Convex Sets and Functions
(part II)

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UMB
1. Convex Functions - Basics and Examples
2. Extrema of Convex Functions
3. Convexity of One-Argument Functions
4. Jensen’s Inequality
Definition

Let $D$ be a subset of a real linear space $L$. A function $f : D \rightarrow \mathbb{R}$ is convex if for every $x, y \in D$ such that $(1 - t)x + ty \in D$ for $t \in [0, 1]$ we have

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y).$$

A function $g : D \rightarrow \mathbb{R}$ is concave if $-g$ is convex at $x$, that is, $g((1 - t)x + ty) \geq (1 - t)g(x) + tg(y)$ for $x, y \in D$.

If $x, y \in D$ implies the strict inequality

$$f((1 - t)x + ty) < (1 - t)f(x) + tf(y),$$

then we say that $f$ is strictly convex.

Similarly, if

$$f((1 - t)x + ty) > (1 - t)f(x) + tf(y),$$

for every $x, y \in D$, then we say that $f$ is strictly concave.
It is useful for the study of convex functions to extend the notion of convex function by allowing $\infty$ as a value. Thus, if a function $f$ is defined on a subset $S$ of a linear space $L$, $f : S \to \mathbb{R}$, the extended-value function of $f$ is the function $\hat{f} : L \to \hat{\mathbb{R}}$ defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \infty & \text{otherwise,} \end{cases}$$

If a function $f : S \to \mathbb{R}$ is convex, where $S \subseteq L$ is a convex set, then its extended-value function $\hat{f}$ satisfies the inequality that defines convexity

$$\hat{f}((1 - t)x + ty) \leq (1 - t)\hat{f}(x) + t\hat{f}(y)$$

for every $x, y \in L$ and $t \in [0, 1]$, if we adopt the convention that $0 \cdot \infty = 0$. 
A extended-value convex function $f : S \rightarrow \hat{\mathbb{R}}$ is properly convex if $f$ is not the constant function defined by $f(x) = \infty$.

The effective domain of a convex function $f : S \rightarrow \hat{\mathbb{R}}$ is the set $\text{Dom}(f) = \{x \in S \mid f(x) < \infty\}$. 
Example

Let \( f : (0, \infty) \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \). The definition domain of \( f \) is clearly convex and we have

\[
    f((1 - t)x_1 + tx_2) = ((1 - t)x_1 + tx_2)^2 = (1 - t)^2 x_1^2 + t^2 x_2^2 + 2(1 - t)tx_1x_2.
\]

Therefore,

\[
 f((1 - t)x_1 + tx_2) - (1 - t)f(x_1) - tf(x_2) = (1 - t)^2 x_1^2 + t^2 x_2^2 + 2(1 - t)tx_1x_2 - (1 - t)x_1^2 - tx_2^2 = -t(1 - t)(x_1 - x_2)^2 \leq 0,
\]

which implies that \( f \) is indeed convex.
Example

The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = |a - xb| \) is convex because

\[
\begin{align*}
    f((1 - t)x_1 + tx_2) &= |a - ((1 - t)x_1 + tx_2)b| \\
    &= |a(1 - t) + at - ((1 - t)x_1 + tx_2)b| \\
    &= |(1 - t)(a - x_1 b) + t(a - x_2 b)| \\
    &\leq |(1 - t)(a - x_1 b)| + |t(a - x_2 b)| \\
    &= (1 - t)f(x_1) + tf(x_2)
\end{align*}
\]

for \( t \in [0, 1] \).
Example

The function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $g(x) = |a - x_1x_2|$ is not convex, in general. Consider, for example the special case $g(x) = |12 - x_1x_2|$. We have

$$f \begin{pmatrix} 6 \\ 2 \end{pmatrix} = f \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 0.$$  

Note that

$$\binom{4}{4} = \frac{1}{2} \binom{6}{2} + \frac{1}{2} \binom{2}{6} \quad \text{and} \quad f \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 > \frac{1}{2} f \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \frac{1}{2} f \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$
Example

Any norm $\nu$ on a real linear space $L$ is convex. Indeed, for $t \in [0, 1]$ we have

$$\nu(tx + (1 - t)y) \leq \nu(tx) + \nu((1 - t)y) = t\nu(x) + (1 - t)\nu(y)$$

for $x, y \in L$. 
Example

Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix. The function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) given by \( f(x) = x' Ax \) is convex if and only if \( A \) is a positive semidefinite matrix.

