A Graph-Theoretical Approach to Boolean Interpolation of Non-Boolean Functions

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Abstract

We introduce a graph-theoretical approach to the study of approximation of non-Boolean functions on Boolean algebra. We show that optimal interpolations of non-Boolean functions by Boolean functions are linked to minimal chromatic decompositions of graphs attached to these functions and we study special vertices in these graphs.

1 Introduction

The purpose of this paper is to develop a graph-theoretical approach for finding optimal Boolean interpolations of non-Boolean functions on Boolean algebras (see [10, 9]). The interest in non-Boolean function has been sparked by work dealing with the applications of set-valued non-Boolean functions in circuit design [3, 8, 1, 2, 6, 5] and [4].

Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra, where B is a set, $\vee$ and $\cdot$ are binary operations on B called disjunction and conjunction, respectively, $'$ is a unary operation, called the complementation operation, and $0, 1$ are two special elements of B, with $0 \neq 1$ such that the usual axioms of Boolean algebras are satisfied as given e.g. in [11].

Elements of B^n, where $n > 1$ will be denoted by capital letters $X, Y, \ldots$, while elements of the algebra $\mathcal{B}$ will be denoted by small letters. *Elementary n-tuples*, that is, members of the set $\{0,1\}^n$, will be designated by $A, B, \ldots$. Boolean functions will be denoted by small letters $f, g, \ldots$. Arbitrary functions will be denoted by capital letters: $F, G, \ldots$.

If $x \in B$ and $a \in \{0,1\}$ we use the notation

$$x^a = \begin{cases} x & \text{if } a = 1 \\ x' & \text{if } a = 0. \end{cases}$$

Note that $a^a = 1$ for $a \in \{0,1\}$. If $X = (x_1, \ldots, x_n) \in B^n$ and $A = (a_1, \ldots, a_n) \in \{0,1\}^n$, define X^A as the conjunction $x_1^{a_1} \cdots x_n^{a_n}$. We have

$$A^C = \begin{cases} 1 & \text{if } A = C \\ 0 & \text{if } A \neq C, \end{cases}$$

for every $A, C \in \{0,1\}^n$. Also, if $A \neq C$, we have $X^A X^C = 0$ for every $X \in B^n$.

The binary operation "+" is defined on B by $x + y = xy' \vee x'y$ for $x, y \in B$. An easy argument by induction on n shows that if $z_1, \ldots, z_n \in B$ such that $z_i z_j = 0$ for $1 \leq i, j \leq n$ and $i \neq j$, then

$$\bigvee_{1 \leq i \leq n} z_i = \sum_{1 \leq i \leq n} z_i.$$

If $X = (x_1, \ldots, x_n) \in B^n$, then we denote $x_1 \vee x_2 \vee \cdots \vee x_n$ by $X^{\vee}$. Also, if $X, Y \in B^n$, $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, we denote by $X + Y$ the n-tuple $(x_1 + y_1, \ldots, x_n + y_n)$.

Lemma 1.1 *Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra.*

1. *For every $X \in B^n$ we have $\bigvee_{A \in \{0,1\}^n} X^A = 1$.*
2. *If $X, Y \in B^n$, then $1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A = (X + Y)^{\vee}$.*

Proof. Since both identities involve only Boolean operations, they can be proven immediately by Müller-Lövenheim Verification Theorem (Theorem 2.13 from [11]) by observing that the functions designated by the expressions from the left and right members are equal for all elementary vectors $C \in \{0,1\}^n$. ∎

It is useful to note that the mapping $d : B^n \times B^n \longrightarrow B$ defined by $d(X, Y) = (X + Y)^{\vee}$ for $X, Y \in B^n$ is a distance on B^n, in the sense defined in [11], p. 313. Namely, it is clear that $d(X,Y) = 0$ if and only if $X = Y$ and that $d(X,Y) = d(Y,X)$. It is easy to verify that for every $x, y, z \in B$ we have

$$x + y \leq (x + z) \vee (z + y).$$

This implies immediately

$$d(X,Y) \leq d(X,Z) \vee d(Z,Y)$$

for $X, Y, Z \in B^n$, which justifies our observation.

