Differential Privacy - I

Prof. Dan A. Simovici

UMB
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Releasing the data and just removing the names does nothing for privacy. If you know their name and a few records, then you can identify that person in the other (private) database.

Vitaly Shmatikov, Professor of Computer Science, University of Texas at Austin
Privacy in data analysis is treated from many perspectives:

- statistics,
- databases,
- philosophy,
- law,
- cryptography,
- theoretical computer science.
The developments in the
- internet,
- database technology, and
- data mining
brought to forefront the issues of privacy and has imposed limitations of the work of data analysts.

In general, data analysts **do not have direct access to raw data** and certain limitations are imposed on the content and number of exploring queries because accurate answers to too many questions will destroy privacy.
exposure of medical records of governor William Weld of Massachusetts;
identification of an user of AOL;
identification of an user of Netflix based on movie ratings;
massive data breaches at TJMaxx and other retailers.
A graduate MIT student, Latanya Sweeney managed to access the medical records of William Weld, the then governor of Massachusetts using poorly public anonymized medical records. In his 2010 UCLA Law Review paper, "Broken Promises of Privacy, (Ohm, 2010) University of Colorado law professor Paul Ohm describes Sweeney’s re-identification of Weld’s hospitalization data as follows:
At the time GIC released the data, William Weld, then Governor of Massachusetts, assured the public that GIC had protected patient privacy by deleting identifiers. In response, then graduate student Sweeney started hunting for the Governors hospital records in the GIC data. She knew that Governor Weld resided in Cambridge, Massachusetts, a city of 54,000 residents and seven ZIP codes. For twenty dollars, she purchased the complete voter rolls from the city of Cambridge, a database containing, among other things, the name, address, ZIP code, birth date, and sex of every voter. By combining this data with the GIC records, Sweeney found Governor Weld with ease. Only six people in Cambridge shared his birth date, only three of them men, and of them, only he lived in his ZIP code. In a theatrical flourish, Dr. Sweeney sent the Governors health records (which included diagnoses and prescriptions) to his office.
The Searches of Mrs. Arnold on AOL

Starting from the public anonymized AOL records of the search history of an user it was possible to identify this user as Ms. Thelma Arnold. Among a list of 20 million Web search queries collected by AOL and released on the Internet is were the searches of user No. 4,417,749. The number was assigned by the company to protect the searchers anonymity, but it was not much of a shield. No. 4417749 conducted hundreds of searches over a three-month period on topics ranging from numb fingers to 60 single men to dog that urinates on everything. Following searches for landscapers in Lilburn, Ga, several people with the last name Arnold and homes sold in shadow lake subdivision Gwinnett County Georgia, it became possible to identify the user as Thelma Arnold, a 62-year-old widow who lives in Lilburn, Ga., frequently researches her friends medical ailments and loves her three dogs.
It might appear that Ms. Arnold fears she is suffering from a wide range of ailments. Her search history includes hand tremors, nicotine effects on the body, dry mouth and bipolar. But in an interview, Ms. Arnold said she routinely researched medical conditions for her friends to assuage their anxieties. Explaining her queries about nicotine, for example, she said: I have a friend who needs to quit smoking and I want to help her do it.
The search data was removed from the AOL site, and AOL apologized for its release. This incident shows how much people unintentionally reveal about themselves when they use search engines and how risky it can be for companies like AOL, Google and Yahoo to compile such data. AOL chief technology officer resigned after a massive dataset of 20 million searches performed by 658,000 people was published for use in research. The data was believed to be anonymized, but revealed sensitive details of the searchers private lives, including Social Security numbers, credit-card numbers, addresses, and, in one case, apparently a searcher’s intent to kill their wife.
Identification based on movie ratings

