# Convex Sets and Convex Functions (part I)

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# Special Subsets in $\mathbb{R}^n$

Let $L$ be a real linear space and let $x, y \in L$. The closed segment determined by $x$ and $y$ is the set

$$[x, y] = \{(1 - a)x + ay \mid 0 \leqslant a \leqslant 1\}.$$

The half-closed segments determined by $x$ and $y$ are the sets

$$[x, y) = \{(1 - a)x + ay \mid 0 \leqslant a < 1\},$$

and

$$(x, y] = \{(1 - a)x + ay \mid 0 < a \leqslant 1\}.$$

The open segment determined by $x$ and $y$ is

$$(x, y) = \{(1 - a)x + ay \mid 0 < a < 1\}.$$

The line determined by $x$ and $y$ is the set

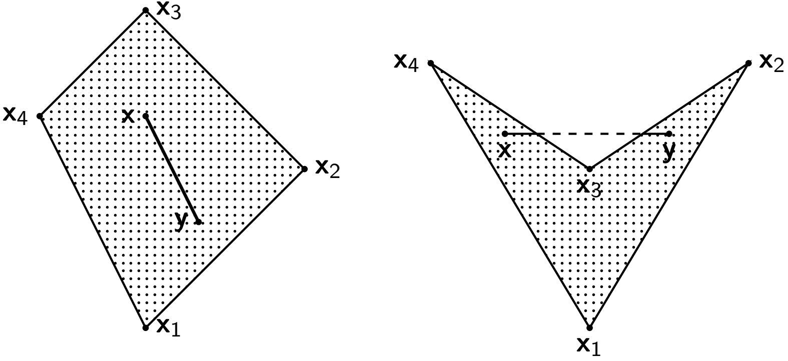
$$\ell_{x,y} = \{(1 - a)x + ay \mid a \in \mathbb{R}\}.$$

### Definition

A subset $C$ of $L$ is convex if we have $[x, y] \subseteq C$ for all $x, y \in C$.

Note that the empty subset and every singleton $\{x\}$ of $L$ are convex.

# Convex vs. Non-convex




(a) (b)

## Example

The set $\mathbb{R}^n_{\geqslant 0}$ of all vectors of $\mathbb{R}^n$ having non-negative components is a convex set called the non-negative orthant of $\mathbb{R}^n$.

### Example

The convex subsets of $(\mathbb{R}, +, \cdot)$ are the intervals of $\mathbb{R}$. Regular polygons are convex subsets of $\mathbb{R}^2$.

### Example

Every linear subspace $T$ of a real linear space $L$ is convex.

## Example

Let $(L, \| \cdot \|)$ be a normed linear space. An open sphere $B(x_0, r) \subseteq L$ is convex.
Indeed, suppose that $x, y \in B(x_0, r)$, that is, $\| x - x_0 \| < r$ and $\| x_0 - y \| < r$.
Let $a \in [0, 1]$ and let $z = (1 - a)x + ay$. We have

$$\begin{aligned} \| x_0 - z \| &= \| x_0 - (1-a)x - ay \| \\ &= \| a(x_0 - y) + (1-a)(x_0 - x) \| \\ &\leqslant a \| x_0 - y \| + (1-a) \| x_0 - x \| < r. \end{aligned}$$

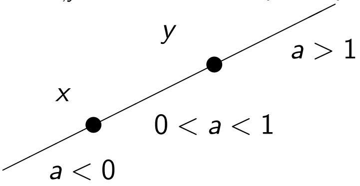
so $z \in B(x_0, r)$.
Similarly, a closed sphere $B[x_0, r]$ is a convex set.

### Theorem

*Let $x, y, z$ be three distinct points in the real linear space $L$ such that $z \in \ell_{x,y}$. Then, one of these points belongs to the open segment determined by the remaining two points.*

# Proof

Since $z \in \ell_{x,y}$, we have $z = (1-a)x + ay$ for some $a \in \mathbb{R}$.




We have $a \notin \{0,1\}$ because the points $x, y, z$ are distinct. If $a > 1$ we have $y = \frac{a-1}{a}x + \frac{1}{a}z$, so $y \in (x,z)$ because $\frac{a-1}{a}, \frac{1}{a} \in (0,1)$. If $0 < a < 1$ we have $z \in (x,y)$. Finally, if $a < 0$, since $x = \left(1 + \frac{a}{1-a}\right) z + \frac{-a}{1-a} y$, we have $x \in (z,y)$.