2 The Graph of a Function over B^n

Let $F : B^n \longrightarrow B$ be a function. Its graph Γ_F has B^n as its set of vertices; an edge (X,Y) exists in Γ_F if

$$F(X) + F(Y) \not\leq (X + Y)^{\vee}.$$

To simplify the notation we denote the relation defined by the graph Γ_F by ρ_F.

Note that if $F_1, F_2 : B^n \longrightarrow B$ are such that $F_2(X) = k + F_1(X)$ for $X \in B^n$ and $k \in B$, then $\Gamma_{F_1} = \Gamma_{F_2}$ because $F_2(X) + F_2(Y) = k + F_1(X) + k + F_1(Y) = F_1(X) + F_1(Y)$ for every $X, Y \in B^n$. Therefore, it is clear that several distinct functions may share the same graph. In particular, $\Gamma_F = \Gamma_{F'}$, where $F'(X) = 1 + F(X)$ for $X \in B^n$.

Definition 2.1 Let $F : B^n \longrightarrow B$ be an arbitrary function (not necessarily Boolean) over the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$.

A *Boolean π-interpolation of F* is a family of Boolean functions $\{f_{\mathcal{C}} \mid \mathcal{C} \in \pi\}$ indexed by a partition π of the set B^n such that for every block $\mathcal{C}$ of π, $f_{\mathcal{C}}(X) = F(X)$ for every $X \in \mathcal{C}$.

An interpolation of the least cardinality is said to be *optimal*. □

Note that every set of the form $F^{-1}(b)$ is an independent set in the graph Γ_F for $b \in B$. Indeed, if $X, Y \in F^{-1}(b)$, then $F(X) + F(Y) = 0 \leq (X + Y)^{\vee}$, so there is no edge between X and Y. Moreover, if $b, c \in B$, then the set $F^{-1}(b) \cup F^{-1}(c)$ is independent if $b + c \leq d(X,Y)$ for every $X \in F^{-1}(b)$ and $Y \in F^{-1}(c)$.

The next theorem is a generalization of Theorem 2.1 from [10]:

Theorem 2.2 *For every independent set I in the graph Γ_F the Boolean function $f_I : B^n \longrightarrow B$ defined by*

$$f_I(X) = \bigvee_{Y \in I} \left(F(Y) \bigvee_{A \in \{0,1\}^n} X^A Y^A \right) \qquad (1)$$

is such that $f_I(X) = F(X)$ for every $X \in I$.

Proof. Suppose $X \in I$. Then,

$$f_I(X) = F(X) \vee \bigvee_{Y \in I - \{X\}} \left(F(Y) \bigvee_{A \in \{0,1\}^n} X^A Y^A \right),$$

and we shall prove that $F(Y) \bigvee_{A \in \{0,1\}^n} X^A Y^A \leq F(X)$. Indeed, $Y \in I - \{X\}$ implies $(X,Y) \notin \rho_F$ because I is an independent set, that is, $F(X) + F(Y) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A$. Therefore, $\bigvee_{A \in \{0,1\}^n} X^A Y^A \leq 1 + F(X) + F(Y) = F(X) + F(Y)'$, so $F(Y) \bigvee_{A \in \{0,1\}^n} X^A Y^A \leq F(Y)\,(F(X) + F(Y)') = F(Y)F(X) \leq F(X)$. Thus, $f_I(X) = F(X)$. ∎

Theorem 2.3 *For every Boolean π-interpolation $\{f_{\mathcal{C}} \mid \mathcal{C} \in \pi\}$ of a function F, the partition π is a chromatic decomposition of the graph Γ_F.*