In a dramatic demonstration of the privacy dangers of databases that collect consumer habits, two researchers from the University of Texas at Austin have shown that a handful of movie ratings can identify a person as easily as a Social Security number.
The researchers – Arvind Narayanan and Vitaly Shmatikov, both from the Department of Computer Sciences at the University of Texas at Austin – claim to have identified two people out of the nearly half million anonymized users whose movie ratings were released by online rental company Netflix last year. The company published the large database as part of its $1 million Netflix Prize, a challenge to the world’s researchers to improve the rental firm’s movie-recommendation engine. While Netflix’s dataset did not include names, instead using an anonymous identifier for each user, the collection of movie ratings – combined with a public database of ratings (IMDb -standing for Internet Movie Database) was enough to identify the people.
Narayanan and Shmatikov identified movie ratings of two of the users in Netflix’s data. Exposing movie ratings that the reviewer thought were private could expose significant details about the person. For example, the researchers found that one of the people had strong – ostensibly private – opinions about some liberal and gay-themed films and also had ratings for some religious films.
Massive data breaches

In the past few years several massive data breaches that have leaked sensitive information on millions of people.

- Recently the head of HM Revenue & Customs, the United Kingdom’s tax agency, resigned after two data discs containing sensitive, yet unencrypted, personal details of 25 million U.K. citizens were lost in the mail.

- Retail giant TJX Companies announced that data thieves had stolen the credit- and debit-card details on, what currently is estimated to be, more than 94 million consumers.
Conclusions

- Privacy research demonstrated that information that a person believes to be benign could be used to identify them in other private databases, a risk understood in privacy and intelligence circles.
- Even as early as decades ago, the U.S. government would classify aggregates of information, (because) you can take unclassified data and put them together to get something that is not unclassified.
Those risks have long pitted privacy advocates against online marketers and other Internet companies seeking to profit from the Internet's unique ability to track the comings and goings of users, allowing for more focused and therefore more lucrative advertising.

The unintended consequences of all that data being compiled, stored and cross-linked is a ticking privacy time bomb.
Differential privacy (DP) is an approach to privacy that guarantees that the distribution of outcomes of a computation that involves a database does not change significantly when an individual record is added or removed from a database.

- DP allows an investigator to learn information about a database without learning anything about an individual.

- DP database mechanisms can make confidential data widely available for accurate data analysis, without resorting to data clean rooms, institutional review boards, data usage agreements, restricted views, or data protection plans.
DP ensures that the ability of an adversary to inflict harm is the same, independent of whether any individual opts in to, or opts out of, the dataset.

DP focuses on the probability of any given output of a privacy mechanism and how this probability can change with the addition or deletion of any row. Thus, we concentrate on pairs of databases differing only in one row, meaning one is a subset of the other and the larger database contains just one additional row.
The **probability simplex** in $\mathbb{R}^m$ is the set $S_m \subseteq \mathbb{R}^m$ defined by

$$S_m = \{ x \in \mathbb{R}^m \mid x_i \geq 0 \text{ for } 1 \leq i \leq m \text{ and } \sum_{i=1}^{m} x_i = 1 \}.$$
A randomized algorithm with domain $A$ and range $B$ is a triplet $\mathcal{M} = (A, B, M)$, where $M : A \rightarrow S_m$ (for $m = |B|$) is a function such that on input $a \in A$, $\mathcal{M}$ produces an output $b = \mathcal{M}(a) \in B$ with the probability $M(a)_b$.

In other words, for every $a \in A$ a randomized algorithm defines a random variable:

$$\xi^\mathcal{M}_a : \left( \begin{array}{c} b_1 \\ M_a(b_1) \\ \vdots \\ b_m \\ M_a(b_m) \end{array} \right),$$

where $m = |B|$. 
A universe is a pair \((X, I)\), where \(X\) is a set of records and \(I\) is a set of types such that each record of \(X\) is associated with exactly one type in \(I\). Thus, there exists a partition \(\tau\) of \(X\) whose blocks \(X_i\) (also known as bins) consist of records of type \(i\).

The number of records of type \(i\) in \(X\) is denoted by \(s(i)\), where the mapping \(s : I \rightarrow \mathbb{N}\) is a histogram.