**Definition**

Let $U$ be a subset of a real linear space $L$ and let $x_1, \ldots, x_k \in U$. A linear combination of $U$, $a_1 x_1 + \cdots + a_k x_k$, where $a_1, \ldots, a_k \in \mathbb{R}$ and $k \geqslant 1$ is:

- an affine combination of $U$ if $\sum_{i=1}^{k} a_i = 1$;
- a non-negative combination of $U$ if $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$;
- a positive combination of $U$ if $a_i > 0$ for $1 \leqslant i \leqslant k$;
- a convex combination of $U$ if it is both a non-negative and an affine combination of $U$.

**Theorem**

*Let $L$ be a real linear space. A subset $C$ of $L$ is convex if and only if any convex combination of elements of $C$ belongs to $C$.*

# Proof

The sufficiency of this condition is immediate. To prove its necessity consider $x_1, \ldots, x_k \in C$ and the convex combination

$$y = a_1 x_1 + \cdots + a_k x_k.$$

We prove by induction on $k \geqslant 1$ that $y \in C$. The base case, $m = 1$ is immediate since in this case $y = a_1 x_1$ and $a_1 = 1$.
For the inductive step, suppose that the statement holds for $k$ and let $y$ be given by $y = a_1 x_1 + \cdots + a_k x_k + a_{k+1} x_{k+1}$, where $a_1 + \cdots + a_k + a_{k+1} = 1$, $a_i \geqslant 0$ and $x_i \in C$ for $1 \leqslant i \leqslant k+1$. We have

$$y = (1 - a_{k+1}) \sum_{i=1}^{k} \frac{a_i}{1 - a_{k+1}} x_i + a_{k+1} x_{k+1}.$$

Since $z = \sum_{i=1}^{k} \frac{a_i}{1-a_{k+1}} x_i$ is a convex combination of $k$ vectors, we have $z \in C$ by the inductive hypothesis, and the equality $y = (1 - a_{k+1}) z + a_{k+1} x_{k+1}$ implies $y \in C$.

### Definition

Let $L, K$ be two linear spaces. A mapping $f : L \longrightarrow K$ is affine when there exists a linear mapping $h : L \longrightarrow K$ and some $b \in K$ such that $f(x) = h(x) + b$ for every $x \in L$

## Theorem

*Let $L, K$ be two linear spaces and let $f : L \longrightarrow K$ be an affine mapping. If $C$ be a convex subset of $L$, then $f(C)$ is a convex subset of $K$. If $D$ is a convex subset of $K$, then $f^{-1}(D) = \{x \in L \mid f(x) \in D\}$ is a convex subset of $L$.*

# Proof

Since $f$ is an affine mapping, we have $f(x) = h(x) + b$, where $h : L \longrightarrow K$ is a linear mapping and $b \in K$ for $x \in L$. Therefore, if $y_1, y_2 \in f(C)$ we can write $y_1 = h(x_1) + b$ and $y_2 = h(x_2) + b$. This, in turn, allows us to write for $a \in [0, 1]$:

$$\begin{aligned}(1-a)y_1 + ay_2 &= (1-a)h(x_1) + (1-a)b + ah(x_2) + ab \\ &= h((1-a)x_1 + ax_2) + b = f((1-a)x_1 + ax_2).\end{aligned}$$

The convexity of $C$ implies $(1-a)x_1 + ax_2 \in C$, so $(1-a)y_1 + ay_2 \in f(C)$, which shows that $f(C)$ is convex.

**Definition**

A subset $C$ of a linear space $L$ is affine subspace if $\ell_{x,y} \subseteq C$ for all $x, y \in C$.

In other words, $C$ is a non-empty affine subspace if every point on the line determined by two members of $C$, $x$ and $y$ belongs to $C$. Note that $C$ is a subspace of $L$ if and only if $0_L \in C$ and $C$ is an affine subspace.

**Example**

The empty set $\emptyset$, every singleton $\{x\}$, and the entire space $L$ are affine subspaces of $L$. Also, every hyperplane $H$ is an affine subspace of $L$.

**Theorem**

*A non-empty subset $C$ of a linear space $L$ is an affine subspace if and only if any affine combination of elements of $C$ belongs to $C$.*

It is immediate to verify that any translation of a linear space $K$ is an affine subspace. The converse is also true as we show next.