Proof. For every $\mathcal{C} \in \pi$ and every $X, Y \in \mathcal{C}$, we have

$$F(X) + F(Y) = f_{\mathcal{C}}(X) + f_{\mathcal{C}}(Y) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A$$

by Theorem 2.1 of [12]. This shows that $(X,Y) \notin \rho_F$, which implies that $\mathcal{C}$ is an independent set. Thus, π is a chromatic decomposition of Γ_F. ∎

Corollary 2.4 *There is a bijection between the Boolean π-interpolations of F and the chromatic decompositions of the graph Γ_F. Further, if μ is a minimal chromatic decomposition of the graph Γ_F, then the corresponding family of Boolean functions $\{f_{\mathcal{C}} \mid \mathcal{C} \in \mu\}$ is an optimal interpolation of F.*

Proof. This statement follows immediately from Theorems 2.2 and 2.3. ∎

The class of functions that we discuss below serves to show that Boolean interpolations for non-Boolean functions are not unique, in general.

Definition 2.5 A function $F : B^n \longrightarrow B$ is *K-evanescible*, where K is a subset of B^n, if $F(X) = 0$ for every $X \in K$. □

A characterization of Boolean K-evanescible functions is given below:

Theorem 2.6 *Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra and let $K \subseteq B^n$. A Boolean function $g : B^n \longrightarrow B$ is K-evanescible if and only if $g(A) \leq 1 + \bigvee_{Z \in K} Z^A$ for every $A \in \{0,1\}^n$.*

Proof. Suppose that g is K-evanescible and take $Z \in K$. Then, since $g(Z) = \bigvee_{A \in B^n} g(A) Z^A$, we have $g(A) Z^A = 0$, so $Z^A \leq 1 + g(A)$. Consequently, $\bigvee_{Z \in K} Z^A \leq 1 + g(A)$, so $g(A) \leq 1 + \bigvee_{Z \in K} Z^A$.

Conversely, suppose that $g(A) \leq 1 + \bigvee_{Z\in K} Z^A$ for every $A \in \{0,1\}^n$. Then, since g is a Boolean function we can write for $X \in K$:

$$\begin{aligned} g(X) &= \bigvee_{A\in\{0,1\}^n} g(A)X^A \\ &\leq \bigvee_{A\in\{0,1\}^n} \left(1 + \bigvee_{Z\in K} Z^A\right) X^A \\ &= \bigvee_{A\in\{0,1\}^n} \left(X^A + \bigvee_{Z\in K} X^A Z^A\right) = 0, \end{aligned}$$

because $X \in K$ implies $\bigvee_{Z\in K} X^A Z^A = X^A$, which shows that g is K-evanescible. ∎

Note that if I is an independent set in the graph Γ_F and f_1, f_2 are two Boolean functions that interpolate F on I, then $f_1 + f_2$ is a Boolean I-evanescible function. Thus, a Boolean interpolant of F for an independent set I of Γ_F is unique if $\bigvee_{Z\in I} Z^A = 1$ for every $A \in \{0,1\}^n$.

Theorem 2.7 *Let $F : B^n \longrightarrow B$ be a function and let I be an independent set in Γ_F. For every Boolean interpolating function $g : B^n \longrightarrow B$ for F on I we have $f_I \leq g$, where f_I is the Boolean function defined in Theorem 2.2.*

Proof. Since g is an interpolating Boolean function for F we can write $g(X) = \bigvee_{C\in\{0,1\}^n} g(C)X^C = F(X)$ for every $X \in I$. Therefore,

$$F(X)X^A = \bigvee_{C\in\{0,1\}^n} g(C)X^C X^A = g(A)X^A \leq g(A)$$

for every $X \in I$. Consequently, $\bigvee_{X\in I} F(X)X^A \leq g(A)$. Since

$$f_I(A) = \bigvee_{X\in I} F(X)X^A$$

for every $A \in \{0,1\}^n$ it follows that $f_I(A) \leq g(A)$ for every $A \in \{0,1\}^n$, so $f_I \leq g$. ∎

3 Boolean and Isolated Vertices in Graphs of Functions

In [12] we extended the classes of upper and lower semi-Boolean functions of one-variable introduced in [9] by introducing the notion of A-Boolean function for $A \in \{0,1\}^n$.

