Homework 4

Posted: November 16th, 2016
Due: November 30th, 2016

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(\mathbf{x}) = x_1$. Consider the problem:

$$\begin{aligned} &\text{minimize } f(\mathbf{x}); \\ &\qquad\qquad \text{subject to } x_1^2 + x_2^2 = 1. \end{aligned}$$

   Derive the dual function and verify its concavity. Find the optimal solution for both the primal and the dual problems and compare their objective values.

2. Consider the problem:

$$\begin{aligned} &\text{minimize } e^{-x}; \\ &\qquad\qquad \text{subject to } -x \leqslant 0. \end{aligned}$$

   Find the Lagrangian, the dual function, and solve the dual problem.

3. Consider the following problem:

$$\begin{aligned} &\text{minimize } (x_1 - 3)^2 + (x_2 - 5)^2 \\ &\qquad\qquad \text{subject to } x_1^2 - x_2 \leqslant 0, \\ &\qquad\qquad\qquad\qquad -x_1 \leqslant 1, \\ &\qquad\qquad\qquad\qquad x_1 + 2x_2 \leqslant 10 \\ &\qquad\qquad\qquad\qquad x_1 \geqslant 0,\ x_2 \geqslant 0. \end{aligned}$$

   Find the optimal solution geometrically and verify it using the Kuhn-Tucker conditions.

4. Suppose that there exist two positive numbers y_1 and y_2 such that

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 &= 0, \\ a_{12}y_1 + a_{22}y_2 &= 0. \end{aligned}$$

   Prove that there exist no numbers x_1, x_2 such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &< 0, \\ a_{21}x_1 + a_{22}x_2 &< 0. \end{aligned}$$