**Theorem**

*Let $D$ be a non-empty affine subspace in a linear space $L$. There exists a translation $t_u$ and a unique subspace $K$ of $L$ such that $D = t_u(K)$.*

# Proof

Let $K = \{x - y \mid x, y \in D\}$ and let $x_0 \in D$. We have $0_L = x_0 - x_0 \in K$ and it is immediate that $K$ is subspace of $L$.
Let $u$ be an element of $D$. We claim that $D = \mathrm{t}_u(K)$. Indeed, if $z \in D$, $z - u \in K$, so $z \in \mathrm{t}_u(K)$, which implies $D \subseteq \mathrm{t}_u(K)$. Conversely, if $x \in \mathrm{t}_u(K)$ we have $x = u + v$ for some $v \in K$ and, therefore, $x = u + s - t$ for some $s, t \in D$, where $v = s - t$. This implies $x \in D$ because $u + s - t$ is an affine combination of $D$.

# Proof (cont'd)

To prove the uniqueness of the subspace $K$ suppose that $D = \mathrm{t}_u(K_1) = \mathrm{t}_{\mathbf{v}}(K_2)$, where both $K_1$ and $K_2$ are subspaces of $L$.
Since $0_L \in K_2$, it follows that there exists $w \in K_1$ such that $u + w = v$.
Similarly, since $0_L \in K_1$, it follows that there exists $t \in K_1$ such that $u = v + t$, which implies $w + t = 0_L$. Thus, both $w$ and $t$ belong to both subspaces $K_1$ and $K_2$.
If $s \in K_1$, it follows that $u + s = v + z$ for some $z \in K_2$. Therefore, $s = (v - u) + z \in K_2$ because $w = v - u \in K_2$. This implies $K_1 \subseteq K_2$.
The reverse inclusion can be shown similarly.

### Definition

Let $D$ be a non-empty affine subspace in a linear space $L$. The dimension of $D$ (denoted by $\dim(D)$) is the dimension of the unique subspace $K$ of $L$ such that $D = \mathrm{t}_u(K)$ for some translation $\mathrm{t}_u$ of $L$.
The dimension of a convex set $C$ is the dimension of the affine space $\mathbf{K}_{\mathrm{aff}}(C)$.

Since $\emptyset$ is an affine subspace of $L$ and there is no subspace of $L$ that can be translated into $\emptyset$, the dimension of $\emptyset$ is set through the special definition $\dim(\emptyset) = -1$.
Let $D, E$ be two affine subspaces in a linear space $L$. The sets $D, E$ are parallel if $E = \mathrm{t}_a(D)$, for some translation $\mathrm{t}_u$ of $L$. In this case we write $D \parallel E$.
It is easy to see that "$\parallel$" is an equivalence relation on the set of affine subspaces of a linear space $L$. Furthermore, each equivalence class contains exacly one subspace of $L$.

# Affine Subspaces and Linear Systems

with solving linear systems.

**Theorem**

*Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{b} \in \mathbb{R}^m$. The set $S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$ is an affine subset of $\mathbb{R}^n$. Conversely, every affine subset of $\mathbb{R}^n$ is the set of solutions of a system of the form $A\mathbf{x} = \mathbf{b}$.*

# Proof

It is immediate that the set of solutions of a linear system is affine. Conversely, let $S$ be an affine subset of $\mathbb{R}^n$ and let $L$ be the linear subspace such that $S = \mathbf{u} + L$. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be a basis of $L^\perp$. We have

$$L = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i'\mathbf{x} = 0 \text{ for } 1 \leqslant i \leqslant m\} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\},$$

where $A$ is a matrix whose rows are $\mathbf{a}_1', \ldots, \mathbf{a}_m'$. By defining $\mathbf{b} = A\mathbf{u}$ we have

$$S = \{\mathbf{u} + \mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \{\mathbf{y} \in \mathbb{R}^n \mid A\mathbf{y} = \mathbf{b}\}.$$

### Definition

A subset $U = \{x_1, \ldots, x_n\}$ of a real linear space $L$ is affinely dependent if $0_L = a_1x_1 + \cdots + a_nx_n$, at least one of the numbers $a_1, \ldots, a_n$ is nonzero, and $\sum_{i=1}^{n} a_i = 1$. If no such affine combination exists, then $x_1, \ldots, x_n$ are affinely independent.