A database is a set of records \(D\) drawn from a universe \(X\).
The blocks of the trace of $\tau$ on $D$, $\tau_D$ consist of records that have the same type $i$. This, in turn, defines a database histogram $s_D : I \rightarrow \mathbb{N}$, where $s_D(i)$ is the number of records of type $i$ contained by $D$. 

\[ X \]
The **histogram** of $D$ is a mapping $s : I \rightarrow \mathbb{N}$, where each entry $s(i) = s_D(i)$ represents the number of records of type $i$ in $D$. Thus, the set of histograms of databases of the universe $\mathcal{X}$ is $\mathbb{N}^I$.

The **norm of a histogram** $s \in \mathbb{N}^I$ of a database $D$ is $\| s \|_1 = \sum_{i=1}^{\| I \|} s(i)$.

- Under this definition $\| s \|_1$ is a measure of the size of the database $D$.
- If $D_1, D_2$ are two databases of the universe $\mathcal{X}$, then $\| s_{D_1} - s_{D_2} \|_1$ measures how many records are different in the two database histograms $s_{D_1}$ and $s_{D_2}$. 
A randomized algorithm \( \mathcal{M} = (\text{DB}(\mathcal{X}), \mathcal{X}, M) \) transforms the members of a database \( D \) of a universe \( \mathcal{X} \) into members of \( \mathcal{X} \) and, therefore into a new database \( D' \). This is a random transformation in general, and the number of records of type \( i \) in the new database is a random variable.

Two databases \( D_1, D_2 \in \mathcal{X}^n \) differ in one record if their symmetric difference consists of two records \( x_1 \in D_1 \) and \( x_2 \in D_2 \), that is, \( D_1 \oplus D_2 = \{ x_1, x_2 \} \).
Definition

Let $\mathcal{M}$ be a randomized algorithm $\mathcal{M} : \mathcal{X}^n \rightarrow B$. $\mathcal{M}$ is $\epsilon$-differentially private if for every two databases $D_1$ and $D_2$ in $\mathcal{X}^n$ such that $|D_1 \oplus D_2| = 2$ we have:

$$P(\mathcal{M}(D_1) \in S) \leq e^\epsilon P(\mathcal{M}(D_2) \in S)$$

for all $S$ in the range of $\mathcal{M}$. 
If the roles of the databases are inverted we have
\( P[\mathcal{M}(D_2) \in S] \leq e^\epsilon P[\mathcal{M}(D_1) \in S] \). Thus, \( \mathcal{M} \) is \( \epsilon \)-differentially private if and only if for every two databases \( D_1, D_2 \) on \( \mathcal{X}^n \) such that \( |D_1 \oplus D_2| \leq 2 \) we have

\[
e^{-\epsilon} \leq \frac{P[\mathcal{M}(D_1) \in S]}{P[\mathcal{M}(D_2) \in S]} \leq e^\epsilon.
\]

(1)

Note that, taking into account that for small values of \( \epsilon \) we have \( e^\epsilon \approx 1 + \epsilon \) and \( e^{-\epsilon} \approx 1 - \epsilon \), Inequality 1 becomes

\[
1 - \epsilon \leq \frac{P[\mathcal{M}(D_1) \in S]}{P[\mathcal{M}(D_2) \in S]} \leq 1 + \epsilon.
\]

The quantity \( \ln \frac{P[\mathcal{M}(D_1) \in S]}{P[\mathcal{M}(D_2) \in S]} \) is the privacy loss incurred by observing the output \( \mathcal{M}(D_1) \).

The definition of differential privacy ensures that seeing \( D_2 \) instead of \( D_1 \) can only increase the probability of any event by at most a small factor.
A more general concept is the notion of $(\epsilon, \delta)$-differential privacy.