Indeed, suppose that \( f \) is convex. For \( x, y \in \mathbb{R}^n \) we have

\[
((1 - t)x + ty)'A((1 - t)x + t y) \leq (1 - t)x' Ax + ty' Ay,
\]

for \( t \in (0, 1) \), which amounts to

\[
(t^2 - t)x' Ax + (t^2 - t)y' Ay + (1 - t)ty' Ax + t(1 - t)x' Ay \leq 0.
\]

Since \( A \) is symmetric, we have \( (y' Ax)' = x' Ay \) and because both terms of the last equality are scalars we have \( y' Ax = x' Ay \). Note that \( t^2 - t \leq 0 \) because \( t \in [0, 1] \). Consequently,

\[
x' Ax + y' Ay - y' Ax - x' Ay \geq 0,
\]

which amounts to \( (x - y)' A(x - y) \geq 0 \), so \( A \) is positive semidefinite. The reverse implication is an exercise!
Local vs. Global Minima

Definition

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. The point \( x_0 \) is a **global minimum** for \( f \) if \( f(x) \geq f(x_0) \) for every \( x \).

The point \( x_1 \) is a **local minimum** for \( f \) if there exists \( \epsilon > 0 \) such that \( f(x) \geq f(x_0) \) for every \( x \in B[x_0, \epsilon] \).

If \( x_1 \) is a local minimum for \( f \) and \( x_0 \) is a global minimum, we have \( f(x_1) \geq f(x_0) \).
Theorem

If \( x_1 \) is a local minimum of a convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), then \( x_1 \) is a global minimum for \( f \).
Proof

Let $x_0$ be a global minimum of $f$ and let $x_1$ be a local minimum. We have $f(x_0) \leq f(x_1)$. Since $x_1$ is a local minimum, there exists $\epsilon$ such that if $\|x_1 - x\| \geq \epsilon$, then $f(x_1) \leq f(x)$.

Let $z = (1 - a)x_1 + ax_0$, where $a \in [0, 1]$. We have $x_1 - z = a(x_1 - x_0)$. By choosing $a$ such that

$$a < \frac{\epsilon}{\|x_1 - x_0\|}$$

we have $\|x_1 - z\| \leq \epsilon$, which implies $z \in B[x_1, \epsilon]$, so $f(z) \geq f(x_1)$ because $x_1$ is a local minimum. By the convexity of $f$ we have

$$f(z) = f((1 - a)x_1 + ax_0) \leq (1 - a)f(x_1) + af(x_0) \leq f(x_1),$$

so $f(z) = f(x_1)$. This, in turn implies

$$f(x_1) \leq (1 - a)f(x_1) + af(x_0),$$

which yields $f(x_1) \leq f(x_0)$, hence $f(x_1) = f(x_0)$. Therefore, the local minimum $x_1$ is also a global minimum.
One-argument convex function

The curve representing $f(x)$ is located under the chord determined by $(u, f(u))$ and $(v, f(v))$. 

\[ f(x) \]

\[ (1 - t)f(x) + tf(y) \]

\[ f((1 - t)x + ty) \]

\[ f(y) \]
Lemma

Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. If $x \in (a, b)$, then

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}.$$
Proof

It is easy to see that $x$ can be regarded as either of the following convex combinations:

$x = \left(1 - \frac{x - a}{b - a}\right) a + \frac{x - a}{b - a} b,$

$= \frac{b - x}{b - a} a + \left(1 - \frac{b - x}{b - a}\right) b.$

The existence of the first convex combination implies

$f(x) = f\left(1 - \frac{x - a}{b - a}\right) a + \frac{x - a}{b - a} b,$

$\leq \left(1 - \frac{x - a}{b - a}\right) f(a) + \frac{x - a}{b - a} f(b),$

which is equivalent to

$f(x) \leq \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b).$

This gives the first inequality of the lemma. The second one can be
Lemma

Let \( f : I \rightarrow \mathbb{R} \) be a function, where \( I \) is an open interval. The following statements are equivalent for \( a < b < c \), where \( a, b, c \in I \):

