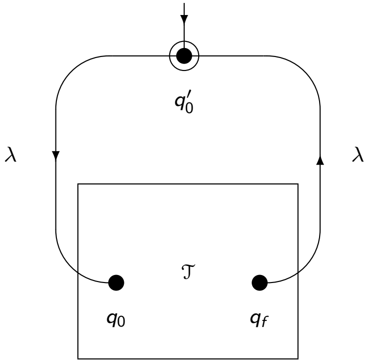
Theorem

If L is a regular language, then L^* is regular.

Proof.

Let $\mathcal{T} = (A, Q, \theta, \{q_0\}, \{q_f\})$ be a transition system such that $L = L(\mathcal{T})$. Define the transition system $\mathcal{T}' = (A, Q \cup \{q_0'\}, \theta', \{q_0'\}, \{q_0'\})$, where $\theta' = \theta \cup \{(q_f, \lambda, q_0'), (q_0', \lambda, q_0)\}$ and q_0' is a new state. □

Proof (cont'd)



Proof (cont'd)

We have $\lambda \in L(\mathcal{T}')$ because q_0' is both the initial and the final state of $\mathcal{T}'$. Further, if $w \in L(\mathcal{T})$, we have $(q_0, w, q_f) \in \theta^*$. Since both triples (q_0', λ, q_0) and (q_f, λ, q_0') belong to θ, we obtain $(q_0', w^k, q_0') \in \theta^*$ for every $k \in \mathbb{N}$, $k \geq 1$. Therefore, $L^k \subseteq L(\mathcal{T}')$ for $k \in \mathbb{N}$, so $L^* \subseteq L(\mathcal{T}')$.

Proof (cont'd)

Conversely, if $u \in L(\mathcal{T}')$, the transition system $\mathcal{T}'$ starts in q_0' and finishes in q_0' while reading the symbols of u. Let m be the number of times the transition system $\mathcal{T}'$ leaves the state q_0' while processing the word u. If $m = 0$, then $u = \lambda$. Otherwise, $m \geq 1$ and $\mathcal{T}'$ passes through the sequence of states: $q_0, \ldots, q_f, q_0', q_0, \ldots, q_f, q_0', \ldots, q_f$, where q_0' occurs m times. Here "passes through" means "enters and then leaves." This implies that we can write $u = u_0 \cdots u_{m+1}$, where $(q_0, u_i, q_f) \in \theta^*$ for $0 \leq i \leq m+1$. Thus, $w \in L^{m+1}$, so $L(\mathcal{T}') \subseteq L^*$. Hence, $L(\mathcal{T}') = L^*$.

Theorem

Let L be a regular language over the alphabet A. For every language K, both the right and the left quotients LK^{-1} and $K^{-1}L$ are regular.

Proof

We first deal with the left quotient. Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system such that $L = L(\mathcal{T})$. Let $Q_K = \{q \in Q \mid (q_0, x, q) \in \theta^* \text{ for some } q_0 \in Q_0 \text{ and } x \in K\}$ and let $\mathcal{T}_K$ be the transition system $\mathcal{T}_K = (A, Q, \theta, Q_K, F)$. The following statements are easily seen to be equivalent:

1. $y \in K^{-1}L$;
2. $xy \in L$ for some $x \in K$;
3. $(q_0, xy, q') \in \theta^*$ for some $q_0 \in Q_0$ and $q' \in F$;
4. there is $q \in Q_K$ such that $(q_0, x, q) \in \theta^*$ and $(q, y, q') \in \theta^*$ for some $q_0 \in Q_0$ and $q' \in F$;
5. $y \in L(\mathcal{T}_K)$.

From these equivalences it follows that $K^{-1}L = L(\mathcal{T}_K)$, and thus $K^{-1}L$ is regular.

Proof (cont'd)

To make the argument for the right quotient, let P_K be the set $P_K = \{q \in Q \mid (q, z, q') \in \theta^* \text{ for some } q' \in F \text{ and } z \in K\}$. Define the transition system $\mathcal{T}^K$ as $\mathcal{T}^K = (A, Q, \theta, Q_0, P_K)$. We have the following equivalent statements:

1. $y \in L(\mathcal{T}^K)$;
2. $(q_0, y, q) \in \theta^*$ for some $q_0 \in Q_0$ and some $q \in P_K$;
3. $(q_0, y, q) \in \theta^*$ and $(q, z, q') \in \theta^*$ for some $q_0 \in Q_0$ and some $q' \in F$;
4. $(q_0, yz, q') \in \theta^*$ for some $q_0 \in Q_0$ and some $q' \in F$ and $z \in K$;
5. $yz \in L$ for some $z \in K$;
6. $y \in LK^{-1}$.

Therefore, $LK^{-1} = L(\mathcal{T}^K)$. This proves that the language LK^{-1} is regular. Note that this property of closure under quotients does not depend on the regularity of K.

Corollary

If $L \subseteq A^*$ is a regular language, then there exists a finite number of distinct left (right) quotients of the form $K^{-1}L$ (of the form LK^{-1}), where $K \subseteq A^*$.

Proof.

Suppose that $L = L(\mathcal{T})$, where $\mathcal{T} = (A, Q, \theta, Q_0, F)$. Note that if K, H are two languages such that $Q_K = Q_H$, then $K^{-1}L = H^{-1}L$. In other words, there are no more distinct left quotients than subsets of Q, which implies that the number of distinct left quotients of L is finite. □

Corollary

If $L \subseteq A^*$ is a regular language, then there exists a finite number of distinct left (right) derivatives of L.

Proof.

Follows immediately from the previous corollary by considering the quotients of L and singleton languages $K = \{x\}$ for $x \in A^*$. □

Corollary

If L is a regular language, then $\mathrm{PREF}(L)$, $\mathrm{SUFF}(L)$, *and* $\mathrm{INFIX}(L)$ *are all regular languages.*

Proof.

Follows from $\mathrm{SUFF}(L) = (A^*)^{-1}L$, $\mathrm{PREF}(L) = L\,(A^*)^{-1}$ and $\mathrm{INFIX}(L) = ((A^*)^{-1}\,L)\,(A^*)^{-1}$. □

Example

Using closure properties, it is easy to verify that if $\rho \subseteq A \times A$, then the language $L_\rho \subseteq A^*$ is regular. Indeed, we can write

$$A^* - L_\rho = \bigcup \{A^* a a' A^* \mid (a, a') \in (A \times A) - \rho\}.$$

Note that each language of the form $A^* a a' A^*$ is regular by. Furthermore, since A is a finite set, the right member of the equality is the union of a finite number of regular languages. Therefore, $A^* - L_\rho$ is regular, so implies that L_ρ is regular.