For $A \in \{0,1\}^n$ we say that $F : B^n \longrightarrow B$ is an *A-Boolean function* if $F(X) + F(Y) \leq 1 + X^A Y^A$ for every $X, Y \in B^n$. The next lemma will allow us to formulate another definition of A-Boolean functions.

Lemma 3.1 *Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra, $F : B^n \longrightarrow B$, $X \in B^n$ and $A \in \{0,1\}^n$. The following conditions are equivalent:*

1. *$X^A F(X) = X^A F(A)$;*
2. *$F(X) + F(A) \leq 1 + X^A$;*
3. *$F(X) + F(Y) \leq 1 + X^A Y^A$ for every $Y \in B^n$.*

Proof. (1) implies (2): We have $X^A(F(X) + F(A)) = X^A F(X) + X^A F(A) = 0$.

(2) implies (3): By the Verification Theorem with respect to Y because for $Y = A$ condition (3) reduces to (2), while for $Y \in \{0,1\}^n - \{A\}$, condition (3) reduces to the identity $F(X) + F(Y) \leq 1$.

(3) implies (1): By condition (3), choosing $Y = A$ we have $F(X) + F(A) \leq 1 + X^A A^A = 1 + X^A$. Thus, $X^A F(X) + X^A F(A) \leq X^A + X^A = 0$, which implies $X^A F(X) + X^A F(A) = 0$. ∎

Definition 3.2 Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra. A function $F : B^n \longrightarrow B$ is *locally Boolean at point* $X \in B^n$ if

$$F(X) = \bigvee_{A\in\{0,1\}^n} F(A)X^A. \tag{2}$$

If F is locally Boolean at X, then we say that X is a *Boolean point* of F. □

It is easy to see that all points $C \in \{0,1\}^n$ are Boolean points for F.

Theorem 3.3 *Let $F : B^n \longrightarrow B$ be a function and let $X \in B^n$, where $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ is a Boolean algebra.*

$X \in B^n$ is a Boolean point of F if and only if any of the equivalent conditions of Lemma 3.1 are satisfied for every $A \in \{0,1\}^n$.

Proof. Suppose that X is a Boolean point of F. By multiplying both sides of the equality (2) by X^A we have $X^A F(X) = X^A F(A)$, which is the first condition of Lemma 3.1.

Conversely, suppose that $X^A F(X) = X^A F(A)$ is satisfied for every $A \in \{0,1\}^n$. Then, by taking the join of these equalities and using Part 1 of Lemma 1.1 we have:

$$F(X) = \bigvee_{A\in\{0,1\}^n} X^A F(X) = \bigvee_{A\in\{0,1\}^n} X^A F(A),$$

which shows that X is a Boolean point of F. ∎

Note that a point $X \in B^n$ is isolated in the graph Γ_F if and only if there is no edge (X, Y) for any $Y \in B^n$, or, equivalently, if

$$F(X) + F(Y) \leq (X + Y)^{\vee}.$$

for every $Y \in B^n - \{X\}$. Therefore, the graph Γ_F of any Boolean function is totally disconnected.

Lemma 3.4 *The following conditions are equivalent:*

1. $F(X) + F(Y) \leq (X+Y)^{\vee}$ *for every* $Y \in B^n$*;*
2. $F(X) + F(Y) \leq 1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A$ *for every* $Y \in B^n$*;*
3. $F(X) + F(Y) \leq 1 + X^A Y^A$ *for every* $Y \in B^n$ *and every* $A \in \{0,1\}^n$.

