On the Ranges of Algebraic Functions in Lattices

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1 Introduction

The development of logical formalisms is paralleled by the development of their algebraic counterparts and the interplay between logic and algebra often plays an inspiring role for both fields. Notable examples of this interaction are the theory of Łukasiewicz-Moisil algebras that was born as an algebraic analogue of the many-valued logics introduced by Łukasiewicz, Epsteins's latticial treatment of Post algebras [Eps60], or Cignoli's study of the connection between Post algebras and Łukasiewicz-Moisil algebras [Cig70]. Algebraic functions in lattices (in Grätzer's sense [Grä79]), which we investigate here, are useful in specifying the semantics of connective symbols in various types of logics.

The purpose of this paper is to study ranges of algebraic functions in lattices and in algebras that are obtained by extending the standard lattice signature by several unary operations.

More specifically, we investigate ranges of ternary Łukasiewicz-Moisil algebras, where we give a characterization of algebraic functions whose ranges are intervals. Among other results we retrieve a canonical form of functions over three-element ternary Łukasiewicz-Moisil algebras, a result due to Gr. C. Moisil, one of the founders of switching theory [Moi57].

Further, we prove that in Artinian or Noetherian lattices the requirement that every algebraic function has an interval as its range implies the distributivity of the lattice.

A lattice $\mathcal{L} = (L, \vee, \cdot)$ is Artinian (Noetherian) if every ascending chain $z_0 \leq z_1 \leq \cdots$ (descending chain $z_0 \geq z_1 \geq \cdots$) is eventually stationary, that is, there is some natural number p such that $z_m = z_p$ for all $m > p$.

A lattice is *distributive* if it satisfies the distributive law, that is, if $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ for every $a, b, c \in L$.

A lattice is *modular* if it satisfies the modular law, that is, $a \geq c$ implies $a \cdot (b \vee c) = (a \cdot b) \vee c$ for every $a, b, c \in L$.

Definition 1.1 The *lattice functions* defined on a lattice $(L, \vee, \cdot)$ are:

1. the *constant functions* of the form f_a, given by $f_a(x_1, \ldots, x_n) = a$;

2. the *projection function* p_i^n, given by $p_i^n(x_1, \dots, x_n) = x_i$;

3. if f, g are lattice functions, the functions $f \vee g$ and $f \cdot g$ defined by:

$$\begin{aligned}(f \vee g)(x_1, \dots, x_n) &= f(x_1, \dots, x_n) \vee g(x_1, \dots, x_n)\\ (f \cdot g)(x_1, \dots, x_n) &= f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)\end{aligned}$$

are lattice functions. $\square$

The *simple lattice functions* have a similar inductive definition that makes use of Parts (2) and (3) of the previous definition. All lattice functions are isotonic (see, for example [Rud01], Proposition 3.3.1).

An *interval* on a lattice $\mathcal{L} = (L, \vee, \cdot)$ is a set of the form:

$$[a, b] = \{x \in L \mid a \le x \le b\},$$

where $a, b \in L$. Clearly an interval $[a, b]$ is not empty if and only if $a \le b$. We denote by $\mathsf{INT}(L)$ the collection of all intervals of the lattice $\mathcal{L} = (L, \vee, \cdot)$.

If $\mathcal{L} = (L, \vee, \cdot, 0, 1)$ is a bounded distributive lattice, $n \in \mathbb{N}$ and $\langle n \rangle = \{1, \dots, n\}$, then every lattice function $f : L^n \longrightarrow L$ can be represented in the form:

$$f(x_1, \dots, x_n) = \bigvee_{S \subseteq \langle n \rangle} (f(\delta_{S1}, \dots, \delta_{Sn}) \cdot x_S) \tag{1}$$

for all $x_1, \dots, x_n \in L$ (see [Rud01], p.48, Lemma 3.1).

Here $x_S = x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_\ell}$, where $S = \{i_1, i_2 \dots, i_\ell\}$, and

$$\delta_{Sh} = \begin{cases} 1 & \text{if } h \in S, \\ 0 & \text{otherwise.} \end{cases}$$

A *De Morgan algebra* is an algebra $\mathcal{D} = (L, \vee, \cdot, N, 0, 1)$ such that $(L, \vee, \cdot, 0, 1)$ is a bounded distributive lattice and N is an involutive dual endomorphism of the underlying bounded lattice; the dual endomorphism N is the *negation* of $\mathcal{D}$.

A *three-valued Łukasiewicz-Moisil algebra* (LM3) is an algebra $(L, \vee, \cdot, N, \phi, 0, 1)$, where $(L, \vee, \cdot, N, 0, 1)$ is a De Morgan algebra and $\phi : L \longrightarrow L$ is an idempotent endomorphism of the underlying lattice, that is, an endomorphism of $(L, \vee, \cdot, 0, 1)$ such that $\phi(\phi(x)) = \phi(x)$ for every $x \in L$. It is assumed that ϕ has the following properties:

$$\phi(x) \cdot N(\phi(x)) = 0; \tag{2}$$

$$\phi(N(\phi(x))) = N(\phi(x)); \tag{3}$$

$$N(\phi(N(x))) \le \phi(x); \tag{4}$$

$$\phi(x) = \phi(y) \text{ and } \phi(N(x)) = \phi(N(y)) \text{ imply } x = y. \tag{5}$$

Alternatively, ϕ is denoted by ϕ_2; the dual endomorphism ϕ_1 is defined by $\phi_1(x) = \phi_2(N(x))$ for $x \in L$. Using this notation, Property (5) can be written as $\phi_1(x) = \phi_1(y)$ and $\phi_2(x) = \phi_2(y)$ imply $x = y$ and is known as *determination principle for* ϕ_1 *and* ϕ_2.

LM3-algebraic functions are defined by extending the definition of lattice functions (Definition 1.1):

Definition 1.2 The *LM3-algebraic functions* defined on a LM3-algebra $(L, \vee, \cdot, N, \phi, 0, 1)$ are:

1. the *constant functions* f_a, where $a \in L$;
2. the *projection functions* p_i^n;
3. for every LM3-algebraic functions f, g, the functions $f \vee g$ and $f \cdot g$;
4. for every LM3-algebraic function f, the functions Nf and ϕf.

