Finite Automata and Regular Languages (part III)

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Transition Systems
Nondeterministic finite automata can be further generalized by allowing transitions between states without reading any input symbol.

**Definition**

A transition system \( T = (A, Q, \theta, Q_0, F) \), where \( A, Q, \) and \( F \) are as in a finite automaton, \( \theta \) is a finite relation, \( \theta \subseteq Q \times A^* \times Q \), called the transition relation of \( T \), and \( Q_0 \) is a nonempty subset of \( Q \) called the set of initial states, and \( F \) is the set of final states. A transition in \( T \) is a triple \( (q, x, q') \in \theta \). We refer to transitions of the form \( (q, \lambda, q') \) as null transitions.
Transition systems are conveniently represented by labeled directed multigraphs. Namely, if $T = (A, Q, \theta, Q_0, F)$ is a transition system, then its graph is a labeled directed multigraph $G(T) = (G, A^*, m)$ that has $Q$ as its set of vertices. Each directed edge $e$ with $s(e) = q, d(e) = q'$, and $m(e) = x$ corresponds to a triple $(q, x, q') \in \theta$, and every such triple is represented by an edge in $G$.

Observe that, unlike the graph of a dfa or an ndfa, the edges of the directed graph of a ts can be labelled with words, including the null word.
Example

The graph of the transition system

\[ \mathcal{T}_1 = (\{a, b, c\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_3\}) \]

where \( \theta \) is given by

\[ \theta = \{(q_0, ab, q_1), (q_0, \lambda, q_2), (q_1, bc, q_2), (q_1, \lambda, q_3), (q_2, c, q_3)\} \]

is shown below:
Extending the transition relation

As we did with the transition functions of dfas and ndfas, we wish to extend the transition relation $\theta$ of a transition system $\mathcal{T}$ to the set $Q \times A^* \times Q$. The extension $\theta^* \subseteq Q \times A^* \times Q$ is given next.

1. For every $q \in Q$ define $(q, \lambda, q) \in \theta^*$.
2. Every triple $(q, x, q') \in \theta$ belongs to $\theta^*$.
3. If $(q, x, q'), (q', y, q'') \in \theta^*$, then $(q, xy, q'') \in \theta^*$.

Note that $(q, w, q') \in \theta^*$ if and only if there is a path in $G(\mathcal{T})$ that begins with $q$ and ends with $q'$ such that the concatenated labels of the directed edges of this path form the word $w$. 
If \( \mathcal{T} \) is the above transition system, then \((q_0, abbcc, q_3) \in \theta^* \) because \((q_0, ab, q_1), (q_1, bc, q_2), (q_2, c, q_3) \in \theta \). Similarly, \((q_0, c, q_3) \in \theta \) because \((q_0, \lambda, q_2), (q_2, c, q_3) \in \theta \).
Definition

Let \( \mathcal{T} = (A, Q, \theta, Q_0, F) \) be a transition system. The language accepted by \( \mathcal{T} \) is

\[
L(\mathcal{T}) = \{ x \in A^* \mid (q_0, x, q) \in \theta^* \text{ for some } q_0 \in Q_0, q \in F \}.
\]

Thus, a word \( x \) belongs to \( L(\mathcal{T}) \) if there is a path in \( G(\mathcal{T}) \) that begins in an initial state \( q_0 \in Q_0 \), labeled by \( x \), such that the path ends in one of the states of \( F \), the set of final states.
Transition systems generalize ndfas

If $\mathcal{M} = (A, Q, \delta, q_0, F)$ is a nondeterministic automaton, define the transition system $\mathcal{T}_\mathcal{M} = (A, Q, \theta, \{q_0\}, F)$, where

$$\theta = \{(q, a, q') \mid q, q' \in Q, a \in A \text{ and } q' \in \delta(q, a)\}.$$  

It can be shown by induction on $|x|$ that $(q, x, q') \in \theta^*$ if and only if $q' \in \delta^*(q, x)$. This implies that $L(\mathcal{T}_\mathcal{M}) = L(\mathcal{M})$. Therefore, every regular language can be accepted by a transition system. Furthermore, any language that can be accepted by a transition system is regular.
Lemma

For every transition system $T = (A, Q, \theta, Q_0, F)$ there exists a transition system $T' = (A, Q', \theta', Q_0, F)$ such that $(q, x, q_1) \in \theta'$ implies $|x| \leq 1$ and $L(T') = L(T)$. 
Proof

Define the relation $\theta'$ and the set $Q'$ as follows:

1. Every state $q \in Q$ also belongs to $Q'$.
2. Every triple $(q, x, q') \in \theta$ such that $|x| \leq 1$ also belongs to $\theta'$.
3. If $t = (q, x, q') \in \theta$ such that $x = a_0 \ldots a_{n-1}$ and $n \geq 2$, add $n - 1$ new states $q_0^t, \ldots q_{n-2}^t$ to $Q'$ and the triples

\[(q, a_0, q_0^t), (q_0^t, a_1, q_1^t), \ldots, (q_{n-2}^t, a_{n-1}, q')\]

...to $\theta'$.