Proof. The equivalence of the first two conditions follows immediately from Part 2 of Lemma 1.1. Using the DeMorgan Law

$$1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A = \bigwedge_{A\in\{0,1\}^n} \left(1 + X^A Y^A\right),$$

we obtain the equivalence of the last two conditions. ∎

Theorem 3.5 *Every isolated point of a function F is a Boolean point of the function.*

Proof. Suppose that $X \in B^n$ is an isolated point of F. Then, by Lemma 3.4 we have $F(X) + F(Y) \leq 1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A$ for every $Y \in B^n$. In particular, taking $Y = A \in \{0,1\}^n$ we have

$$F(X) + F(A) \leq 1 + X^A$$

for every $A \in \{0,1\}^n$, which is one of the equivalent conditions of Lemma 3.1. ∎

If B is a finite Boolean algebra that has k atoms, then $|B^n| = 2^{kn}$. Thus, it is easier to test whether $X \in B^n$ is a Boolean point for a function F than to test whether X is an isolated point in Γ_F (2^k vs. 2^{kn}) tests), especially if B has a large number of atoms. In computing the graph Γ_F, we begin by identifying the Boolean points and, then, determine which of these are isolated points.

Example 3.6 Consider the 4-element Boolean algebra $\mathcal{B} = (\{0,1,c,c'\}, \vee, \cdot, ', 0, 1)$ and a function $F : B \longrightarrow B$. We saw that $0, 1$ are Boolean points. We claim 0 is an isolated point in the graph Γ_F if and only if $F(0) = cF(c') + c'F(c)$; similarly, 1 is an isolated point in Γ_F if $F(1) = cF(c) + c'F(c')$.

An element a is an isolated point in Γ_F if and only if the all of the following conditions are satisfied:

$$\begin{aligned} F(a) + F(0) &\leq a, \\ F(a) + F(1) &\leq a + 1, \\ F(a) + F(c) &\leq a + c, \\ F(a) + F(c') &\leq a + c'. \end{aligned}$$

It is clear that $a = 0$ trivially satisfies the first two conditions. Thus, 0 is an isolated point if and only if $F(0) + F(c) \leq c$ and $F(0) + F(c') \leq c'$. These conditions imply $F(0)c' = F(c)c'$ and $F(0)c = F(c')c$, respectively. Adding the last two equalities yields $F(0) = cF(c') + c'F(c)$.

Conversely, if $F(0) = cF(c') + c'F(c)$, then $F(0) + F(c) = cF(c') + cF(c) \leq c$ and $F(0) + F(c') = c'F(c) + c'F(c') \leq c'$, which shows that 0 is an isolated point. The argument for 1 is similar.

It is not difficult to see that c is an isolated point if and only if $F(c) = cF(1) + c'F(0)$, hence c' is such a point if $F(c') = cF(0) + c'F(1)$. Further, if both c and c' are isolated, then so are 0 and 1.

Consider, for example, the function F defined by $F(0) = 0$, $F(c) = c'$, $F(c') = F(1) = 1$. None of the vertices of Γ_F is isolated. Indeed, the graph of this function is shown in Figure 1.

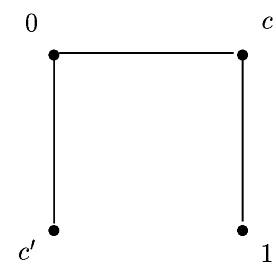

Figure 1. The Graph Γ_F

Since the graph is not the totally disconnected graph, F is not a Boolean function; however, the chromatic number of this graph is 2, because the sets $\mathcal{C}_0 = \{0,1\}$ and $\mathcal{C}_1 = \{c,c'\}$ are maximal independent sets. The Boolean interpolating functions are:

$$\begin{aligned} f_{\mathcal{C}_0}(x) &= F(0)(x^0 0^0 \vee x^1 0^1) \vee F(1)(x^0 1^0 \vee x^1 1^1) = x, \\ f_{\mathcal{C}_1}(x) &= F(c)(x^0 c^0 \vee x^1 c^1) \vee F(c')(x^0 (c')^0 \vee x^1 (c')^1) \\ &= c'(x'c' \vee xc) \vee 1(x'c \vee xc') \\ &= x'c' \vee x'c \vee xc' = x' \vee xc' = x' \vee c'. \end{aligned}$$