□

The axioms of the three-valued Łukasiewicz-Moisil algebra imply:

$$x \cdot N(x) = \phi(x) \cdot N(x) \tag{6}$$
$$\phi(x) \vee N(x) = 1 \tag{7}$$
$$x \leq \phi(x), \tag{8}$$

as it is shown in [BFGR91], pp. 131-133.

Let $C(L)$ be the set of complemented elements of a three-valued Łukasiewicz-Moisil algebra. It can be shown that $C(L) = \phi(L)$ and $(L, \vee, \cdot, N, \phi, 0, 1)$ is a Boolean algebra in which N plays the role of the complement.

Lemma 1.3 *Let $(L, \vee, \cdot, N, \phi, 0, 1)$ be a three-valued Łukasiewicz-Moisil algebra. We have:*

$$\phi(x) \vee \phi(N(x)) = 1 \tag{9}$$
$$N(\phi(x)) \leq \phi(N(x)) \tag{10}$$
$$N(\phi(N(x))) \leq x \tag{11}$$
$$x \cdot N(\phi(x)) = 0, \tag{12}$$

for every $x \in L$.

Proof. The first part of the lemma follows from (7) and (8).

Since N is involutive we have $N(N(\phi(x)) = \phi(x)$, so, by Equality (9), we have $N(N(\phi(x)) \vee \phi(N(x)) = 1$. This yields $N(\phi(x)) \leq \phi(N(x))$, that is, Equality (10).

Replacing x by $N(x)$ in Inequality (8) gives $N(x) \leq \phi(N(x))$. This implies $N(\phi(N(x)) \leq N(N(x)) = x$, which is Inequality (11).

Finally, the Equality (2) combined with the Inequality (8) gives:

$$x \cdot N(\phi(x)) \leq \phi(x) \cdot N(\phi(x)) = 0,$$

which implies

$$x \cdot N(\phi(x)) = 0.$$

■

2 The Algebraic Functions of ternary Łukasiewicz-Moisil Algebras

To study the properties of the algebraic functions of LM3 we investigate the submonoid of the monoid of transformations of the set L generated by I, N, and ϕ, where I is the identity mapping of the set L.

Theorem 2.1 *Let $(L, \vee, \cdot, N, \phi, 0, 1)$ be an LM3 algebra. The submonoid $\mathcal{M}(I, N, \phi)$ of the monoid of transformations of the set L generated I, N, and ϕ consists of $I, N, \phi, N\phi, \phi N, N\phi N$.*

Proof. We define inductively the sets of transformations of L, $\mathcal{F}_k$ for $k \geq 1$ as:

$$\begin{aligned} \mathcal{F}_1 &= \{I, N, \phi\} \\ \mathcal{F}_{k+1} &= \{fg \mid f \in \mathcal{F}_k \text{ and } g \in \mathcal{F}_1\} \\ &\quad \cup \{gf \mid f \in \mathcal{F}_k \text{ and } g \in \mathcal{F}_1\}. \end{aligned}$$

Since $I \in \mathcal{F}_k$ it is clear that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for $k \geq 1$. Observe that

$$\mathcal{F}_2 = \{I, N, \phi, N\phi, \phi N\},$$

because $\phi^2 = \phi$ and $N^2 = I$.

For $\mathcal{F}_3$ we have:

$$\mathcal{F}_3 = \{I, N, \phi, N\phi, \phi N, N\phi N\},$$

because $\phi N \phi = N\phi$ by Axiom (3). Finally, $\mathcal{F}_4 = \mathcal{F}_3$ because $\phi N \phi N = N \phi N$ and $N\phi N \phi = \phi$. Thus, $\mathcal{F}_k = \mathcal{F}_3$ for $k \geq 3$ and the submonoid generated by I, N, ϕ equals $\mathcal{F}_3$. ∎

Theorem 2.2 *Let $(L, \vee, \cdot, N, \phi, 0, 1)$ be an LM3 algebra. For every LM3-algebraic function $f : L^n \longrightarrow L$ there exists a family of elements $\{a_0\} \cup \{a_i \mid i \in I\} \subseteq L$ such that*

$$f(x_1, \ldots, x_n) = a_0 \vee \bigvee_{i \in I} a_i \cdot g_i(x_1, \ldots, x_n),$$

for $(x_1, \ldots, x_n) \in L^n$, where each g_i is a conjunction of the form

$$g_i(x_1, \ldots, x_n) = \prod_{\ell=1}^{q_i} t_i^\ell(x_{i_\ell}),$$

such that $t_i^1, \ldots, t_i^{q_i} \in \mathcal{M}(I, N, \phi)$ and $\{x_{i_1}, \ldots, x_{i_{q_i}}\} \subseteq \{x_1, \ldots, x_n\}$.

Proof. The argument is by structural induction on the definition of LM3-algebraic functions.

The basis case has two subcases. In the first subcase let f be a constant function whose value is a_0. Then,

$$f(x_1, \ldots, x_n) = a_0,$$

for $(x_1,\ldots,x_n) \in L_n$.

In the second case let f be a projection, say $f(x_1,\ldots,x_n) = x_k$ for $x_1,\ldots,x_n \in L$. The function f can be written as

$$f(x_1,\ldots,x_n) = 1 \cdot I(x_k),$$

for $(x_1,\ldots,x_n) \in L_n$.

Suppose that the statement holds for the LM3-algebraic functions $f, h : L^n \longrightarrow L$, that is,

$$\begin{aligned} f(x_1,\ldots,x_n) &= a_0 \vee \bigvee_{i\in I} a_i \cdot g_i(x_1,\ldots,x_n) \\ k(x_1,\ldots,x_n) &= b_0 \vee \bigvee_{i\in I} b_i \cdot h_i(x_1,\ldots,x_n) \end{aligned}$$

for $(x_1,\ldots,x_n) \in L_n$.

