Inner Products and Norms
(part I)

Prof. Dan A. Simovici

UMB

Definition of Linear Spaces

Definition

Let L be a nonempty set and let $(\mathbb{F}, \{0, +, -, \cdot, \})$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

An $\mathbb{F}$-*linear space* is a triple $(L, +, \cdot)$ such that $(L, \{0_L, +, -\})$ is an Abelian group and $\cdot : \mathbb{F} \times L \longrightarrow L$ is an operation such that the following conditions are satisfied:

1. $a \cdot (b \cdot x) = (a \cdot b) \cdot x$,
2. $1 \cdot x = x$,
3. $a \cdot (x + y) = a \cdot x + a \cdot y$, and
4. $(a + b) \cdot x = a \cdot x + b \cdot x$

for every $a, b \in F$ and $x, y \in L$.

If $\mathbb{F} = \mathbb{R}$, then we refer to L as a *real linear space*; for $\mathbb{F} = \mathbb{C}$ we say that L is a *complex linear space*.

Notations

- The commutative binary operation of L is denoted by the same symbol "$+$" as the corresponding operation of the field $\mathbb{F}$.
- The multiplication by a scalar, $\cdot : \mathbb{F} \times L \longrightarrow L$ is also referred to as an *external operation* since its two arguments belong to two different sets, $\mathbb{F}$ and L.
- The neutral additive element 0_L of L is referred to as the *zero element* of L; every linear space must contain at least this element.
- To simplify the notation, we will simply denote a linear space $(L, +, \cdot)$ by L.

Vectors

Let S be a nonempty set and let $n \in \mathbb{N}$ be a number such that $n \geqslant 1$. A *vector of length n over S* is a function $\mathbf{v} : \{1, \ldots, n\} \longrightarrow S$, that is a sequence of length n of elements of S. We denote $\mathbf{v}$ by

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

where $v_i = \mathbf{v}(i)$ is the i^{th} *component* of $\mathbf{v}$ for $1 \leqslant i \leqslant n$. The set of vectors of length n over S will be denoted by S^n.

Example

The set $\mathbb{R}^n$ of vectors of length n over $\mathbb{R}$ is a real linear space under the definitions

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \text{ and } a \cdot \mathbf{x} = \begin{pmatrix} a \cdot x_1 \\ \vdots \\ a \cdot x_n \end{pmatrix}$$

of the operations $+$ and $\cdot$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

In this linear space, the zero of the Abelian group is the n-tuple

$$\mathbf{0}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Example

The set of infinite sequences of real numbers $\mathbf{Seq}_\infty(\mathbb{R})$ can be organized as a real linear space by defining the addition of two sequences

$$\mathbf{x} = (x_0, x_1, \dots) \text{ and } \mathbf{y} = (y_0, y_1, \dots)$$

as $\mathbf{x} + \mathbf{y} = (x_0 + y_0, x_1 + y_1, \dots)$, and the multiplication by $c\mathbf{x}$ as $c\mathbf{x} = (cx_0, cx_1, \dots)$ for $c \in \mathbb{R}$.

Example

The set of complex-valued functions defined on a set S is a real linear space. The addition of functions is given by $(f + g)(s) = f(s) + g(s)$, and the multiplication of a function with a real number is defined by $(af)(s) = af(s)$ for $s \in S$ and $a \in \mathbb{R}$.

Example

Let S be a set. A function $f : S \longrightarrow \mathbb{R}$ is *bounded* if there exists $k \in \mathbb{R}$ such that $|f(x)| \leqslant k$. The set of bounded functions defined on S is denoted by $\text{bound}(S)$. It is easy to see that the sum of two bounded functions on S is again bounded and the product af of a bounded function f with $a \in \mathbb{R}$ is also bounded. Thus, the set of functions bounded on S is a real linear space.

Relations on $\mathbb{R}^n$

1. $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for $1 \leqslant i \leqslant n$;
2. $\mathbf{x} \geq \mathbf{y}$ if $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$;
3. $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$ for $1 \leqslant i \leqslant n$.

If $\mathbf{x} \geqslant \mathbf{0}_n$, we say that $\mathbf{x}$ is *non-negative*; if $\mathbf{x} \geq \mathbf{0}_n$, $\mathbf{x}$ is said to be *semi-positive*, and if $\mathbf{x} > \mathbf{0}_n$, then $\mathbf{x}$ is *positive*. The relation "$\geqslant$" is a partial order on $\mathbb{R}^n$; the other two relations, "$\geq$" and "$>$" are strict partial orders on the same set because they lack reflexivity.