**Definition**

Let $\mathcal{M}$ be a randomized algorithm $\mathcal{M} : \mathcal{X}^n \rightarrow B$. $\mathcal{M}$ is $(\epsilon, \delta)$-differentially private if for every two databases $D_1$ and $D_2$ in $\mathcal{X}^n$ such that $|D_1 \oplus D_2| = 2$ and for all $S$ in the range of $\mathcal{M}$ we have:

$$P(\mathcal{M}(D_1) \in S) \leq e^\epsilon P(\mathcal{M}(D_2) \in S) + \delta.$$
Example

Consider a database $D$ of individuals draw from a population $X$ that may or may not smoke. We present a technique that allows us to estimate the fraction $p$ of individuals who smoke by using a randomized survey that preserves the privacy of individual responders. The individuals are instructed to answer yes or no when questioned about their smoking as follows:

1. flip a coin;
2. if tail, then respond truthfully;
3. if head, then flip a second coin and respond “yes” if head and “no” if tail.
Let $a$ be an individual who smokes. We have the following scenario:

1. If the first coin toss produces a tail, $a$ responds truthfully and the answer is yes.
2. If the first coin toss produces head, a second coin is tossed and the answer is dictated by tossing of the coin: if head the individual will respond yes (which is the truth); if tail the answer will be no (which is not truthful).

The distribution of the answer is

$$\xi_a^M : \begin{pmatrix} \text{yes} & \text{no} \\ 3/4 & 1/4 \end{pmatrix}.$$
Example cont’d

If \(a\) does not smoke the distribution of the answer is

\[
\xi_a \cdot \begin{pmatrix}
\text{yes} & \text{no} \\
1/4 & 3/4
\end{pmatrix}.
\]

Noise is introduced in this experiment through the spurious yes and no answers obtained by coin tossing.
A yes answer is not incriminating because this answer occurs with probability at least $1/4$ regardless whether the respondent smokes. This provides plausible deniability to participants.
If $p$ is the fraction of individuals who smoke, the expected number of yes answers in the histogram $s'$ is $n = \frac{1}{4}(1 - p) + \frac{3}{4}p = \frac{1}{4} + \frac{p}{2}$. Thus, $p$ can be estimated as $2n - \frac{1}{2}$.

The expected number of no answers in $s'$ is $\frac{1}{4}p + \frac{3}{4}(1 - p)$.

The randomized algorithm discussed above has $(\ln 3, 0)$ privacy. Note that the range of $\mathcal{M}$ is the set \{yes, no\}.
If $D_1, D_2$ are such that $\{x_1, x_2\} = D_1 \oplus D_2$, where $x_1 \in D_1 - D_2$ and $x_2 \in D_2 - D_1$ we may have the following four cases:

1. $\mathcal{M}(x_1) = \mathcal{M}(x_2) = \text{yes}$;
2. $\mathcal{M}(x_1) = \mathcal{M}(x_2) = \text{no}$;
3. $\mathcal{M}(x_1) = \text{yes}$ and $\mathcal{M}(x_2) = \text{no}$;
4. $\mathcal{M}(x_1) = \text{no}$ and $\mathcal{M}(x_2) = \text{yes}$.
Example cont’d

In the first two cases, we have

\[
\frac{P[M(D_1) \in S]}{P[M(D_2) \in S]} = 1
\]

for \( S = \{yes\} \) or \( S = \{no\} \).

In the third case, four subcases are possible:

1. both \( x_1 \) and \( x_2 \) are smokers;
2. neither \( x_1 \) nor \( x_2 \) are smokers;
3. \( x_1 \) is a smoker, but \( x_2 \) is not a smoker;
4. \( x_1 \) is not a smoker but \( x_2 \) is one.
Example cont’d

The probabilities the first subcase are:

\[
\begin{align*}
\frac{P(M(x_1) = \text{no})}{P(M(x_2) = \text{no})} &= \frac{1/4}{1/4} = 1; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_1) = \text{no})} &= \frac{3/4}{1/4} = 3; \\
\frac{P(M(x_2) = \text{no})}{P(M(x_2) = \text{yes})} &= \frac{1/4}{3/4} = \frac{1}{3}; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_2) = \text{yes})} &= \frac{3/4}{3/4} = 1.
\end{align*}
\]