It is immediate that $f \cdot k$ and $f \vee k$ can be written in a similar way by the distributivity of $\vee$ over $\cdot$. For ϕf we can write

$$\phi(f(x_1,\ldots,x_n)) = \phi(a_0) \vee \bigvee_{i\in I} \phi(a_i) \cdot \phi(g_i(x_1,\ldots,x_n)),$$

because ϕ is an endomorphism. If $g(x_1,\ldots,x_n) = \prod_{\ell=1}^{q_i} t_i^\ell(x_{i_\ell})$, then

$$\phi(g(x_1,\ldots,x_n)) = \prod_{\ell=1}^{q_i} \phi t_i^\ell(x_{i_\ell}),$$

and every function ϕt_i^ℓ belongs to $\mathcal{M}(I,\phi,N)$.

Finally, since N is a dual endomorphism, we have

$$\begin{aligned} N(f(x_1,\ldots,x_n)) &= N\left(a_0 \vee \bigvee_{i\in I} a_i \cdot g_i(x_1,\ldots,x_n)\right) \\ &= N(a_0) \cdot \prod_{i\in I}(N(a_i) \vee N(g_i(x_1,\ldots,x_n)) \\ &= N(a_0) \cdot \prod_{i\in I}(N(a_i) \vee N(\prod_{\ell=1}^{q_i} t_i^\ell(x_{i_\ell})) \\ &= N(a_0) \cdot \prod_{i\in I}(N(a_i) \vee \bigvee_{\ell=1}^{q_i} N t_i^\ell(x_{i_\ell})). \end{aligned}$$

Applying distributivity to the previous equality leads to the desired conclusion for Nf. ∎

Lemma 2.3 *The set $\mathcal{C}^x$ of conjunctions of the form $\prod_{i=1}^{\ell} f_i(x)$, where $f_i \in \mathcal{M}(I,N,\phi)$ for $1 \le i \le \ell$ and $\ell \ge 1$ consists of:*

$$0, x, N(x), \phi(x), N(\phi(x)), \phi(N(x)), N(\phi(N(x))), \phi(x)\cdot N(x), \phi(x)\cdot\phi(N(x)).$$

Proof. Define inductively the chain of sets of conjunctions $\mathcal{C}_k$ (for $k \geq 1$) as:

$$\begin{aligned} \mathcal{C}_1^x &= \{f(x) \mid f \in \mathcal{M}(I, N, \phi)\} \\ \mathcal{C}_{k+1}^x &= \mathcal{C}_k^x \cup \mathcal{C}_1^x \cdot \mathcal{C}_k^x. \end{aligned}$$

Clearly, $C^x = \bigcup\{\mathcal{C}_k^x \mid k \geq 1\}$. By Theorem 2.1 the set $\mathcal{C}_1^x$ is:

$$\mathcal{C}_1^x = \{x, N(x), \phi(x), N(\phi(x)), \phi(N(x)), N(\phi(N(x)))\}.$$

To compute the set $\mathcal{C}_2^x$ we need to consider the equalities:

$$x \cdot N(x) = \phi(x) \cdot N(x) \qquad (Equality\ (6)),$$

$$x \cdot \phi(x) = x, \tag{13}$$

$$x \cdot N(\phi(x)) = 0, \qquad (Equality\ (12)),$$

$$x \cdot \phi(N(x)) = \phi(x) \cdot N(x), \tag{14}$$

$$x \cdot N(\phi(N(x))) = N(\phi(N(x))). \tag{15}$$

Equality (13) follows from Inequality (8). Substituting $N(x)$ for x in Equality (6) gives $N(x) \cdot x = \phi(N(x)) \cdot x$; but $N(x) \cdot x = \phi(x) \cdot N(x)$, again by Equality (6). Thus, we have shown Equality (14).

Inequality (11) implies Equality (15).

Let us compute now the conjunctions involving $N(x)$. Applying N to Inequality (8) and taking into account that N is a dual endomorphism yields $N(\phi(x)) \leq N(x)$, or, equivalently, $N(x) \cdot N(\phi(x)) = N(\phi(x))$.

The same Inequality (8) implies $N(x) \cdot \phi(N(x)) = N(x)$.

Inequality (8) and Equality (2) give:

$$N(x) \cdot N(\phi(N(x))) \leq \phi(N(x)) \cdot N(\phi(N(x))) = 0.$$

Thus, we have shown that:

$$N(x) \cdot N(\phi(x)) = N(\phi(x)) \tag{16}$$

$$N(x) \cdot \phi(N(x)) = N(x) \tag{17}$$

$$N(x) \cdot N(\phi(N(x))) = 0. \tag{18}$$

To compute the conjunctions involving $\phi(x)$ we can use Axiom (2):

$$\phi(x) \cdot N(\phi(x)) = 0.$$

Also, that by (4) we have $\phi(x) \cdot N(\phi(N(x))) = N(\phi(N(x)))$.

Conjunctions involving $N(\phi(x))$ are used in the following equalities:

$$\begin{aligned} N(\phi(x)) \cdot \phi(N(x)) &= N(\phi(x) \\ &\quad \text{by Inequality (10)} \end{aligned} \tag{19}$$

$$\begin{aligned} N(\phi(x)) \cdot N(\phi(N(x))) &= N(\phi(x) \vee \phi(N(x))) \\ &= N(1) \\ &\quad \text{by Inequality (9)} \\ &= 0 \end{aligned} \tag{20}$$

Finally, we observe that:

$$\phi(N(x)) \cdot N(\phi(N(x))) = 0,$$

by Equality (2). Thus, $\mathcal{C}_2^x$ is the set:

$$\begin{aligned}\mathcal{C}_2^x &= \{0, x, N(x), \phi(x), N(\phi(x)), \phi(N(x)), N(\phi(N(x))),\\ &\quad \phi(x) \cdot N(x), \phi(x) \cdot \phi(N(x))\}.\end{aligned}$$