1. \((c - a)f(b) \leq (b - a)f(c) + (c - b)f(a)\);  
2. \( \frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \);  
3. \( \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{c-b} \).
Proof

(i) is equivalent to (ii): Suppose that (i) holds. Then we have

$$(c - a)f(b) - (c - a)f(a) \leq (b - a)f(c) + (c - b)f(a) - (c - a)f(a),$$

which is equivalent to

$$(c - a)(f(b) - f(a)) \leq (b - a)(f(c) - f(a)). \tag{1}$$

By dividing both sides by $(b - a)(c - a) > 0$ we obtain Inequality (ii).

Conversely, note that (ii) implies Inequality (1). By adding $(c - a)f(a)$ to both sides of this inequality we obtain (i).

In a similar manner it is possible to prove the equivalence between (i) and (iii).
Theorem

Let $I$ be an open interval and let $f : I \rightarrow \mathbb{R}$ is a function. Each of the conditions of the previous Lemma is equivalent to the convexity of $f$ for $a < b < c$ and $a, b, c \in I$. 
Proof

Let \( f : I \to \mathbb{R} \) be a convex function and let \( a, b, c \in I \) be such that \( a < b < c \). Define \( t = \frac{c-b}{c-a} \). Clearly \( 0 < t < 1 \) and by the convexity property,

\[
\begin{align*}
    f(b) &= f(at + (1-t)c) \leq tf(a) + (1-t)f(c) \\
    &= \frac{c-b}{c-a}f(a) + \frac{b-a}{c-a}f(c),
\end{align*}
\]

which yields the first inequality of Lemma 10. Conversely, suppose that the first inequality of Lemma 10 is satisfied. Choose \( a = x \), \( c = y \) and \( b = tx + (1-t)y \) for \( t \in (0,1) \). We have \((c-a)f(b) = (y-x)f(tx + (1-t)y)\) and \((b-a)f(c) + (c-b)f(a) = (1-t)(y-x)f(y) + t(y-x)f(x)\) Taking into account that \( y > x \), we obtain the inequality 

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),
\]

which means that \( f \) is convex.
Theorem

Let $I$ be an open interval and let $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function. The function $g(x, h)$ defined for $x \in I$ and $h \in \mathbb{R} - \{0\}$ as

$$g(x, h) = \frac{f(x + h) - f(x)}{h}$$

is increasing with respect to each of its arguments.
Proof

We need to examine three cases: $0 < h_1 < h_2$, $h_1 < h_2 < 0$, and $h_1 < 0 < h_2$.

In the first case choose $a = x$, $b = x + h_1$ and $c = x + h_2$ in the second inequality of Lemma 10, where all three numbers $x, x + h_1$ and $x + h_2$ belong to $I$. We obtain \[ \frac{f(x + h_1) - f(x)}{h_1} \leq \frac{f(x + h_2) - f(x)}{h_2}, \] which shows that $g(x, h_1) \leq g(x, h_2)$.

If $h_1 < h_2 < 0$, choose $a = x + h_1$, $b = x + h_2$ and $c = x$ in the last inequality of Lemma 10. This yields: \[ \frac{f(x) - f(x + h_1)}{-h_1} \leq \frac{f(x) - f(x + h_2)}{-h_2}, \] that is $g(x, h_1) \leq g(x, h_2)$.