In the second subcase we have:

\[
\begin{align*}
\frac{P(M(x_1) = \text{no})}{P(M(x_2) = \text{no})} &= \frac{3/4}{3/4} = 1; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_1) = \text{no})} &= \frac{3/4}{1/4} = 3; \\
\frac{P(M(x_2) = \text{no})}{P(M(x_2) = \text{yes})} &= \frac{1/4}{3/4} = \frac{1}{3}; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_2) = \text{yes})} &= \frac{1/4}{1/4} = 1.
\end{align*}
\]
For the third subcase ($x_1$ is a smoker, but $x_2$ is not a smoker) we can write:

\[
\begin{align*}
\frac{P(M(x_1) = \text{no})}{P(M(x_2) = \text{no})} &= \frac{1/4}{1/4} = 1; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_2) = \text{no})} &= \frac{3/4}{1/4} = 3; \\
\frac{P(M(x_1) = \text{no})}{P(M(x_2) = \text{yes})} &= \frac{1/4}{3/4} = 1/3; \\
\frac{P(M(x_1) = \text{yes})}{P(M(x_2) = \text{yes})} &= \frac{3/4}{3/4} = 1.
\end{align*}
\]
Finally, for the fourth subcase (\(x_1\) is not a smoker but \(x_2\) is one):

\[
\begin{align*}
P(M(x_1)=\text{no}) & = \frac{3}{4} \quad \frac{1}{4} = 3; \\
P(M(x_2)=\text{no}) & = \frac{1}{4} \quad \frac{1}{4} = 1; \\
P(M(x_1)=\text{yes}) & = \frac{3}{4} \quad \frac{3}{4} = \frac{1}{3}; \\
P(M(x_2)=\text{yes}) & = \frac{1}{4} \quad \frac{3}{4} = \frac{1}{3}. 
\end{align*}
\]
A more general randomization algorithm can be developed using a binary tree, that is, a tree where every vertex with the exception of the leaves has two descendants. These descendants correspond to results of flipping a coin, that is, to head and tail.
Definition

Let $\mathcal{X}$ be a universe and let $f : \mathcal{X}^n \rightarrow \mathbb{R}^d$ be a function. The $L_1$-sensitivity of $f$ is the smallest number $S(f)$ such that for all $D, \tilde{D} \in \mathcal{X}^n$ which differ in a single entry we have

$$\| f(D) - f(\tilde{D}) \|_1 \leq S(f) d(D, \tilde{D}),$$

where $d$ is the Hamming distance on $\mathcal{X}^n$.

In particular, if $D, D'$ are two databases that differ in one position, $\| f(D) - f(\tilde{D}) \|_1 \leq S(f)$. 
Example

If $\mathcal{X} = \{0, 1\}$ and $f(D) = \sum_{i=1}^{n} x_i$, then the sensitivity of $f$ is 1.
Example

Suppose that a domain $\mathcal{X}$ has been partitioned into $d$ bins $X_1, \ldots, X_n$. The histogram $s : \mathcal{X}^n \rightarrow \mathbb{R}^d$ that computes the number of points that fall into each bin has sensitivity 2 independent of $d$ because changing one point in a database can change at most two of these points: one bin loses a point and another gains one.
Definition

A random variable has the Laplace \((\mu, b)\) distribution if its probability density function is

\[
h(x|\mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}
\]

for \(x \in \mathbb{R}\), where \(b > 0\).

In particular, the \textit{Laplace distribution} centered at 0 with scale 1 is the distribution with the probability density given by

\[
h(x|0, b) = \frac{1}{2b} e^{-\frac{|x|}{b}}.
\]

This function will be denoted simply by \(h_b(x)\).
The Laplace distribution can be regarded as a symmetric version of the exponential distribution. It is easy to see that $\int_{\mathbb{R}} \text{Lap}(x|\mu, b) \, dx = 1$. The mean of this distribution is $\mu$, while the variance is $2b^2$. The probability density function for the Laplace distribution with $\mu = 0$ and $b = 1$ is shown below.
The privacy preserving Laplace mechanism computes the query $f(D) \in \mathbb{R}^k$ and perturbs each coordinate $f(D)_i$ with noise drawn from a Laplace distribution $Y_i$.