The equalities:

$$\begin{array}{l}
x \cdot \phi(x) \cdot N(x) = x \cdot N(x) = \phi(x) \cdot N(x), \text{ by Equality (6),}\\
x \cdot \phi(x) \cdot \phi(N(x)) = x \cdot \phi(N(x)) = \phi(x) \cdot N(x), \text{ by Equality (14)}\\
N(x) \cdot \phi(x) \cdot N(x) = \phi(x) \cdot N(x),\\
N(x) \cdot \phi(x) \cdot \phi(N(x)) = N(x) \cdot \phi(x), \text{ by Inequality (8),}\\
\phi(x) \cdot \phi(x) \cdot N(x) = \phi(x) \cdot N(x),\\
\phi(x) \cdot \phi(x) \cdot \phi(N(x)) = \phi(x) \cdot \phi(N(x))\\
N(\phi(x)) \cdot \phi(x) \cdot N(x) = 0, \text{ by Axiom (2)}\\
N(\phi(x)) \cdot \phi(x) \cdot \phi(N(x)) = 0, \text{ by Axiom (2)}\\
\phi(N(x)) \cdot \phi(x) \cdot N(x) = \phi(x) \cdot N(x), \text{ by Inequality (8)}\\
\phi(N(x)) \cdot \phi(x) \cdot \phi(N(x)) = \phi(x) \cdot \phi(N(x)),\\
N(\phi(N(x))) \cdot \phi(x) \cdot N(x) = 0, \text{ by Equality (12)}\\
N(\phi(N(x))) \cdot \phi(x) \cdot \phi(N(x)) = 0, \text{ by Axiom (2),}
\end{array}$$

allow us to conclude that $\mathcal{C}_3^x \subseteq \mathcal{C}_2^x$, and therefore, that $\mathcal{C}_3^x = \mathcal{C}_2^x$.

Assuming that $\mathcal{C}_k^x = \mathcal{C}_2^x$, it follows that $\mathcal{C}_{k+1}^x = \mathcal{C}_2^x \cup \mathcal{C}_1^x \cdot \mathcal{C}_2^x = \mathcal{C}_3^x = \mathcal{C}_2^x$, which is precisely the equality we intended to prove. ∎

Theorem 2.4 *Let $(L, \vee, \cdot, N, \phi, 0, 1)$ be an LM3 algebra. For every algebraic function $f : L \longrightarrow L$ there exist $a_1, \dots, a_9 \in L$ such that:*

$$\begin{aligned}f(x) &= a_1 \cdot \phi(x) \cdot N(x) \vee a_2 \cdot \phi(x) \cdot \phi(N(x)) \vee a_3 \cdot N(\phi(N(x)))\\ &\vee a_4 \cdot \phi(N(x)) \vee a_5 \cdot N(\phi(x)) \vee a_6 \cdot \phi(x) \vee a_7 \cdot N(x) \vee a_8 \cdot x \vee a_9,\end{aligned} \tag{21}$$

for every $x \in L$.

Proof. By Theorem 2.2 an LM3-algebraic function $f : L \longrightarrow L$ can be written as:

$$f(x) = a_0 \vee \bigvee_{i \in I} a_i \cdot g_i(x),$$

where each g_i is a conjunction:

$$g_i(x) = \prod_{\ell=1}^{q_i} t_i^\ell(x),$$

such that $t_i^1, \dots, t_i^{q_i} \in \mathcal{M}(I, N, \phi)$. In view of Lemma 2.3 we obtain the desired Equality (21). ∎

We retrieve now a result obtained in [Moi57], which highlights the significance of Theorem 2.4. Moisil obtained the formula contained by the next corollary starting from the known completeness of L_3.

Corollary 2.5 *Let $L_3 = \{0, \frac{1}{2}, 1\}$ be the three-element LM algebra. Every function $f : L_3 \longrightarrow L_3$ can be written as:*

$$f(x) = f\left(\frac{1}{2}\right) \cdot \phi(x) \cdot \phi(N(x)) \vee f(1) \cdot N(\phi(N(x))) \vee f(0) \cdot N(\phi(x)), \tag{22}$$

for $x \in L_3$.

Proof. Suppose first that f is an algebraic function. Then, Theorem 2.4 yields:

$$\begin{aligned} f(0) &= a_4 \vee a_5 \vee a_7 \vee a_9, \\ f(1) &= a_3 \vee a_6 \vee a_8 \vee a_9, \\ f\left(\frac{1}{2}\right) &= (a_1 \vee a_7 \vee a_8) \cdot \frac{1}{2} \vee a_2 \vee a_4 \vee a_6 \vee a_9. \end{aligned}$$

In particular, the algebraic functions for which

$$a_1 = a_4 = a_6 = a_7 = a_8 = a_9 = 0$$

satisfy $f(0) = a_5$, $f(1) = a_3$, $f(\frac{1}{2}) = a_2$ and, since a_5, a_3, a_2 can be chosen arbitrarily, it follows that these functions comprise in fact all functions $f : L_3 \longrightarrow L_3$. By substituting these values in Equality (21) we obtain the desired Equality (22). ∎

Note that the range of f fails to be an interval if and only if $\mathrm{Ran}(f) = \{0, 1\}$, that is, if and only if $\{a_5, a_3, a_2\} = \{0, 1\}$.

2.1 Equations in LM3

We now return to the study of LM3 algebras. To resolve the equation $f(x) = 0$, where f is an algebraic function of an LM3 algebra we use the determination principle by requiring that $\phi(f(x)) = 0$ and $\phi(N(f(x)) = 1$.

Clearly, $a_9 = 0$ is a necessary condition of consistency for $f(x) = 0$.

Let $b_i = \phi(a_i)$ and $c_i = \phi(N(a_i))$ for $1 \leq i \leq 9$. Note that $b_i \vee c_i = \phi(a_i) \vee \phi(N(a_i)) = 1$ by Equality (9). Further, let $X = \phi(x)$ and $Y = \phi(N(x))$. Note that both belong to the center of the algebra and, by Equality (9), we have $X \vee Y = 1$. The elements $N(X)$ and $N(Y)$ are denoted by X' and Y', respectively.

Lemma 2.6 *Let f be an algebraic function of an LM3 algebra. We have the equalities:*

$$\begin{aligned} \phi(f(x)) &= (b_1 \vee b_2) \cdot X \cdot Y \vee b_3 \cdot Y' \vee (b_4 \vee b_7) \cdot Y \\ &\quad \vee b_5 \cdot X' \vee (b_6 \vee b_8) \cdot X \vee b_9, \\ \phi(N(f(x))) &= (c_2 \vee X' \vee Y') \cdot (c_3 \cdot c_8 \vee Y) \\ &\quad \cdot (c_4 \vee Y') \cdot (c_5 \cdot c_7 \vee X) \cdot (c_6 \vee X') \cdot c_9. \end{aligned}$$

for every $x \in L_3$.