In the last case, $h_1 < 0 < h_2$, begin by noting that the last two inequalities of Lemma 10 imply

\[ \frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}. \]

By taking $a = x + h_1$, $b = x$, and $c = x + h_2$ in this inequality we obtain

\[ \frac{f(x) - f(x + h_1)}{-h_1} \leq \frac{f(x + h_2) - f(x)}{h_2}, \]
To prove the monotonicity in the first argument let $x_1, x_2$ be in $I$ such that $x_1 < x_2$ and let $h$ be a number such that both $x_1 + h$ and $x_2 + h$ belong to $I$. Since $g$ is monotonic in its second argument we have

$$g(x_1, h) = \frac{f(x_1 + h) - f(x_1)}{h} \leq \frac{f(x_2 + h) - f(x_1)}{h + (x_2 - x_1)}$$

and

$$\frac{f(x_2 + h) - f(x_1)}{h + (x_2 - x_1)} = \frac{f(x_1) - f(x_2 + h)}{-h - (x_2 - x_1)} = \frac{f((x_2 + h) - h - (x_2 - x_1)) - f(x_2 + h)}{-h - (x_2 - x_1)} \leq \frac{f((x_2 + h) - h) - f(x_2 + h)}{-h} = \frac{f(x_2 + h) - f(x_2)}{h},$$

which proves the monotonicity in its first argument.
Convexity of functions of $n$ arguments can be expressed as convexity of one-argument functions.

**Theorem**

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. The function $f$ is convex if and only if the function $\phi_{x,h} : \mathbb{R} \rightarrow \mathbb{R}$ given by $\phi_{x,h}(t) = f(x + th)$ is a convex function for every $x$ and $h$ in $\mathbb{R}^n$. 
Proof

Suppose that $f$ is convex. We have

$$
\phi_{x,h}(ta + (1 - t)b) = f(x + (ta + (1 - t)b)h) \\
= f(t(x + ah) + (1 - t)(x + bh)) \\
\leq t\phi_{x,h}(a) + (1 - t)\phi_{x,h}(b),
$$

which shows that $\phi_{x,h}$ is indeed convex. The converse implication follows in a similar manner.
Lemma

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function such that $g(0_n) = 0$. We have $-g(x) \leq g(x)$ for $x \in \mathbb{R}^n$.

Proof: Note that if $x \neq 0_n$, $0_n \in [-x, x]$. Since $g$ is clearly mid-point convex, we have

$$0 = g(0_n) = g\left(\frac{-x + x}{2}\right) \leq \frac{1}{2}g(-x) + \frac{1}{2}g(x),$$

which implies the desired inequality.
Theorem

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function. The function \( f \) is continuous at every \( x_0 \in (a, b) \).
Proof

Let $g : (a - x_0, b - x_0) \rightarrow \mathbb{R}$ be defined as $g(x) = f(x + x_0) - f(x_0)$. It is clear that $g$ is convex on $(a - x_0, b - x_0)$, $0 \in (a - x_0, b - x_0)$, and $g(0) = 0$; also, $g$ is continuous in 0 if and only if $f$ is continuous at $x_0$. For $x \in (a - x_0, b - x_0)$ let

$$s(x) = \begin{cases} 
1 & \text{if } x \leq 0, \\
-1 & \text{if } x < 0.
\end{cases}$$

If $|x| < \delta$, then the convexity of $g$ implies

$$g(x) = g \left( \frac{|x|}{\delta} s(x) \delta + \left( 1 - \frac{|x|}{\delta} \right) 0 \right) \leq \frac{|x|}{\delta} g(s(x) \delta) + \left( 1 - \frac{|x|}{\delta} \right) g(0) = \frac{|x|}{\delta} g(s(x) \delta).$$
Therefore, $$g(x) \leq \frac{1}{\delta} \max\{g(-\delta), g(\delta)\}|x|$$. The convexity of $$g$$ implies that $$-g(-x) \leq g(x)$$ by the previous lemma, so
$$|g(x)| \leq \frac{1}{\delta} \max\{g(-\delta), g(\delta)\}|x|$$. If $$\lim_{n \to \infty} x_n = 0$$, where $$(x_n)$$ is a sequence in $$(a - x_0, b - x_0)$$, then $$\lim_{n \to \infty} g(x_n) = 0 = g(0)$$, so $$g$$ is continuous in 0. This implies that $$f$$ is continuous in $$x_0$$. 
Convexity of One-Argument Functions

Definition

Let $f : I \longrightarrow \mathbb{R}$ be a convex function where $I$ is an interval of $\mathbb{R}$ and let $a \in I$. The function $f$ is **left-differentiable** if the limit

$$
f'(a-) = \lim_{x \to a, x < a} \frac{f(x) - f(a)}{x - a}
$$

exists. In this case, the value of the limit is known as the **left derivative** of $f$ in $a$.