**Theorem**

*Let* $U = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *be a finite subset of a real linear space* $L$*. The set* $U$ *is affinely independent if and only if the set* $V = \{x_1 - x_n, \ldots, x_{n-1} - x_n\}$ *is linearly independent.*

# Proof

Suppose that $U$ is affinely independent but $V$ is linearly dependent; that is, $0_L = b_1(x_1 - x_n) + \cdots + b_{n-1}(x_{n-1} - x_n)$ such that not all numbers $b_i$ are 0. This implies $b_1 x_1 + \cdots + b_{n-1} x_{n-1} - \left(\sum_{i=1}^{n-1} b_i\right) x_n = \mathbf{0}$, which contradicts the affine independence of $U$.

# Proof (cont'd)

Conversely, suppose that $V$ is linearly independent but $U$ is not affinely independent. In this case, $0_L = a_1 x_1 + \cdots + a_n x_n$ such that at least one of the numbers $a_1, \ldots, a_n$ is nonzero and $\sum_{i=1}^{n} a_i = 0$. This implies $a_n = -\sum_{i=1}^{n-1} a_i$, so $0_L = a_1(x_1 - x_n) + \cdots + a_{n-1}(x_{n-1} - x_n)$. Observe that at least one of the numbers $a_1, \ldots, a_{n-1}$ must be distinct from 0 because otherwise we would have $a_1 = \cdots = a_{n-1} = a_n = 0$. This contradicts the linear independence of $V$, so $U$ is affinely independent.

### Definition

The subset $U = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is in general position if its points are affinely independent, or equivalently, if the set $V = \{x_1 - x_n, \ldots, x_{n-1} - x_n\}$ is linearly independent.

### Corollary

*The maximal size of an affinely independent set of vectors in $\mathbb{R}^n$ is $n+1$.*

**Proof:** Since the maximal size of a linearly independent set in $\mathbb{R}^n$ is $n$, it follows that the maximal size of an affinely independent set in $\mathbb{R}^n$ is $n+1$.

### Example

Let $\mathbf{x}_1, \mathbf{x}_2$ be vectors in $\mathbb{R}^2$. The line that passes through $\mathbf{x}_1$ and $\mathbf{x}_2$ consists of all vectors $\mathbf{x}$ such that $\mathbf{x} - \mathbf{x}_1$ and $\mathbf{x} - \mathbf{x}_2$ are collinear; that is, $a(\mathbf{x} - \mathbf{x}_1) + b(\mathbf{x} - \mathbf{x}_2) = 0$ for some $a, b \in \mathbb{R}$ such that $a + b \neq 0$. Thus, we have $\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$, where $a_1 = \frac{a}{a+b}$, $a_2 = \frac{b}{a+b}$ and $a_1 + a_2 = 1$, so $\mathbf{x}$ is an affine combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. On other hand, the segment of line contained between $\mathbf{x}_1$ and $\mathbf{x}_2$ is consists of convex combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$.

### Theorem

*The intersection of any collection of convex sets in a real linear space is a convex set.*
*The intersection of any collection of affine subspaces in a real linear space is an affine subspace.*

# Proof

Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a collection of convex sets and let $C = \bigcap \mathcal{C}$.
Suppose that $x_1, \ldots, x_k \in C$, $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$, and $a_1 + \cdots + a_k = 1$.
Since $x_1, \ldots, x_k \in C_i$, it follows that $a_1x_1 + \cdots + a_kx_k \in C_i$ for every $i \in I$.
Thus, $a_1x_1 + \cdots + a_kx_k \in C$, which proves the convexity of $C$.
The argument for the affine subspaces is similar.

**Corollary**

*The families of convex sets and non-empty affine subspaces in a real linear space $L$ are closure systems.*

**Proof:** This statement follows immediately by observing that $L$ itself is both a convex set and an affine subspace.

The convex hull (or the convex closure) of a subset $U$ of $L$ is the intersection $\mathbf{K}_{\mathrm{conv}}(U)$ of all closed sets that contain the set $U$.
Similarly, the affine hull of $U$, denoted by $\mathbf{K}_{\mathrm{aff}}(U)$, is the intersection of all affine sets that contain $U$.

It is immediate that $\mathbf{K}_{\mathrm{conv}}(U)$ consists of all convex combinations of elements of $U$, $\mathbf{K}_{\mathrm{aff}}(U)$ consists of all affine combinations of the same elements, and

$$\mathbf{K}_{\mathrm{conv}}(U) \subseteq \mathbf{K}_{\mathrm{aff}}(U) \subseteq \langle U \rangle$$

because each convex combination is also an affine combination and each affine combination is a linear combination.