Proof. We have

$$\begin{aligned}
\phi(f(x)) &= b_1 \cdot \phi(x) \cdot \phi(N(x)) \vee b_2 \cdot \phi(x) \cdot \phi(N(x)) \vee b_3 \cdot N(\phi(N(x))) \\
&\quad \vee b_4 \cdot \phi(N(x)) \vee b_5 \cdot N(\phi(x)) \vee b_6 \cdot \phi(x) \vee b_7 \cdot \phi(N(x)) \\
&\quad \vee b_8 \cdot \phi(x) \vee b_9 \\
&= (b_1 \vee b_2) \cdot \phi(x) \cdot \phi(N(x)) \vee b_3 \cdot N(\phi(N(x))) \vee (b_4 \vee b_7) \cdot \phi(N(x)) \\
&\quad \vee b_5 \cdot N(\phi(x)) \vee (b_6 \vee b_8) \cdot \phi(x) \vee b_9,
\end{aligned}$$

which gives the first equality of the lemma.

Starting from the equality:

$$\begin{aligned}
N(f(x)) &= (N(a_1) \vee N(\phi(x)) \vee x) \cdot (N(a_2) \vee N(\phi(x)) \vee N(\phi(N(x)))) \\
&\quad \cdot (N(a_3) \vee \phi(N(x))) \cdot (N(a_4) \vee N(\phi(N(x)))) \\
&\quad \cdot (N(a_5) \vee \phi(x)) \cdot (N(a_6) \vee N(\phi(x))) \\
&\quad \cdot (N(a_7) \vee x) \cdot (N(a_8) \vee N(x)) \cdot N(a_9),
\end{aligned}$$

by applying ϕ to both sides one gets:

$$\begin{aligned}
\phi(N(f(x))) &= (c_1 \vee N(\phi(x)) \vee \phi(x)) \cdot (c_2 \vee N(\phi(x)) \vee N(\phi(N(x)))) \\
&\quad \cdot (c_3 \vee \phi(N(x))) \cdot (c_4 \vee N(\phi(N(x)))) \\
&\quad \cdot (c_5 \vee \phi(x)) \cdot (c_6 \vee N(\phi(x))) \\
&\quad \cdot (c_7 \vee \phi(x)) \cdot (c_8 \vee \phi(N(x))) \cdot c_9 \\
&= (c_2 \vee N(\phi(x)) \vee N(\phi(N(x)))) \cdot (c_3 \cdot c_8 \vee \phi(N(x))) \\
&\quad \cdot (c_4 \vee N(\phi(N(x)))) \cdot (c_5 \cdot c_7 \vee \phi(x)) \\
&\quad \cdot (c_6 \vee N(\phi(x))) \cdot c_9,
\end{aligned}$$

Replacing $\phi(x)$, $\phi(N(x))$ by X and Y, and $N(X)$, $N(Y)$ by X' and Y', respectively we obtain the second equality of the lemma. ∎

Thus, we have the system of equations in X, Y:

$$\begin{aligned}
&(b_1 \vee b_2) \cdot X \cdot Y \vee b_3 \cdot Y' \vee (b_4 \vee b_7) \cdot Y \vee b_5 \cdot X' \vee (b_6 \vee b_8) \cdot X \vee b_9 = 0 \\
&(c_2 \vee X' \vee Y') \cdot (c_3 \cdot c_8 \vee Y) \cdot (c_4 \vee Y') \cdot (c_5 \cdot c_7 \vee X) \cdot (c_6 \vee X') \cdot c_9 = 1 \\
&X \vee Y = 1,
\end{aligned}$$

where $X' = N(X)$, $Y' = N(Y)$, and $b_i \vee c_i = 1$ for $1 \leq i \leq n$.

Taking into account the consistency conditions $b_9 = 0$ and $c_9 = 1$, the system can be written as a single equation:

$$\begin{aligned}
&(b_1 \vee b_2) \cdot X \cdot Y \vee b_3 \cdot Y' \vee (b_4 \vee b_7) \cdot Y \\
&\vee b_5 \cdot X' \vee (b_6 \vee b_8) \cdot X \\
&\vee c_2' \cdot X \cdot Y \vee (c_3' \vee c_8') \cdot Y' \vee (c_4' \cdot Y) \vee (c_5' \vee c_7') \cdot X' \\
&\vee c_6' \cdot X \vee X' \cdot Y' = 0.
\end{aligned} \tag{23}$$

Note that $b_i \vee c_i' = b_i \vee b_i' c_i' = b_i$ for $1 \leq i \leq 9$.

Equation (23) can be written as:

$$(b_1 \vee b_2)XY \vee (b_3 \vee c_8')Y' \vee (b_4 \vee b_7)Y \vee (b_5 \vee c_7')X' \vee (b_6 \cup b_8)X \vee X'Y' = 0,$$

or, equivalently,

$$\begin{array}{l}(b_1 \vee b_2 \vee b_4 \vee b_7 \vee b_6 \vee b_8)XY \vee (b_3 \vee c_8' \vee b_6 \vee b_8)XY' \\ \vee(b_4 \vee b_7 \vee b_5 \vee c_7')X'Y \vee X'Y' = 0.\end{array}$$

This equality can be further simplified by absorbing c_i':

$$\begin{array}{l}(b_1 \vee b_2 \vee b_4 \vee b_7 \vee b_6 \vee b_8)XY \vee (b_3 \vee b_6 \vee b_8)XY' \\ \vee(b_4 \vee b_7 \vee b_5)X'Y \vee X'Y' = 0.\end{array}$$

The consistency condition for this equation is:

$$(b_1 \vee b_2 \vee b_4 \vee b_7 \vee b_6 \vee b_8) \cdot (b_3 \vee b_6 \vee b_8) \cdot (b_4 \vee b_7 \vee b_5) = 0.$$

If $B = b_6 \vee b_8$ and $C = b_4 \vee b_7$ the last condition can be written as

$$(b_1 \vee b_2 \vee C \vee B) \cdot (b_3 \vee B) \cdot (C \vee b_5) = 0,$$

which can be written as:

$$[B \vee b_3 \cdot (b_1 \vee b_2 \vee C)] \cdot (b_5 \vee C) = 0.$$

Applying distributivity we get:

$$b_5 \cdot B \vee B \cdot C \vee b_3 \cdot b_5 \cdot (b_1 \vee b_2) \vee b_3 \cdot C = 0.$$

This equality is equivalent to the conditions:

$$\begin{array}{rcl}(b_6 \vee v_8) \cdot (b_4 \vee b_5 \vee b_7) & = & 0, \\ b_3 \cdot (b_1 \cdot b_5 \vee b_2 \cdot b_5 \vee b_4 \vee b_7) & = & 0.\end{array}$$

We obtained the general equation (23) that allows us to extract the values of $\phi(x)$ and $\phi(N(x))$. However, obtaining the value of x is possible when specific properties of ϕ and N allow it. One such situation is the result of Theorem 2.9.