Similarly, $f$ is **right-differentiable** if the limit

$$
f'(a+) = \lim_{x \to a, x > a} \frac{f(x) - f(a)}{x - a}
$$

exists. In this case, the value of the limit is known as the **right derivative** of $f$ in $a$. 
Convexity of One-Argument Functions

Definition
A function $f : I \rightarrow \mathbb{R}$ is differentiable in $a$, where $a \in I$, if $f'(a-) = f'(a+)$. In this case we write $f'(a) = f'(a+) = f'(a-)$. If $f$ is convex and differentiable on $I$ if $f'$ is increasing.
Example

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$. We have:

$$f'(0-) = \lim_{x \to 0, x < 0} \frac{|x|}{x} = -1$$

and

$$f'(0+) = \lim_{x \to 0, x > 0} \frac{|x|}{x} = 1.$$ 

Note that if both $f'(a-)$ and $f'(a+)$ exist and are finite, then $f$ is continuous in $a$. 
Theorem

Let $f : (a, b) \rightarrow \mathbb{R}$ be convex function on $[a, b]$. If $x, y \in (a, b)$ and $x < y$, then

$$f'(x-) \leq f'(x+) \leq f'(y-) \leq f'(y+).$$
If \( a < x < b \) we have

\[
\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}.
\]

Since \( \lim_{a \to x, a < x} \frac{f(x)-f(a)}{x-a} \leq \lim_{b \to x, b > x} \frac{f(b)-f(x)}{b-x} \), it follows that \( f'(x-) \leq f'(x^+) \); similarly, \( f'(y-) \leq f'(y+) \).

Let \( t \in (x, y) \). By the same previous result we have:

\[
\frac{f(t) - f(x)}{t - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(t)}{y - t}.
\]

The first inequality implies

\[
f'(x+) = \lim_{t \to x, t > x} \frac{f(t) - f(x)}{t - x} \leq \frac{f(y) - f(x)}{y - x},
\]

while the second yields

\[
\frac{f(y) - f(x)}{y - x} \leq \lim_{t \to y, t < y} \frac{f(y) - f(t)}{y - t} = f'(y-),
\]

so \( f'(x+) \leq f'(y-) \).
Theorem

If $f : I \rightarrow \mathbb{R}$ is a function and $f'$ is increasing, then $f$ is convex.
Proof

For \( x, y \in I \) the inequality

\[
f((1 - a)x + ay) \leq (1 - a)f(x) + af(y)
\]

is immediate for \( a = 0 \) and \( a = 1 \). So suppose that \( 0 < a < 1 \). By the Mean Value theorem we have

\[
f((1 - a)x + ay) = f'(\xi_1)a(y - x),
\]

\[
f(y) - f((1 - a)x + ay) = f'(\xi_2)(1 - a)(y - x),
\]

where \( x < \xi_1 < (1 - a)x + ay < \xi_2 < y \).

Since \( f'(\xi_1) \leq f'(\xi_2) \), the last equalities yield the convexity property (multiply first by \((1 - a)\), second by \(a\) and subtract).
Corollary

If \( f : I \rightarrow \mathbb{R} \) is twice differentiable function, then \( f \) is convex if and only if \( f''(x) \geq 0 \) for every \( x \in (a, b) \). If \( f''(x) > 0 \) for \( x \in I \), then \( f \) is strictly convex.
Table: Examples of convex or concave functions.

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<td>$e^x$</td>
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Theorem (Jensen’s Theorem)

Let $f$ be a function that is convex on an interval $I$. If $t_1, \ldots, t_n \in [0, 1]$ are $n$ numbers such that $\sum_{i=1}^{n} t_i = 1$, then

$$f \left( \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} t_i f(x_i)$$

for every $x_1, \ldots, x_n \in I$. 