The dimension $\dim(D)$ of an affine subspace $D$ is the dimension of the unique subspace $K$ of $L$ such that $D = \mathrm{t}_u(K)$ for some translation $\mathrm{t}_u$.

**Theorem**

*For an affine subspace $D$ of $\mathbb{R}^n$ we have $\dim(D) = m - 1$ if and only if $m$ is the largest non-negative integer such that there exists an affinely independent set of $m$ elements of $D$.*

# Proof

Suppose that $\dim(D) = m - 1$ and $D = t_{\mathbf{u}}(K)$, where $K$ is a subspace of $\mathbb{R}^n$ of dimension $m - 1$ and $\mathbf{u} \notin K$. Let $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_{m-1}\}$ be a basis of $K$.

The set that consists of $m$ vectors of $D$

$$\mathbf{x}_1 = \mathbf{u} + \mathbf{y}_1, \ldots, \mathbf{x}_{m-1} = \mathbf{u} + \mathbf{y}_{m-1}, \mathbf{x}_m = \mathbf{u}$$

is affinely independent because the set $\{\mathbf{x}_1 - \mathbf{x}_m, \ldots, \mathbf{x}_{m-1} - \mathbf{x}_m\}$ is linearly independent.

There is no affinely independent set in $D$ that consists of more than $m$ point because this would entail the existence in $K$ of a basis that consists of more than $m - 1$ vectors.

Let $S$ be a non-empty subset of $\mathbb{R}^n$. If $\mathbf{0}_n \in \mathbf{K}_{\mathrm{aff}}(S)$, it follows that $\mathbf{K}_{\mathrm{aff}}(S)$ is a subspace of $\mathbb{R}^n$ that coincides with the subspace $\langle S \rangle$ generated by $S$, and $\dim(\mathbf{K}_{\mathrm{aff}}(S)) = \dim(\langle S \rangle)$.

Theorem

**(Stone's Theorem)** *Let $L$ be a real linear space and let $A, B$ be two disjoint convex subsets of $L$. There exists a partition $\pi = \{C, D\}$ of $L$ such that $C$ and $D$ are convex, $A \subseteq C$ and $B \subseteq D$.*

# Proof

Let $\mathcal{E} = \{E \in \mathcal{P}(L) \mid E \text{ is convex}, A \subseteq E, B \cap E = \emptyset\}$. Clearly, $\mathcal{E} \neq \emptyset$ because $A \in \mathcal{E}$. The collection $\mathcal{E}$ is partially ordered by set inclusion, so by Zorn's Lemma, it contains a maximal element $C$, which is clearly convex and disjoint from $B$. We need to show only that $D = L - C$ is convex.
If $D = B$, $D$ is convex and the argument is complete. Therefore, we assume that $B \subset D$, so the set $D - B$ is non-empty.

# Proof (cont'd)

If $D$ were not convex, then we would have $x, z$ in $D$ such that $[x, z] \cap C \neq \emptyset$, so we would have $y \in (x, z) \cap C$, that is $y = (1 - c)x + cz$ for some $c \in (0, 1)$.
Note that we cannot have both $x \in B$ and $z \in B$ because this would imply that $C \cap B \neq \emptyset$. Thus, at least one of $x$ and $z$ must not belong to $B$.
Suppose for now that neither $x$ nor $z$ belong to $B$.

# Proof (cont'd)

We claim that there is a point $p \in C$ such that $(p, x) \cap B \neq \emptyset$ and a point $q \in C$ such that $(q, z) \cap B \neq \emptyset$. Equivalently, if for all $p \in C$, $(p, x) \cap B = \emptyset$, or for all $q \in C$, $(q, z) \cap B = \emptyset$, then $C$ is not a maximal convex set that contains $A$ and is disjoint from $B$. Assume that for all $p \in C$, $(p, x) \cap B = \emptyset$. Then, $C \subseteq \mathbf{K}_{\mathrm{conv}}(\{x\} \cup C\}$. Since $x \notin B$, $\mathbf{K}_{\mathrm{conv}}(\{x\} \cup C)$ is disjoint from $B$, which contradicts the maximality of $C$. Therefore, there exists $p \in C$ such that $(p, x) \cap B \neq \emptyset$. Similarly, there exists $q \in C$ such that $(q, z) \cap B \neq \emptyset$.