2.2 Finite LM Algebras

A result obtained by Gr. C. Moisil in [Moi41] and by R. Cignoli in [Cig69] shows that every finite algebra LMn is a direct product of subalgebras of $L_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. Since L_3 has only the subalgebras L_2 and L_3 it follows that every finite algebra in LM3 is a direct product of algebras L_2 and L_3.

Thus, every algebraic function on L is the product of algebraic functions on L_2 and L_3, so the range of a function is an interval if and only if the ranges of the component functions are intervals, or equivalently, if the range of every L_3 component is an interval. Thus, the problem of identifying the algebraic functions whose ranges are intervals is reduced to the case $L = L_3$, where we can take advantage of the fact that $C(L_3) = L_2$.

If we try to generalize this approach by imposing directly the condition $C(L) = L_2$ we could use the equivalent characterization of algebras that satisfy this condition (Theorem 1.6, p. 288 in [BFGR91]), namely L being a subalgebra of L_n, or L a chain. Thus, for the case of current interest, L must be a finite chain.

2.3 The Ranges of Algebraic Functions of LM3

To determine the range of an algebraic function f of LM3 we need to find those y for which the equation $f(x) = y$ is consistent. This amounts to the consistency of the equations $\phi(f(x)) = \phi(y)$ and $\phi(N(\phi(x))) = \phi(N(y))$, by the determination principle.

Let $Z = \phi(y)$ and $T = \phi(N(y))$. We have $Z \vee T = 1$. Thus, by Lemma 2.6, we have the system $\mathcal{S}$ of equations in X, Y:

$$\begin{array}{l}(b_1 \vee b_2) \cdot X \cdot Y \vee b_3 \cdot Y' \vee (b_4 \vee b_7) \cdot Y \vee b_5 \cdot X' \vee (b_6 \vee b_8) \cdot X \vee b_9 = Z \\ (c_2 \vee X' \vee Y') \cdot (c_3 \cdot c_8 \vee Y) \cdot (c_4 \vee Y') \cdot (c_5 \cdot c_7 \vee X) \cdot (c_6 \vee X') \cdot c_9 = T \\ X \vee Y = 1,\end{array}$$

where $X' = N(X)$, $Y' = N(Y)$, and $b_i \vee c_i = 1$ for $1 \le i \le n$.

Lemma 2.7 *In an arbitrary Boolean algebra the system $\mathcal{S}$ is consistent if and only if*

$$\begin{array}{l}(b_1' \cdot b_2' \cdot b_4' \cdot b_6' \cdot b_7' \cdot b_8' \cdot b_9' \vee c_2' \vee c_4' \vee c_6' \vee c_9') \\ \cdot(b_3' \cdot b_6' \cdot b_8' \cdot b_9' \vee c_3' \vee c_6' \vee c_8' \vee c_9') \\ \cdot(b_4' \cdot b_5' \cdot b_7' \cdot b_9' \vee c_4' \vee c_5' \vee c_7' \vee c_9') \cdot Z \cdot T \\ \vee c_2 \cdot c_3 \cdot c_4 \cdot c_5 \cdot c_6 \cdot c_7 \cdot c_8 \cdot c_9 \cdot Z \cdot T' \vee \\ (b_1 \vee b_2 \vee b_4 \vee b_6 \vee b_7 \vee b_8 \vee b_9) \cdot (b_3 \vee b_6 \vee b_8 \vee b_9) \\ \cdot(b_4 \vee b_5 \vee b_7 \vee b_9) \cdot Z'T \vee Z'T' = 0.\end{array} \tag{24}$$

Proof. The system $\mathcal{S}$ can be written as a single equation:

$$\begin{array}{l}[(b_1 \vee b_2) \cdot X \cdot Y \vee b_3 \cdot Y' \vee (b_4 \vee b_7) \cdot Y \vee b_5 \cdot X' \vee (b_6 \vee b_8) \cdot X \vee b_9] \cdot Z' \vee \\ (b_1' \cdot b_2' \vee X' \vee Y') \cdot (b_3' \vee Y) \cdot (b_4' \cdot b_7' \vee Y') \cdot (b_5' \vee X) \cdot (b_6' \cdot b_8' \vee X') \cdot b_9' \cdot Z \\ \vee (c_2 \vee X' \vee Y') \cdot (c_3 \cdot c_8 \vee Y) \cdot (c_4 \vee Y') \cdot (c_5 \cdot c_7 \vee X) \cdot (c_6 \vee X') \cdot c_9 T' \\ \vee [c_2' \cdot X \cdot Y \vee (c_3' \vee c_8') \cdot Y' \vee c_4' \cdot Y \vee (c_5' \vee c_7') \cdot X' \vee c_6' \cdot X \vee c_9'] \cdot T \\ \vee X' \cdot Y' = 0.\end{array}$$