The argument is by induction on \( n \), where \( n \geq 2 \). The basis step, \( n = 2 \), follows immediately from the definition of convex functions. Suppose that the statement holds for \( n \), and let \( u_1, \ldots, u_n, u_{n+1} \) be \( n + 1 \) numbers such that \( \sum_{i=1}^{n+1} u_i = 1 \). We have

\[
f(u_1x_1 + \cdots + u_{n-1}x_{n-1} + u_n x_n + u_{n+1} x_{n+1})
\]

\[
= f \left( u_1 x_1 + \cdots + u_{n-1} x_{n-1} + (u_n + u_{n+1}) \frac{u_n x_n + u_{n+1} x_{n+1}}{u_n + u_{n+1}} \right).
\]

By the inductive hypothesis, we can write

\[
f(u_1x_1 + \cdots + u_{n-1}x_{n-1} + u_n x_n + u_{n+1} x_{n+1})
\]

\[
\leq u_1 f(x_1) + \cdots + u_{n-1} f(x_{n-1}) + (u_n + u_{n+1}) f \left( \frac{u_n x_n + u_{n+1} x_{n+1}}{u_n + u_{n+1}} \right).
\]

Next, by the convexity of \( f \), we have

\[
f \left( \frac{u_n x_n + u_{n+1} x_{n+1}}{u_n + u_{n+1}} \right) \leq \frac{u_n}{u_n + u_{n+1}} f(x_n) + \frac{u_{n+1}}{u_n + u_{n+1}} f(x_{n+1}).
\]

Combining these inequalities gives desired conclusion.
If $f$ is a concave function and $t_1, \ldots, t_n \in [0, 1]$ are $n$ numbers such that $\sum_{i=1}^{n} t_i = 1$, then

$$f \left( \sum_{i=1}^{n} t_i x_i \right) \geq \sum_{i=1}^{n} t_i f(x_i).$$
Example

We saw that the function \( f(x) = \ln x \) is concave. Therefore, if \( t_1, \ldots, t_n \in [0, 1] \) are \( n \) numbers such that \( \sum_{i=1}^{n} t_i = 1 \), then

\[
\ln \left( \sum_{i=1}^{n} t_i x_i \right) \geq \sum_{i=1}^{n} t_i \ln x_i.
\]

This inequality can be written as

\[
\ln \left( \sum_{i=1}^{n} t_i x_i \right) \geq \ln \prod_{i=1}^{n} x_i^{t_i},
\]

or equivalently

\[
\sum_{i=1}^{n} t_i x_i \geq \prod_{i=1}^{n} x_i^{t_i},
\]

for \( x_1, \ldots, x_n \in (0, \infty) \).
In the special case where $t_1 = \cdots = t_n = \frac{1}{n}$, we have the inequality that relates the arithmetic to the geometric average on $n$ positive numbers:

$$\frac{x_1 + \cdots + x_n}{n} \geq \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}.$$  \hfill (2)
Weighted Means

Let \( w = (w_1, \ldots, w_n) \in \mathbb{R}^n \) be such that \( \sum_{i=1}^{n} w_i = 1 \). For \( r \neq 0 \), the \textit{w-weighted mean of order} \( r \) of a sequence of \( n \) positive numbers \( x = (x_1, \ldots, x_n) \in \mathbb{R}_{>0}^n \) is the number

\[
\mu_w^r(x) = \left( \sum_{i=1}^{n} w_i x_i^r \right)^{\frac{1}{r}}.
\]

Of course, \( \mu_w^r(x) \) is not defined for \( r = 0 \); we will give as special definition

\[
\mu_w^0(x) = \lim_{r \to 0} \mu_w^r(x).
\]

We have

\[
\lim_{r \to 0} \ln \mu_w^r(x) = \lim_{r \to 0} \frac{\ln \sum_{i=1}^{n} w_i x_i^r}{r} = \lim_{r \to 0} \frac{\sum_{i=1}^{n} w_i x_i^r \ln x_i}{\sum_{i=1}^{n} w_i x_i^r} = \sum_{i=1}^{n} w_i \ln x_i = \ln \prod_{i=1}^{n} x_i^{w_i}.
\]