Example

Let C be the set of real-valued continuous functions defined on $\mathbb{R}$,

$$C = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}.$$

Define $f + g$ by $(f + g)(x) = f(x) + g(x)$ and $(a \cdot f)(x) = a \cdot f(x)$ for $x \in \mathbb{R}$.
The triple $(C, +, \cdot)$ is a real linear space.
Similarly, the set $C[a, b]$ of real-valued continuous functions on $[a, b]$ is a real linear space.

Inner Products on Complex Linear Spaces

Definition

Let L be a complex linear space. An *inner product* on L is a function $\wp : L \times L \longrightarrow \mathbb{C}$ that has the following properties:

1. $\wp(x, y) = \overline{\wp(y, x)}$ for $x, y \in L$;
2. $\wp(x + y, z) = \wp(x, z) + \wp(y, z)$ for $x, y, z \in L$;
3. $\wp(ax, y) = a\wp(x, y)$ for $x, y \in L$ and $a \in \mathbb{C}$;
4. (x, x) is a non-negative real number for $x \in L$;
5. $(x, x) = 0$ implies $x = 0_L$.

The pair $(L, \wp)$ is called an *inner product space*.

Inner Products in Real and Complex Linear Spaces

Observe that for $a \in \mathbb{C}$ we have

$$\wp(x, ay) = \bar{a}\wp(x, y)$$

because

$$\wp(x, ay) = \overline{\wp(ay, x)} = \bar{a}\overline{\wp(y, x)} = \bar{a}\wp(x, y).$$

If L is a real linear space we assume that $\wp(x, y) \in \mathbb{R}$ for $x, y \in L$. Thus, for an inner product on a real linear space we have the symmetry property $\wp(x, y) = \wp(y, x)$ for $x, y \in L$.

Example

Let $a_1, \ldots, a_n$ are n real, positive numbers. The function $\wp : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\wp(\mathbf{x}, \mathbf{y}) = a_1 x_1 y_1 + a_2 x_2 y_2 + \cdots + a_n x_n y_n$ is an inner product on $\mathbb{R}^n$, as the reader can easily verify.
If $a_1 = \cdots = a_n = 1$, we have the *Euclidean inner product*:

$$\wp(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_n y_n = \mathbf{y}'\mathbf{x} = \mathbf{x}'\mathbf{y}.$$

To simplify notations we denote an inner product $\wp(x, y)$ by (x, y) when there is no risk of confusion.

Example

An inner product on $\mathbb{R}^{n \times n}$, the real linear space of matrices of format $n \times n$, can be defined as $(X, Y) = \mathit{trace}(XY')$ for $X, Y \in \mathbb{R}^{n \times n}$. Similarly, for the complex linear space $\mathbb{C}^{n \times n}$, an inner product can be defined as $(X, Y) = \mathit{trace}(XY^{\mathsf{H}})$.

An Inner Product on $\mathbb{R}^n$

A fundamental property of the inner product defined on $\mathbb{R}^n$ is the equality

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A'\mathbf{y}), \tag{1}$$

which holds for every matrix $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Indeed, we have

$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n}(A\mathbf{x})_i y_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j y_i = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} y_i \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} y_i = (\mathbf{x}, A'\mathbf{y}).\end{aligned}$$

An Inner Product on $\mathbb{C}^n$

For the complex linear space $\mathbb{C}^n$ we can define an inner product as

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y_i} = \bar{\mathbf{y}}\mathbf{x}.$$

In this case we have

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^{\mathsf{H}}\mathbf{y}), \tag{2}$$

Indeed, if $A \in \mathbb{C}^{n \times n}$, then

$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} (A\mathbf{x})_i \overline{y_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \overline{y_i} = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} \overline{y_i} \\ &= \sum_{j=1}^{n} x_j \overline{\sum_{i=1}^{n} \overline{a_{ij}} y_i} = (\mathbf{x}, A^{\mathsf{H}}\mathbf{y})\end{aligned}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

The Cauchy-Schwarz Inequality

Theorem

Let $(L, (\cdot, \cdot))$ *be an inner product* $\mathbb{F}$*-linear space. For* $x, y \in L$ *we have*

$$|(x, y)|^2 \leqslant (x, x)(y, y).$$

Proof

We discuss the complex case. If $a, b \in \mathbb{C}$ we have

$$\begin{aligned}(ax+by, ax+by) &= a\bar{a}(x,x) + a\bar{b}(x,y) + b\bar{a}(y,x) + b\bar{b}(y,y) \\ &= |a|^2(x,x) + 2\Re(a\bar{b}(x,y)) + |b|^2(y,y) \geqslant 0.\end{aligned}$$

Let $a = (y,y)$ and $b = -(x,y)$. We have

$$(y,y)^2(x,x) - 2(y,y)|(x,y)|^2 + |(x,y)|^2(y,y) \geqslant 0,$$

hence

$$(y,y)|(x,y)|^2 \leqslant (y,y)^2(x,x).$$

If $y = 0$ the inequality obviously holds. If $y \neq 0$, the inequality follows.