# Proof (cont'd)

Let $u \in (p, x) \cap B$ and let $v \in (q, z) \cap B$. We have

$$u = (1-a)p + ax \text{ and } v = (1-b)q + bz$$

for some $a, b \in (0, 1)$. Since

$$\begin{aligned} x &= \frac{1}{a}u - \frac{1-a}{a}p, \\ z &= \frac{1}{b}v - \frac{1-b}{b}q, \end{aligned}$$

we have

$$y = \frac{1-c}{a}u - \frac{(1-c)(1-a)}{a}p + \frac{c}{b}v - \frac{c(1-b)}{b}q,$$

or. equivalently

$$\frac{1-c}{a}u + \frac{c}{b}v = y + \frac{(1-c)(1-a)}{a}p + \frac{c(1-b)}{b}q. \tag{1}$$

# Proof (cont'd)

Observe that

$$\frac{1-c}{a} + \frac{c}{b} = 1 + \frac{(1-c)(1-a)}{a} + \frac{c(1-b)}{b}. \tag{2}$$

Let $k$ be the value of either side of the equality. Since the coefficients that occur in both sides of Equality (1) are non-negative, by dividing both sides of this equality by $k$ we obtain a convex combination of $u$ and $v$ equal to a convex combination of $y, p$ and $q$. Thi contradicts that $C$ and $B$ are disjoint. Therefore, $D$ is convex.

Suppose now that $x \in B$ and $z \notin B$. The role played previously by $u$ will be played by $x$ and the previous argument is applicable with $x = u$.

### Theorem

*Every affine subset $S$ of $\mathbb{R}^n$ is the intersection of a finite collections of hyperplanes.*

**Proof:** $S$ can be written as $S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Therefore, $\mathbf{x} \in S$ if and only if $\mathbf{a}_i'\mathbf{x} = b_i$, where $\mathbf{a}_i$ is the $i^{\text{th}}$ row of $A$. Thus, $S = \bigcap_{i=1}^m H_{\mathbf{a}_i, b_i}$.

The class of convex set is closed with respect to scalar multiplications and translations. In other words, it is immediate that if $C$ is a subset of a real linear space, then $\mathrm{h}_r(C) = rC = \{rx \mid x \in C\}$ is a convex set for $r \in \mathbb{R}$; also, for $b \in L$ the set $\mathrm{t}_b(C) = C + b = \{x + b \mid x \in C\}$ is convex.
The Minkowski sum of two subsets $C_1, C_2$ of $\mathbb{R}^n$ is the set

$$C_1 + C_2 = \{\mathbf{x}_1 + \mathbf{x}_2 \mid \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}.$$

**Theorem**

*If $C_1, C_2$ are convex subsets of a real linear space $L$, their Minkowski sum $C_1 + C_2$ is a convex subset of $L$.*

# Proof

Let $x, y \in C_1 + C_2$. We have $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. Therefore, for $c \in [0, 1]$ we have

$$\begin{aligned}(1-a)x + ay &= (1-a)(x_1 + x_2) + a(y_1 + y_2) \\ &= (1-a)x_1 + ay_1 + (1-a)x_2 + ay_2 \in C_1 + C_2,\end{aligned}$$

because $(1-a)x_1 + ay_1 \in C_1$ and $(1-a)x_2 + ay_2 \in C_2$ because of the convexity of $C_1$ and $C_2$.

If $C_1, \ldots, C_m$ are convex sets and $r_1, \ldots, r_m$ then the set $r_1 C_1 + \cdots + r_m C_m$ is convex.

**Theorem**

*Let $C$ be a convex subset of a real linear space $L$. If $r_1, r_2 \in \mathbb{R}_{\geqslant 0}$, then we have*

$$(r_1 + r_2)C = r_1 C + r_2 C.$$

# Proof

If at least one of $r_1, r_2$ is 0 the equality obviously holds; therefore, assume that both $r_1$ and $r_2$ are positive.
Let $z \in r_1 C + r_2 C$. There exists $x, y \in C$ such that $z = r_1 x + r_2 y$, and therefore,

$$z = (r_1 + r_2)\left(\frac{r_1}{r_1 + r_2}x + \frac{r_1}{r_1 + r_2}y\right).$$

Since $C$ is convex, $\frac{r_1}{r_1+r_2}x + \frac{r_1}{r_1+r_2}y \in C$, which implies $z \in (r_1 + r_2)C$, so $r_1 C + r_2 C \subseteq (r_1 + r_2)C$. The reverse inclusion is immediate and makes no use of the convexity of $C$.