Thus, if we define \( \mu_w^0(x) = \prod_{i=1}^{n} x_i^{w_i} \), the weighted mean of order \( r \)
For $w_1 = \cdots = w_n = \frac{1}{n}$, we have

$$
\begin{align*}
\mu_{w}^{-1}(x) &= \frac{n x_1 \cdots x_n}{x_2 \cdots x_n + \cdots + x_1 \cdots x_{n-1}} \\
&= \frac{nx_1 \cdots x_n}{x_2 \cdots x_n + \cdots + x_1 \cdots x_{n-1}} \\
&= \text{(the harmonic average of } x), \\
\mu_{w}^{0}(x) &= (x_1 \cdots x_n)^{\frac{1}{n}} \\
&= (x_1 \cdots x_n)^{\frac{1}{n}} \\
&= \text{(the geometric average of } x), \\
\mu_{w}^{1}(x) &= \frac{x_1 + \cdots + x_n}{n} \\
&= \frac{x_1 + \cdots + x_n}{n} \\
&= \text{(the arithmetic average of } x).
\end{align*}
$$

**Theorem**

If $p < r$, we have $\mu_{w}^{p}(x) \leq \mu_{w}^{r}(x)$. 
Proof

There are three cases depending on the position of 0 relative to $p$ and $r$. In the first case, suppose that $r > p > 0$. The function $f(x) = x^{r/p}$ is convex, so by Jensen’s inequality applied to $x_1^p, \ldots, x_n^p$, we have

$$
\left( \sum_{i=1}^{n} w_i x_i^p \right)^{r/p} \leq \sum_{i=1}^{n} w_i x_i^r,
$$

which implies

$$
\left( \sum_{i=1}^{n} w_i x_i^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} w_i x_i^r \right)^{1/r},
$$

which is the inequality of the theorem.

If $r > 0 > p$, the function $f(x) = x^{r/p}$ is again convex because $f''(x) = \frac{r}{p} \left( \frac{r}{p} - 1 \right) x^{r/p - 2} \geq 0$. Thus, the same argument works as in the previous case.
Finally, suppose that $0 > r > p$. Since $0 < \frac{r}{p} < 1$, the function $f(x) = x^{\frac{r}{p}}$ is concave. Thus, by Jensen’s inequality,

$$\left( \sum_{i=1}^{n} w_i x_i^p \right)^{\frac{r}{p}} \geq \sum_{i=1}^{n} w_i x_i^r.$$ 

Since $\frac{1}{r} < 0$, we obtain again

$$\left( \sum_{i=1}^{n} w_i x_i^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} w_i x_i^r \right)^{\frac{1}{r}}.$$
The **indicator function** of a subset $S$ of a set $Z$ is the function $I_S : Z \rightarrow \hat{\mathbb{R}}$ defined by

$$I_S(z) = \begin{cases} 
0 & \text{if } z \in S, \\
\infty & \text{if } z \notin S.
\end{cases}$$

**Theorem**

A set $S \subseteq \mathbb{R}^n$ is convex if and only if its indicator function $I_S$ is convex.
Jensen’s Inequality

Proof

If \( I_S \) is convex, we have \( I_S(tx + (1 - t)y) \leq tI_S(x) + (1 - t)I_S(y) \) for every \( x, y \in S \). Therefore, if \( x, y \in S \) we have \( I_S(x) = I_S(y) = 0 \), which implies \( I_S(tx + (1 - t)y) = 0 \), so \( tx + (1 - t)y \in S \). Thus, \( S \) is convex.

Conversely, suppose that \( S \) is convex. We need to prove that

\[
I_S(tx + (1 - t)y) \leq tI_S(x) + (1 - t)I_S(y). \tag{3}
\]

If at least one of \( x \) or \( y \) does not belong to \( S \), Inequality (3) is satisfied. The remaining case occurs when we have both \( x \in S \) and \( y \in S \), in which case, \( tx + (1 - t)y \in S \) and \( I_S(x) = I_S(y) = I_S(tx + (1 - t)y) = 0 \), and, again, Inequality (3) is satisfied.