**Definition**

Given a query $f : DB(\mathcal{X}^n) \rightarrow \mathbb{R}^k$, the Laplace mechanism is defined as

$$M(D, f, \epsilon) = f(D) + (Y_1, \ldots, Y_k)$$

where $Y_1, \ldots, Y_k$ are independent, identically distributed Laplace variables from $\text{Lap} \left( \frac{S(f)}{\epsilon} \right)$. 
Example

Suppose $D \in \{0, 1\}^n$ and the user wishes to learn $f(D) = \sum_{i=1}^{n} x_i$, that is, the total number of 1s in $D$. If we add random Laplace noise $Y \sim Lap(1/\epsilon)$ (that is, a Laplace random variable with the parameter $b = \frac{1}{\epsilon}$ and the probability density function $h(x) = \frac{\epsilon}{2} e^{-\epsilon|x|}$), then the algorithm will produce $T(D)$, where $T(D) = \sum_{i=1}^{n} x_i + Y$. Note that $T(D) = t$ is equivalent to

$$Y = t - \sum_{i=1}^{n} x_i = t - f(D).$$
Let $D$ and $\tilde{D}$ be two databases that differ in a single entry. We have:

\[
\frac{P(T(D) = t)}{P(T(\tilde{D}) = t)} = \frac{h(t - f(D))}{h(t - f(\tilde{D}))} \leq e^{\epsilon |f(D) - f(\tilde{D})|} \leq e^\epsilon
\]

because the two databases $D$ and $\tilde{D}$ differ in a single entry (which means that the sums $f(D)$ and $f(D')$ differ by 1) which shows that we have $\epsilon$-privacy.
Definition

Given a query $f : \text{DB}(\mathcal{X}^n) \rightarrow \mathbb{R}^k$, the Laplace mechanism is defined as

$$\mathcal{M}(D, f, \epsilon) = f(D) + (Y_1, \ldots, Y_k)$$

where $Y_1, \ldots, Y_k$ are independent, identically distributed Laplace variables from $\text{Lap} \left( \frac{\text{GS}^n(f)}{\epsilon} \right)$.

Note that for the Laplace density function $h$ we have:

$$\frac{h_b(y)}{h_b(y')} = \frac{e^{-\frac{|y|}{b}}}{e^{-\frac{|y'|}{b}}} = e^{\frac{|y'| - |y|}{b}} \leq e^{\frac{|y - y'|}{b}}$$

for $y, y' \in \mathbb{R}$. 
Example

Suppose $D \in \{0, 1\}^n$ and the user wishes to learn $f(D) = \sum_{i=1}^{n} x_i$, that is, the total number of 1s in $D$.

If we add random Laplace noise $Y \sim Lap(1/\epsilon)$ (that is, a Laplace random variable with the parameter $b = \frac{1}{\epsilon}$ and the probability density function $h(x) = \frac{\epsilon}{2} e^{-\epsilon|x|}$), then the algorithm will produce $T(D)$, where

$$T(D) = \sum_{i=1}^{n} x_i + Y.$$ 

Note that $T(D) = t$ is equivalent to

$$Y = t - \sum_{i=1}^{n} x_i = t - f(D).$$

Let $D$ and $\tilde{D}$ be two databases that differ in a single entry. We have:

$$P(T(D) = t) = \frac{h(t - f(D))}{h(t - f(\tilde{D}))}.$$
Noise that must be added to a querying algorithm can be calibrated to achieve differential privacy according to the sensitivity of a query. If we have a querying mechanism $\mathcal{M}(D, f, \epsilon) = f(D) + \mathbf{y}$, where the noise $\mathbf{y}$ is drawn from $(Y_1, \ldots, Y_k)$, the density function of $(Y_1, \ldots, Y_k)$ at $\mathbf{y}$ is proportional to $e^{-\frac{\|\mathbf{y}\|_1}{b}}$. Thus, for all $t \in \mathbb{R}^d$ we have

$$\frac{P(\mathbf{z} + \mathbf{Y} = t)}{P(\tilde{\mathbf{z}} + \mathbf{Y} = t)} \in \{e^{\frac{\|\mathbf{z} - \tilde{\mathbf{z}}\|_1}{b}}, e^{-\frac{\|\mathbf{z} - \tilde{\mathbf{z}}\|_1}{b}},\}$$