### Definition

Let $L$ be a real linear space. A cone in $L$ is a non-empty set $C \subseteq L$ such that $x \in C$ and $a \in \mathbb{R}_{\geqslant 0}$ imply $ax \in C$.

### Example

Let $L$ be a real linear space and let $S$ be a non-empty subset of $L$. The set

$$C_S = \{ax \mid a \geqslant 0 \text{ and } x \in S\}$$

is cone contained by every other cone that contains $S$.

### Example

The set $(\mathbb{R}_{\geqslant 0})^n$ is a pointed cone.

### Theorem

*Let $L$ be a real linear space and let $C \in L$ be a cone. $C$ is convex if and only if $x + y \in C$ for $x, y \in \mathbb{R}$.*

# Proof

Let $C$ be a convex cone. If $x, y \in C$ and $a \in (0,1)$, then $\frac{1}{a}x \in C$ and $\frac{1}{1-a}y \in C$. Therefore, by convexity we have

$$x + y = a\frac{1}{a}x + (1-a)\frac{1}{1-a}y \in C.$$

Conversely, let $C$ be a cone such that $x, y \in C$ imply $x + y \in C$. For $u, v \in C$ and $a \in [0,1]$ let $z_a = au + (1-a)v$. Since $C$ is a cone, $au \in C$ and $(1-a)v \in C$, hence $z_a = au + (1-a)v \in C$. Therefore, $C$ is convex.

Example

Let $U$ be a non-empty subset of a real linear space $L$. The set of all non-negative combinations of $U$ is a convex cone that is included in every convex cone that contains $U$.

### Theorem

*The intersection of any collection of cones (convex cones) in a real liner space $L$ is a cone (a convex cone).*

### Corollary

*The families of cones (convex cones) in a real linear space L is a closure system.*

**Proof:** This statement follows immediately by observing that $\mathbb{R}^n$ itself is cone (a convex cone).

We denote the closure operator corresponding to the family of cones by $\mathbf{K}_{\mathrm{cone}}$.

**Theorem**

*Let $S$ be a non-empty subset of a real linear space $L$. We have* $\mathbf{K}_{cone}(S) = \{ax \mid a \geqslant 0 \text{ and } x \in S\}$.

**Proof:** Since $\{ax \mid a \geqslant 0 \text{ and } x \in S\}$ is a cone that contains $S$, $\mathbf{K}_{\mathrm{cone}}(S) \subseteq \{ax \mid a \geqslant 0 \text{ and } x \in S\}$.
Conversely, since $S \subseteq \mathbf{K}_{\mathrm{cone}}(S)$, if $x \in S$ it follows that $ax \in \mathbf{K}_{\mathrm{cone}}(S)$ for every $a \geqslant 0$, so $\{ax \mid a \geqslant 0 \text{ and } x \in S\} \subseteq \mathbf{K}_{\mathrm{cone}}(S)$.

### Definition

Let $C$ be a non-empty convex subset of a real linear space $L$. An extreme point of $C$ is a point $x \in C$ such that if $x \in [u, v]$ and $u, v \in C$, then $u = v = x$.

### Theorem

*Let $C$ be a non-empty convex subset of a real linear space $L$. A point $x \in C$ is an extreme point of $C$ if the set $C - \{x\}$ is convex.*

# Proof

Suppose that $C - \{x\}$ is a convex set for $x \in C$ and that $x \in [u, v]$, where $u, v \in C$.

If $x$ is distinct from both $u$ and $v$, then $u, v$ belong to the convex set $C - \{x\}$, which yields the contradiction $x \in C - \{x\}$. Thus, $x$ is an extreme point of $C$.

Conversely, suppose that $x$ is an extreme point of $C$. Let $u, v \in C - \{x\}$, so $u \neq x$ and $y \neq x$. If $x \in [u, v]$, we obtain a contradiction since this implies $u = v = x$. Therefore $[u, v] \subseteq C - \{x\}$, so $C - \{x\}$ is convex.

The set of extreme points of a convex set $C$ is denoted by $\text{extr}(C)$.