The consistency condition for a Boolean equation $F(X, Y) = 0$ is $F(1, 1) \cdot F(1, 0) \cdot F(0, 1) \cdot F(0, 0) = 0$. In the case of this equation we obtain:

$$\begin{array}{l}[(b_1 \vee b_2 \vee b_4 \vee b_7 \vee b_6 \vee b_8 \vee b_9) \cdot Z' \vee b_1' \cdot b_2' \cdot b_4' \cdot b_7' \cdot b_6' \cdot b_8' \cdot b_9' \cdot Z \\ \qquad \vee c_2 \cdot c_4 \cdot c_6 \cdot c_9 \cdot T' \vee (c_2' \vee c_4' \vee c_6' \vee c_9') \cdot T] \\ \cdot[(b_3 \vee b_6 \vee b_8 \vee b_9) \cdot Z' \vee b_3' \cdot b_6' \cdot b_8' \cdot b_9' \cdot Z \vee c_3 \cdot c_8 \cdot c_6 \cdot c_9 \cdot T' \\ \qquad \vee (c_3' \vee c_8' \vee c_6' \vee c_9') \cdot T] \\ \cdot[(b_4 \vee b_7 \vee b_5 \vee b_9) Z' \vee b_4' \cdot b_7' \cdot b_5' \cdot b_9' \cdot Z \vee c_4 \cdot c_5 \cdot c_7 \cdot c_9 \cdot T' \\ \qquad \vee (c_4' \vee c_5' \vee c_7' \vee c_9') \cdot T] \\ \vee Z' \cdot T' = 0.\end{array}$$

The coefficient of $Z \cdot T$ is:

$$(b_1' \cdot b_2' \cdot b_4' \cdot b_6' \cdot b_7' \cdot b_8' \cdot b_9' \vee c_2' \vee c_4' \vee c_6' \vee c_9') \cdot (b_3' \cdot b_6' \cdot b_8' \cdot b_9' \vee c_3' \vee c_6' \vee c_8' \vee c_9') \cdot (b_4' \cdot b_5' \cdot b_7' \cdot b_9' \vee c_4' \vee c_5' \vee c_7' \vee c_9').$$

The coefficient of $Z \cdot T'$ is:

$$(b_1' \cdot b_2' \cdot b_4' \cdot b_6' \cdot b_7' \cdot b_8' \cdot b_9' \vee c_2 \cdot c_4 \cdot c_6 \cdot c_9) \cdot (b_3' \cdot b_6' \cdot b_8' \cdot b_9' \vee c_3 \cdot c_6 \cdot c_8 \cdot c_9) \cdot (b_4' \cdot b_5' \cdot b_7' \cdot b_9' \vee c_4 \cdot c_5 \cdot c_7 \cdot c_9).$$

Since $b_i \vee c_i = 1$, we have $b'_i \cdot c'_i = 0$, hence $c'_i \leq b_i$, or, equivalently, $b_i \vee c'_i = b_i$. Therefore, the coefficient of $Z' \cdot T$ reduces to:

$$(b_1 \vee b_2 \vee b_4 \vee b_6 \vee b_7 \vee b_8 \vee b_9) \cdot (b_3 \vee b_6 \vee b_8 \vee b_9) \cdot (b_4 \vee b_5 \vee b_7 \vee b_9).$$

Furthermore, we have $b'_i \leq c_i$, hence $b'_i \cdot d \vee c_i = (b'_i \vee c_i) \cdot (d \vee c_i) = c_i \cdot (d \vee c_i) = c_i$, then $b'_i \cdot b'_j \cdot d \vee c_i \cdot c_j = (b'_i \cdot b'_j \cdot d \vee c_i) \cdot (b'_i \cdot b'_j \cdot d \vee c_j) = c_i \cdot c_j$, etc. This shows that the coefficient of $Z \cdot T'$ reduces to $c_2 \cdot c_3 \cdot c_4 \cdot c_5 \cdot c_6 \cdot c_7 \cdot c_8 \cdot c_9$. Therefore, the consistency condition reduces to Equation (24). ∎

Lemma 2.8 *Let L be an LM3 algebra, $f : L \longrightarrow L$ an algebraic function and let $y \in L$. Then, a necessary condition for y to be in $\mathrm{Ran}(f)$ is that $Z = \phi(y)$ and $T = \phi(N(y))$ satisfy the condition of Lemma 2.7.*

Proof. We have $y = f(x)$ for some x, hence $(X, Y) = (\phi(x), \phi(N(x))$ is a solution of the system $\mathcal{S}$. ∎

The condition in Lemma 2.8 is not sufficient because, although it implies the existence of a solution (X, Y) to the system $\mathcal{S}$, this solution is not necessarily of the form $(\phi(x), \phi(N(x))$. However, we can prove sufficiency in a particular case:

Theorem 2.9 *Let $L_3 = \{0, \frac{1}{2}, 1\}$ be the three-element LM algebra and let $f : L_3 \longrightarrow L_3$ be an arbitrary function, as described in Corollary 2.5. Then, a necessary and sufficient condition for an element $y \in L_3$ to be in $\mathrm{Ran}(f)$ is that $Z = \phi(y)$ and $T = \phi(N(y))$ satisfy the equality:*

(25) $(b'_2 \vee c'_2) \cdot (b'_3 \vee c'_3) \cdot (b'_5 \vee c'_5) \cdot Z \cdot T \vee c_2 \cdot c_3 \cdot c_5 \cdot Z \cdot T' \vee b_2 \cdot b_3 \cdot b_5 \cdot Z \cdot T' \vee Z' \cdot T' = 0.$

Proof. The arbitrary function $f : L_3 \longrightarrow L_3$ has the representation given in Corollary 2.5, which is the representation (21) given in Theorem 2.4 with $a_i = 0$ for $i \in \{1, 4, 6, 7, 8, 9\}$. These values for a_i imply $b_i = 0$ and $c_i = 1$ for the same values of i. In view of Lemma 2.8, $Z = \phi(y)$ and $T = \phi(N(y))$ should satisfy the condition of Lemma 2.7, which now reduces to Equality (25).