Thus, to release a perturbed value $f(D)$ while satisfying $\epsilon$-differential privacy it suffices to add Laplace noise with standard deviation $\frac{S(f)}{\epsilon}$ in each coordinate.
Definition

Let $\mathcal{M}_i$ be two private algorithms that are $\epsilon_i$ differentially private, for $i = 1, 2$. The composition of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the algorithm $\mathcal{M}_{1,2}$ defined as

$$\mathcal{M}_{1,2}(D) = (\mathcal{M}_1(D), \mathcal{M}_2(D))$$
Suppose that for $i = 1, 2$ $\mathcal{M}_i$ are private $\epsilon_i$ differential algorithms, respectively and that $D, D'$ are databases that differ in one position. We have

\[
P(\mathcal{M}_1(D) = S_1) \leq e^{\epsilon_1} P(\mathcal{M}_1(D') = S_1),
\]
\[
P(\mathcal{M}_2(D) = S_2) \leq e^{\epsilon_1} P(\mathcal{M}_2(D') = S_2).
\]

Then,

\[
P[(\mathcal{M}_1(D) = S_1), (\mathcal{M}_2(D) = S_2)] = P[(\mathcal{M}_1(D) = S_1)]P[(\mathcal{M}_2(D) = S_2)]
\]
\[
\leq e^{\epsilon_1} P[(\mathcal{M}_1(D') = S_1)] e^{\epsilon_2} P[(\mathcal{M}_2(D') = S_2)]
\]
\[
= e^{\epsilon_1 + \epsilon_2} P[(\mathcal{M}_1(D') = S_1)] P[(\mathcal{M}_2(D') = S_2)]
\]

hence the composition is $(\epsilon_1 + \epsilon_2)$-private.
Lemma

If $Y \sim \text{Lap}(b)$, then $P(|Y| > tb) = e^{-t}$.

Proof.

This fact follows immediately from

$$P(|Y| > tb) = P(Y > tb) + P(Y < -tb)$$

$$= 1 - \int_{-tb}^{tb} h_b(t) \, dt$$

$$= 1 - 2 \int_0^{tb} \frac{1}{2b} e^{-\frac{t}{b}} \, dt = e^{-t}.$$
Theorem

Let \( f : DB(X^n) \rightarrow \mathbb{R}^k \) and let \( y = M(D, f, \epsilon) \), where \( M \) is a \( k \)-dimensional Laplace mechanism. For every \( \delta \in (0, 1] \) we have:

\[
P \left( \| f(x) - y \|_\infty \geq \ln \frac{k}{\delta} \cdot \frac{S(f)}{\epsilon} \right) \leq \delta.
\]
Proof

We have

\[
P \left( \| f(x) - y \|_\infty \geq \ln \frac{k}{\delta} \cdot \frac{S(f)}{\epsilon} \right) \\
= P \left( \max_{1 \leq i \leq k} |Y_i| \geq \ln \frac{k}{\delta} \cdot \frac{S(f)}{\epsilon} \right) \\
\leq k \cdot P \left( \max_{1 \leq i \leq k} |Y_i| \geq \ln \frac{k}{\delta} \cdot \frac{S(f)}{\epsilon} \right) \\
= k \cdot \frac{\delta}{k} = \delta,
\]

where the inequality follows from the fact that each $Y_i$ is distributed $\text{Lap} \left( \frac{S(f)}{\epsilon} \right)$ and from the previous Lemma.
Example

Suppose that we have a list of 10,000 potential name and we wish to compute which first name were the most common in a national census. Take

\[ k = 10,000, \delta = 0.05, \text{ and } \frac{S(f)}{\epsilon} = 1. \]

Note that the sensitivity of this query is 1 because every person may have only one first name. By the theorem, we can calculate the frequency of all 10,000 names with (1,0)-differential privacy and with the probability 95%, no estimate will be off am additive error of \( \ln \frac{10000}{0.05} \approx 12.2 \).