### Example

Let $B[\mathbf{x}_0, r]$ be a closed sphere of radius $r$ in $\mathbb{R}^n$. Each point $\mathbf{x}$ located on the circumference of this sphere, that is, each point $\mathbf{x}$ such that $\| \mathbf{x}_0 - \mathbf{x} \| = r$ is an extreme point of $B[\mathbf{x}_0, r]$.
Indeed, suppose that $a\mathbf{u} + (1-a)\mathbf{v} = \mathbf{x}$ for some $a \in (0,1)$ and $\| \mathbf{x}_0 - \mathbf{u} \| = \| \mathbf{x}_0 - \mathbf{v} \| = r$. Then, by Supplement **??**, we have $\mathbf{u} = \mathbf{v} = \mathbf{x}$.

## Example

An open sphere $B(\mathbf{x}_0, r)$ in $\mathbb{R}^n$ has no extreme points for if $\mathbf{x} \in B(\mathbf{x}_0, r)$. Indeed, let $\mathbf{u}$ be a vector such that $\mathbf{u} \neq \mathbf{0}_n$ and let $\mathbf{x}_1 = \mathbf{x} + a\mathbf{u}$ and $\mathbf{x}_2 = \mathbf{x} - a\mathbf{u}$, where $a > 0$. Observe that if $a < \frac{r - \|\mathbf{x} - \mathbf{x}_0\|}{\|\mathbf{u}\|}$, we have

$$\| \mathbf{x}_1 - \mathbf{x}_0 \| = \| \mathbf{x} + a\mathbf{u} - \mathbf{x}_0 \| \leqslant \| \mathbf{x} - \mathbf{x}_0 \| + a \| \mathbf{u} \| < r$$

and

$$\| \mathbf{x}_2 - \mathbf{x}_0 \| = \| \mathbf{x} - a\mathbf{u} - \mathbf{x}_0 \| \leqslant \| \mathbf{x} - \mathbf{x}_0 \| + a \| \mathbf{u} \| < r,$$

and we have both $\mathbf{x}_1 \in B(\mathbf{x}_0, r)$ and $\mathbf{x}_2 \in B(\mathbf{x}_0, r)$. Since $\mathbf{x} = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2$, $\mathbf{x}$ is not an extreme point.

Example

The extreme points of the cube $[0, 1]^n$ are all its $2^n$ "corners" $(a_1, \ldots, a_n) \in \{0, 1\}^n$.

### Definition

Let $C$ be a convex set in a real linear space $L$. A convex subset $F$ of $C$ is a face of $C$ if for every open segment $(u, v) \subseteq C$ such that at least one of $u, v$ is not in $F$ we have $(u, v) \cap F = \emptyset$.
If $F \neq C$, we say that $F$ is a proper face of $C$.
A $k$-face of $C$ is a face $F$ of $C$ such that $\dim(F) = k$.

A convex subset $F$ is a face of $C$ if $u, v \in C$ and $(u, v) \cap F \neq \emptyset$ implies $u \in F$ and $v \in F$, which is equivalent to $[u, v] \subseteq F$.
Note that if $F = \{x\}$ is a face of $C$ if and only if $x \in \text{extr}(C)$. An convex subset $C$ is a face of itself.

**Theorem**

*If $F$ is a face of a convex set $C$, then $F = \mathbf{K}_{\mathit{aff}}(F) \cap C$.*

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# Proof

If $z \in \mathbf{K}_{\mathrm{aff}}(F) \cap C$, we have $x = a_1 y_1 + \cdots + a_k y_k$, where $\sum_{i=1}^{k} a_i = 1$ and $y_1, \ldots, y_k \in F$. If all $a_i$ are non-negative, then it is immediate that $x \in F$. Otherwise, let $b = -\sum\{a_i \mid a_i < 0\}$ and let

$$\begin{aligned} u &= \frac{1}{1+b}\sum\{a_i y_i \mid a_i \geqslant 0\} \\ v &= -\frac{1}{b}\sum\{a_i y_i \mid a_i < 0\}. \end{aligned}$$

We have $x \in C, v \in C$, and

$$u = \frac{1}{1+b}x + \frac{b}{1+b} \in [x, v] \cap F.$$

Since $F$ is a face, we have $u \in F$. Thus, $\mathbf{K}_{\mathrm{aff}}(F) \cap C \subseteq F$. The reverse inclusion is immediate.