Seminorms

The notions of seminorm and norm formalize the notion of vector length.

Definition

Let L be an $\mathbb{F}$-linear space. A *seminorm* on L is a mapping $\nu : L \longrightarrow \mathbb{R}_{\geqslant 0}$ that satisfies the following conditions:

1. $\nu(x + y) \leqslant \nu(x) + \nu(y)$ (subadditivity), and
2. $\nu(ax) = |a|\nu(x)$ (positive homogeneity),

for $x, y \in L$ and every scalar a.

By taking $a = 0$ in the second condition of the definition we have $\nu(0_L) = 0$ for every seminorm on a real or complex space.

Theorem

If L is a real linear space and $\nu : L \longrightarrow \mathbb{R}$ is a seminorm on L, then $\nu(x - y) \geqslant |\nu(x) - \nu(y)|$ for $x, y \in L$.

Proof.

We have $\nu(x) \leqslant \nu(x - y) + \nu(y)$, so $\nu(x) - \nu(y) \leqslant \nu(x - y)$. Since $\nu(y - x) + \nu(x) \leqslant \nu(y)$ and $\nu(y - x) = \nu(x - y)$, we have the inequalities

$$\nu(x) - \nu(y) \leqslant \nu(x - y) \leqslant \nu(y) - \nu(x),$$

which imply the inequality of the theorem. □

Definition

Let L be a real or complex linear space. A *norm* on L is a seminorm $\nu : L \longrightarrow \mathbb{R}$ such that $\nu(x) = 0$ implies $x = 0$ for $x \in L$.
The pair (L, ν) is referred to as a *normed linear space*.

Inner products on linear spaces induce norms on these spaces.

Theorem

Let $(L, (\cdot, \cdot))$ *be an inner product* $\mathbb{F}$*-linear space. The function* $\| \cdot \|: L \longrightarrow \mathbb{R}_{\geqslant}$ *defined by* $\| x \| = \sqrt{(x, x)}$ *is a norm on* L*.*

Proof

We present the argument for inner product $\mathbb{C}$-linear spaces.
It is immediate from the properties of the inner product that $\parallel x \parallel$ is a real non-negative number and that $\parallel ax \parallel = |a| \parallel x \parallel$ for $a \in \mathbb{C}$.
Note that

$$\begin{aligned}
\parallel x + y \parallel^2 &= (x + y, x + y) \\
&= (x, x) + (x, y) + (y, x) + (y, y) \\
&= \parallel x \parallel^2 + 2\Re(x, y) + \parallel y \parallel^2 \\
&\quad \text{(because } (y, x) = \overline{(x, y)}) \\
&\leqslant \parallel x \parallel^2 + 2|(x, y)| + \parallel y \parallel^2 \\
&\quad \text{(because } \Re(y, x) \leqslant |(x, y)|) \\
&\leqslant \parallel x \parallel^2 + 2 \parallel x \parallel \parallel y \parallel + \parallel y \parallel^2 \\
&\quad \text{(be the Cauchy-Schwarz Inequality)} \\
&= (\parallel x \parallel + \parallel y \parallel)^2,
\end{aligned}$$

which produces the needed inequality.

The Cauchy-Schwarz Inequality can now be formulated using the norm.

Corollary

Let $(L, (\cdot, \cdot))$ be an inner product $\mathbb{F}$-linear space. For $x, y \in L$ we have

$$|(x, y)| \leqslant \| x \| \cdot \| y \| .$$

The Polarization Identity

Theorem

Let $\| \cdot \|$ be a norm on a complex linear space L that is generated by the inner product $(\cdot, \cdot)$. We have the complex polarization identity*:*

$$(x, y) = \frac{1}{4} \left(\| x + y \|^2 - \| x - y \|^2 - i \| x - iy \|^2 + i \| x + iy \|^2 \right)$$

for $x, y \in L$.