The ts $\mathcal{T}'$ clearly satisfies the conditions of the lemma, since $(q, x, q') \in \theta^*$ if and only if $(q, x, q') \in \theta'^*$.
Theorem

For every transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ there exists a deterministic finite automaton $M$ such that $L(\mathcal{T}) = L(M)$.

Proof.

By the previous Lemma we can assume that $(q, x, q') \in \theta$ implies $|x| \leq 1$. Define the deterministic finite automaton $M = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$, where the initial state of $M$ is

$$Q'_0 = \{ q \in Q \mid (q_0, \lambda, q) \in \theta^* \text{ for some } q_0 \in Q_0 \},$$

the set of final states is $F' = \{ S \mid S \subseteq Q, S \cap F \neq \emptyset \}$, and the function $\Delta$ is defined by

$$\Delta(S, a) = \{ q' \in Q \mid (q, a, q') \in \theta^* \text{ for some } q \in S \},$$

for every $S \subseteq Q$ and $a \in A$. 

\qed
It is not difficult to verify, by induction on $|x|$, that

$$
\Delta^*(Q'_0, x) = \{ q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0 \},
$$

for $x \in A^*$. For the basis case, $|x| = 0$, so the above equality becomes

$$
Q'_0 = \{ q' \in Q \mid (q_0, \lambda, q') \in \theta^* \text{ for some } q_0 \in Q_0 \},
$$

which holds by the definition of $Q'_0$. 

Proof (cont’d)

Suppose that the equality holds for words of length $n$, and let $y$ be a word of length $n + 1$. We can write $y = xa$, so

$$\Delta^*(Q'_0, y) = \Delta^*(Q'_0, xa)$$
$$= \Delta(\Delta^*(Q'_0, x), a)$$
$$= \Delta(\{q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0\}, a)$$
(by the inductive hypothesis)
$$= \{r \in Q \mid (q', a, r) \in \theta^*, \text{ for some } q' \text{ such that}$$
$$(q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0\}$$
(by the definition of $\Delta$)
$$= \{r \in Q \mid (q_0, xa, r) \in \theta^* \text{ for some } q_0 \in Q_0\}$$
(by the definition of $\theta^*$)
$$= \{r \in Q \mid (q_0, y, r) \in \theta^* \text{ for some } q_0 \in Q_0\},$$

which concludes our inductive argument.
Proof (cont’d)

From this it follows that $L(M) = L(J)$. By definition, $x \in L(M)$ if and only if $\Delta^*(Q'_0, x) \in F'$. This is equivalent to $\Delta^*(Q'_0, x) \cap F \neq \emptyset$. This is equivalent to the existence of a state $q' \in F'$ such that $(q_0, x, q') \in \theta^*$ for some $q_0 \in Q_0$, and this is equivalent to $x \in L(J)$. 
Corollary

The class of languages that are accepted by transition systems is the class $\mathcal{R}$ of regular languages.
Definition

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The $\lambda$-closure is the mapping $K_\mathcal{T} : \mathcal{P}(Q) \rightarrow \mathcal{P}(Q)$ given by

$$K_\mathcal{T}(S) = \{ q \in Q \mid (s, \lambda, q) \in \theta^* \text{ for some } s \in S \}.$$ 

The set $K_\mathcal{T}(S)$ comprises the states in $S$ plus all the states that can be reached from a state in $S$ using a series of $\lambda$-transitions.
Theorem

Let $T = (A, Q, \theta, Q_0, F)$ be a transition system. The $\lambda$-closure of $T$ has the following properties.

1. $S \subseteq K_T(S)$;
2. $S \subseteq S'$ implies $K_T(S) \subseteq K_T(S')$;
3. $K_T(K_T(S)) = K_T(S)$,

for every $S, S' \in \mathcal{P}(Q)$. 
Proof

Since \((s, \lambda, s) \in \theta^*\) it is immediate that \(S \subseteq K_J(S)\) for every \(S \in \mathcal{P}(Q)\). The second part of the theorem is a direct consequence of the definition of \(K_J\).