Note that for both $\mathcal{C}_0 = \{0,1\}$ and $\mathcal{C}_1 = \{c,c'\}$ the interpolating functions are unique. ∎

Example 3.7 Consider the binary function $F : B^2 \longrightarrow B$, where B is the four-element Boolean algebra introduced in Example 3.6 and F is the binary function specified by the

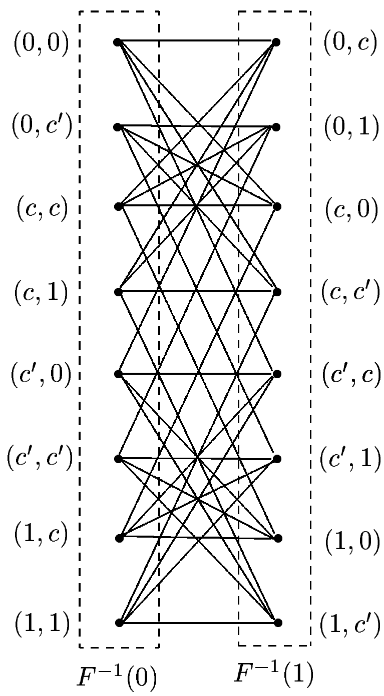

Figure 2. Γ_F **for the function** $F : B^2 \longrightarrow B$ **of Example 3.7**

following table:

$F(x,y)$	0	c	c'	1
0	0	1	0	1
c	1	0	1	0
c'	0	1	0	1
1	1	0	1	0

The graph Γ_F is shown in Figure 2. The Boolean functions that correspond to the maximal independent sets $\mathcal{C}_0 = F^{-1}(0)$ and $\mathcal{C}_1 = F^{-1}(1)$ are $f_{\mathcal{C}_0}(X) = 0$ and

$$\begin{aligned} f_{\mathcal{C}_1}(X) &= \bigvee_{F(Y)=1} \bigvee_{A \in \{0,1\}^n} X^A Y^A \\ &= \bigvee_{A \in \{0,1\}^n} \bigvee_{F(Y)=1} X^A Y^A \\ &= \bigvee_{A \in \{0,1\}^n} X^A \bigvee_{F(Y)=1} Y^A. \end{aligned}$$

A direct verification shows that

$$\begin{aligned} \bigvee_{F(Y)=1} Y^{(0,0)} &= c' \\ \bigvee_{F(Y)=1} Y^{(0,1)} &= 1 \\ \bigvee_{F(Y)=1} Y^{(1,0)} &= 1 \\ \bigvee_{F(Y)=1} Y^{(1,1)} &= c'. \end{aligned}$$

Thus,

$$f_{\mathcal{C}_1}(X) = X^{(0,0)}c' \vee X^{(0,1)} \vee X^{(1,0)} \vee X^{(1,1)}c'.$$

∎

4 Conclusions and Open Problems

The graph Γ_F of an arbitrary function $F : B^n \longrightarrow B$ over a Boolean algebra defined in this paper is a tool for determining Boolean interpolations using the independent sets in such graphs. An algorithm that finds maximal independent sets using Boolean techniques is given in [7].

If K is a maximal independent set and $X \in B^n$ denote by b_X^K a Boolean variable, where

$$b_X^K = \begin{cases} 1 & \text{if } X \in K \\ 0 & \text{otherwise} \end{cases}$$

Then, the K is a maximal independent set in Γ_F if the family $\{b_X^K\}$ satisfies the system of Boolean equations

$$b_X^K = \prod \{ \left(b_Y^K\right)' \mid F(X) + F(Y) \not\leq (X+Y)^{\vee} \}$$

for $X \in B^n$. Such systems can be solved using a branching technique, as discussed in [7]. We are now developing a specialized algorithm that begins with the specification of the non-Boolean function in a tabular form and generates the maximal independent sets of the graph Γ_F.

Graphs of special classes of functions such as generalized Boolean functions (see [13, 14, 15]), or chain-valued functions, etc. should also be investigated.

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