Proof

$$
\begin{aligned}
\| x+y \|^2 &= (x+y, x+y) = \| x \|^2 + \| y \|^2 + (x,y) + (y,x) \\
&= (x+y, x+y) = \| x \|^2 + \| y \|^2 + (x,y) + \overline{(x,y)} \\
&= \| x \|^2 + \| y \|^2 + 2\Re(x,y).
\end{aligned}
$$

Replacing y by $-y$ we have

$$\| x-y \|^2 = \| x \|^2 + \| y \|^2 - 2\Re(x,y),$$

hence $\| x+y \|^2 - \| x-y \|^2 = 4\Re(x,y)$. Replacing now y by iy we obtain

$$\| x+iy \|^2 - \| x-iy \|^2 = 4\Re(x,iy).$$

Note that

$$(x,iy) = -i(x,y) = -i(\Re(x,y) + i\Im(x,y)) = \Im(x,y) - i\Re(x,y),$$

so $\Re(x,iy) = \Im(x,y)$, which implies

$$\| x+iy \|^2 - \| x-iy \|^2 = 4\Im(x,iy).$$

Proof (cont'd)

Taking into account the previous equalities we can write

$$\begin{aligned} & \| x+y \|^2 - \| x-y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2 \\ = \;& 4\Re(x,y) + 4i\Im(x,y) = 4(x,y), \end{aligned}$$

which is the desired identity.

Corollary

Let $\| \cdot \|$ be a norm on a real linear space L that is generated by the inner product $(\cdot, \cdot)$. We have the polarization identity

$$(x, y) = \frac{1}{4} \left(\| x + y \|^2 - \| x - y \|^2 \right)$$

for $x, y \in L$.

Parallelogram Equality

Theorem

Let $\| \cdot \|$ *be a norm on a linear space* L *that is generated by the inner product* $(\cdot, \cdot)$. *We have the* parallelogram equality*:*

$$\| x \|^2 + \| y \|^2 = \frac{1}{2} \left(\| x + y \|^2 + \| x - y \|^2 \right)$$

for $x, y \in L$.

Proof

By applying the definition of $\| \cdot \|$ and the properties of the inner product we can write:

$$
\begin{aligned}
&\frac{1}{2}\left(\| x + y \|^2 + \| x - y \|^2\right) \\
&\quad = \frac{1}{2}\left((x + y, x + y) + (x - y, x - y)\right) \\
&\quad = \frac{1}{2}\left((x, x) + 2(x, y) + (y, y) + (x, x) - 2(x, y) + (y, y)\right) \\
&\quad = \| x \|^2 + \| y \|^2,
\end{aligned}
$$

which concludes the proof.

Lemma

Let $p, q \in \mathbb{R} - \{0, 1\}$ *be two numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *Then, for every* $a, b \in \mathbb{R}_{\geqslant 0}$, *we have*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where the equality holds if and only if $a = b^{-\frac{1}{1-p}}$.

Proof

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = x^p - px + p - 1,$$

where $p > 1$. Note that $f(1) = 0$ and that $f'(x) = p(x^{p-1} - 1)$. This implies that f has a minimum in $x = 1$ and, therefore, $x^p - px + p - 1 \geqslant 0$ for $x \in [0, \infty)$. Substituting $ab^{-\frac{1}{p-1}}$ for x yields the desired inequality.

Discrete Hölder's Inequality

Theorem

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ nonnegative numbers, and let p and q be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

Proof

Define the numbers

$$x_i = \frac{a_i}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}} \text{ and } y_i = \frac{b_i}{\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}}$$

for $1 \leqslant i \leqslant n$. Lemma 19 applied to x_i, y_i yields

$$\frac{a_i b_i}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}} \leqslant \frac{1}{p}\frac{a_i^p}{\sum_{i=1}^n a_i^p} + \frac{1}{q}\frac{b_i^q}{\sum_{i=1}^n b_i^q}.$$

Adding these inequalities, we obtain

$$\sum_{i=1}^n a_i b_i \leqslant \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}.$$

Theorem

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ real numbers and let p and q be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leqslant \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}.$$

Proof

By a previous theorem we have:

$$\sum_{i=1}^{n} |a_i||b_i| \leqslant \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}.$$

The needed equality follows from the fact that

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \sum_{i=1}^{n} |a_i||b_i|.$$

The Cauchy-Schwarz Inequality

Corollary

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ real numbers. We have

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leqslant \sqrt{\sum_{i=1}^{n} |a_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |b_i|^2}.$$

The inequality follows immediately by taking $p = q = 2$.