Note that Parts (i) and (ii) imply \(K_J(S) \subseteq K_J(K_J(S))\). Let \(q \in K_J(K_J(S))\). There is a state \(s \in S\) and a state \(r \in K_J(S)\) such that \((s, \lambda, r) \in \theta^*\) and \((r, \lambda, q) \in \theta^*\). By the definition of \(\theta^*\) we obtain \((s, \lambda, q) \in \theta^*\), so \(q \in K_J(S)\). This implies \(K_J(K_J(S)) \subseteq K_J(S)\), which gives the last part of the theorem.
Definition

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. A $K_\mathcal{T}$-closed subset of $Q$ is a set $S$ such that $S \subseteq Q$ and $K_\mathcal{T}(S) = S$. 
Example

Consider the transition system

$$\mathcal{T} = (\{a, b\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_2, q_3\})$$

whose graph is shown

![Diagram of the transition system]
The closed subsets of $Q$ are $\emptyset$, $\{q_3\}$, $\{q_1, q_2, q_3\}$, and $Q$ itself.
Theorem

Let $T = (A, Q, \theta, Q_0, F)$ be a transition system and let $M = (A, \mathcal{P}(Q), \Delta, Q_0', F')$ be the dfa constructed in earlier. The set $\Delta(S, a)$ is a $K_T$-closed set of states for every subset $S$ of $Q$ and $a \in A$. 
To prove the theorem it suffices to show that $K_J(\Delta(S, a)) \subseteq \Delta(S, a)$. Let $p \in K_J(\Delta(S, a))$. There is $p_1 \in \Delta(S, a)$ such that $(p_1, \lambda, p) \in \theta^*$. The definition of $\Delta(S, a)$ implies the existence of $q \in S$ such that $(q, a, p_1) \in \theta^*$. Thus, $(q, a, p) \in \theta^*$, so $p \in \Delta(q, a)$. 
Corollary

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system, and let $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$ be the constructed DFA. The accessible states of the DFA $\mathcal{M}$ are $K_{\mathcal{T}}$-closed subsets of $Q$.

Proof.

The initial state $Q'_0$ of $\mathcal{M}$ is obviously closed. If $Q'$ is an accessible state of $\mathcal{M}$, then $Q' = \Delta(S, a)$ for some $S \subseteq Q$. Therefore $Q'$ is closed. $\square$
Theorem

Let \( T = (A, Q, \theta, Q_0, F) \) be a transition system, and let \( M = (A, \mathcal{P}(Q), \Delta, Q'_0, F') \) be the dfa constructed earlier. Then, \( \Delta(S, a) = K_T(\{q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S\}) \), where \( S \) is an accessible state of \( M \).
Let $s \in S$. Note that if $(s, a, q) \in \theta$ and $(q, \lambda, q_1) \in \theta^*$, then by the definition of $\theta^*$ we have $(s, a, q_1) \in \theta^*$. Therefore, $K_T(\{q \in Q \mid (s, a, q) \in \theta\}) \subseteq \Delta(S, a)$.

To prove the converse inclusion, $\Delta(S, a) \subseteq K_T(\{q \in Q \mid (s, a, q) \in \theta\})$, let $(s, a, q_1) \in \theta^*$. Then, there is a path in $G(T)$ that begins with $s$ and ends with $q_1$ such that the concatenated labels of the directed edges of this path form the word $a$. This implies the existence of the states $p, p' \in Q$ such that $(s, \lambda, p) \in \theta^*$, $(p, a, p_1) \in \theta$, and $(p_1, \lambda, q_1) \in \theta^*$. Since $S$ is $K_T$-closed it follows that $p \in S$ and this gives the desired conclusion.
An Algorithm for Constructing a dfa corresponding to a ts

**Input:** A transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$.

**Output:** An accessible dfa $\mathcal{M}_1$ such that $L(\mathcal{M}_1) = L(\mathcal{T})$.

**Method:** Compute the increasing sequence of collections of subsets of $Q$, $Q_0, \ldots, Q_i, \ldots$, where

\[
\begin{align*}
Q_0 &= \{ Q_0' \} \\
Q_{i+1} &= Q_i \cup \{ U \in \mathcal{P}(Q) \mid U = \Delta(S, a) \text{ for some } S \in Q_i \text{ and } a \in A \}.
\end{align*}
\]
the computation of $U = \Delta(S, a)$ can be done by computing first the set $W = \{q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S\}$ and then $U = K_T(W)$. Stop when $Q_{i+1} = Q_i$. The set $Q_i$ is the set of accessible states of $M$. Output $M' = ACC(M)$, the accessible component of $M$. 
For the transition system
the transition system is;