Minkowski's Inequality

Theorem

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be 2n nonnegative numbers. If $p \geqslant 1$, we have

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

If $p < 1$, the inequality sign is reversed.

Proof

For $p = 1$, the inequality is immediate. Therefore, we can assume that $p > 1$. Note that

$$\sum_{i=1}^{n}(a_i + b_i)^p = \sum_{i=1}^{n} a_i(a_i + b_i)^{p-1} + \sum_{i=1}^{n} b_i(a_i + b_i)^{p-1}.$$

By Hölder's inequality for p, q such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \sum_{i=1}^{n} a_i(a_i + b_i)^{p-1} &\leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i + b_i)^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i + b_i)^{p}\right)^{\frac{1}{q}}. \end{aligned}$$

Similarly, we can write

$$\sum_{i=1}^{n} b_i(a_i + b_i)^{p-1} \leqslant \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i + b_i)^{p}\right)^{\frac{1}{q}}.$$

Proof (cont'd)

Adding the last two inequalities yields

$$\sum_{i=1}^{n}(a_i + b_i)^p \leqslant \left(\left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \right) \left(\sum_{i=1}^{n}(a_i + b_i)^p \right)^{\frac{1}{q}},$$

which is equivalent to the desired inequality

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}.$$

Theorem

For $p \geqslant 1$, the function $\nu_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by

$$\nu_p(\mathbf{x}) = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

is a norm on the linear space $(\mathbb{R}^n, +, \cdot)$.

Proof

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Minkowski's inequality applied to the nonnegative numbers $a_i = |x_i|$ and $b_i = |y_i|$ amounts to

$$\left(\sum_{i=1}^{n} (|x_i| + |y_i|)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}.$$

Since $|x_i + y_i| \leqslant |x_i| + |y_i|$ for every i, $1 \leqslant i \leqslant n$, we have

$$\left(\sum_{i=1}^{n} (|x_i + y_i|)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}},$$

that is, $\nu_p(\mathbf{x} + \mathbf{y}) \leqslant \nu_p(\mathbf{x}) + \nu_p(\mathbf{y})$. Thus, ν_p is a norm on $\mathbb{R}^n$.

Example

Consider the mappings $\nu_1, \nu_\infty : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\nu_1(\mathbf{x}) = |x_1| + |x_2| + \cdots + |x_n| \text{ and } \nu_\infty(\mathbf{x}) = \max\{|x_1|, |x_2|, \ldots, |x_n|\},$$

for every $\mathbf{x} \in \mathbb{R}^n$. Both ν_1 and ν_∞ are norms on $\mathbb{R}^n$.

To verify that ν_∞ is a norm we start from the inequality $|x_i + y_i| \leqslant |x_i| + |y_i| \leqslant \nu_\infty(\mathbf{x}) + \nu_\infty(\mathbf{y})$ for $1 \leqslant i \leqslant n$. This in turn implies

$$\nu_\infty(\mathbf{x}+\mathbf{y}) = \max\{|x_i + y_i| \mid 1 \leqslant i \leqslant n\} \leqslant \nu_\infty(\mathbf{x}) + \nu_\infty(\mathbf{y}),$$

which gives the desired inequality.
This norm is a limit case of the norms ν_p. Indeed, let $\mathbf{x} \in \mathbb{R}^n$ and let $M = \max\{|x_i| \mid 1 \leqslant i \leqslant n\} = |x_{l_1}| = \cdots = |x_{l_k}|$ for some $l_1, \ldots, l_k$, where $1 \leqslant l_1, \ldots, l_k \leqslant n$. Here $x_{l_1}, \ldots, x_{l_k}$ are the components of $\mathbf{x}$ that have the maximal absolute value and $k \geqslant 1$. We can write:

$$\lim_{p\to\infty} \nu_p(\mathbf{x}) = \lim_{p\to\infty} M \left(\sum_{i=1}^{n} \left(\frac{|x_i|}{M} \right)^p \right)^{\frac{1}{p}} = \lim_{p\to\infty} M(k)^{\frac{1}{p}} = M,$$

which justifies the notation ν_∞.

We will frequently use the alternative notation $\| \mathbf{x} \|_p$ for $\nu_p(\mathbf{x})$. We refer to the norm ν_2 as the *Euclidean norm*.

Example

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be a unit vector in the sense of the Euclidean norm. We have $|x_1|^2 + |x_2|^2 = 1$. Since x_1 and x_2 are real numbers we can write $x_1 = \cos\alpha$ and $x_2 = \sin\alpha$. This allows us to write

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$