To prove sufficiency, suppose that $(Z, T) = (\phi(y), \phi(N(y))$ satisfies Equality (25) which is a specialization of the equality of Lemma 2.7 to the Boolean algebra $L_2 = \{0, 1\}$ and $b_i = 0, c_i = 1$ for $i \notin \{2, 3, 5\}$. It follows that the corresponding reduced system $\mathcal{S}$ (studied in Lemma 2.7) has a solution (X, Y). If $X = Y = 1$, then $X = \phi(\frac{1}{2})$ and $Y = \phi(\frac{1}{2})$; if $X = 1, Y = 0$, then $X = \phi(1), Y = \phi(N(1))$; if $X = 0, Y = 1$, then $X = \phi(0), Y = \phi(N(0))$. Thus, in all cases we have found an element $x \in L_3$ such that $\phi(f(x)) = \phi(y)$ and $\phi(N(f(x))) = \phi(N(y))$, hence $f(x) = y$. ∎

Theorem 2.10 *The range of a function $f : L_3 \longrightarrow L_3$, where $L_3 = \{0, \frac{1}{2}, 1\}$ is the three-valued LM algebra, is an interval if and only if*

$$(b_2 \vee c'_2) \cdot (b'_3 \cdot c'_5 \vee b'_5 \cdot c'_3) \vee b'_2 \cdot c'_3 \cdot c'_5 \vee c'_2 \cdot b'_3 \cdot b'_5 = 0.$$

Proof. The range $f(L_3)$ of an algebraic function fails to be an interval if and only if both $1 \in f(L_3)$ and $0 \in f(L_3)$, but $1/2 \notin f(L_3)$. In view of Lemma 2.7 this happens if and only if Equation (24) has the solutions $(Z = 1, T = 0)$ and $(Z = 0, T = 1)$, but fails to have the solution $(Z = 1, T = 1)$. This is equivalent to the conditions:

$$\begin{aligned} K_1 &= (b_2' \vee c_2') \cdot (b_3' \vee c_3') \cdot (b_5' \vee c_5') = 1 \\ K_2 &= c_2 \cdot c_3 \cdot c_5 = 0 \\ K_3 &= b_2 \cdot b_3 \cdot b_5 = 0 \end{aligned}$$

We conclude that $f(L_3)$ is an interval if $K_1 = 0$, or $K_2 = 1$, or $K_3 = 1$; equivalently, $f(L_3)$ is an interval if $K_1 \cdot K_2' \cdot K_3' = 0$, which amounts to

$$(b_2' \vee c_2') \cdot (b_3' \vee c_3') \cdot (b_5' \vee c_5') \cdot (c_2' \vee c_3' \vee c_5') \cdot (b_2' \vee b_3' \vee b_5') = 0.$$

Grouping conveniently the factors, this equality can be written in the form:

$$(b_2' \vee c_2') \cdot [c_3' \vee b_3' \cdot (c_2' \vee c_5')] \cdot [b_5' \vee c_5' \cdot (b_2' \vee b_3')] = 0,$$

or equivalently, taking into account that $b_i' \cdot c_i' = 0$.

$$(b_2' \vee c_2') \cdot (c_3' \cdot b_5' \vee c_3' \cdot c_5' \cdot b_2' \vee b_3' \cdot b_5' \cdot c_2' \vee b_3' \cdot c_5') = 0,$$

which is the desired equality because $(p \vee q) \cdot (p \cdot r \vee q \cdot s) = p \cdot r \vee q \cdot s$. ∎

3 A Characterization of Distributive Lattices

The sets of values of lattice functions on bounded distributive lattices are intervals of the lattice. This result belongs to R. L. Goodstein [Goo67] (cf. Corollary 3.5., page 55 of [Rud01]). The main result of this section is to show that for an Artinian or Noetherian lattice the inverse is also true. To this end, we shall prove the following statement:

Theorem 3.1 *Let $(L, \vee, \cdot)$ be an Artinian or Noetherian lattice. If for every lattice function $f : L \longrightarrow L$ the set of values belongs to $\mathsf{INT}(L)$, then L is a bounded distributive lattice.*

Proof. To prove that L is bounded consider the projection p_1^1, whose set of values is L. Since L is an interval $[z, u]$, z is the least and u is the largest element of L, so L is bounded lattice. As usual, we denote the least and greatest elements of L by $0, 1$, respectively.

Let a, b be two arbitrary elements of L. Suppose that L is Artinian and let $\phi, \psi : L \longrightarrow L$ be the lattice functions

$$\begin{aligned} \phi(x) &= (a \cdot x) \vee (b \cdot x) \\ \psi(x) &= x \cdot (a \vee b). \end{aligned}$$

for $x \in L$. We have $\phi(x) \leq \psi(x)$ for $x \in L$. Since L is bounded and the sets of values of ϕ, ψ are intervals, both sets equal $[0, a \vee b]$. We shall prove that $\phi(x) = \psi(x)$, thereby proving the distributivity of L.

Suppose that $\psi(u) = \psi(v)$, that is, $u \cdot (a \vee b) = v \cdot (a \vee b)$. Thus, $u \cdot (a \vee b) \cdot a = v \cdot (a \vee b) \cdot a$, which yields $u \cdot a = v \cdot a$. Similarly, $u \cdot b = v \cdot b$. Thus, we may conclude that $\phi(u) = \phi(v)$.

Since $\psi(x) \in \mathrm{Ran}(\phi)$ there exists $x_1 \in L$ such that $\psi(x) = \phi(x_1)$, hence $\psi(x) \leq \psi(x_1)$. The process can now be repeated for x_1; this yields an element $x_2 \in L$ such that $\psi(x_1) = \phi(x_2) \leq \psi(x_2)$, etc. Thus, we obtain a sequence $x = x_0, x_1, \ldots, x_n, x_{n+1}, \ldots$ such that:

$$\begin{array}{rcl} \psi(x_0) & = & \phi(x_1) \leq \psi(x_1) \\ \psi(x_1) & = & \phi(x_2) \leq \psi(x_2) \\ & \vdots & \\ \psi(x_n) & = & \phi(x_{n+1}) \leq \psi(x_{n+1}) \\ & \vdots & \end{array}$$

and since L is Artinian, there is p such that:

$$\psi(x_0) < \cdots < \psi(x_{p-1}) < \psi(x_p) = \psi(x_{p+1}) = \cdots .$$

Taking into account
(26)

$$\begin{array}{l} \psi(x_0) = \phi(x_1) \leq \psi(x_1) = \phi(x_2) \leq \psi(x_2) = \cdots \\ = \psi(x_{p-2}) = \phi(x_{p-1}) \leq \psi(x_{p-1}) = \phi(x_p) \leq \psi(x_p) = \phi(x_{p+1}) \leq \psi(x_{p+1}), \end{array}$$

the equality $\psi(x_p) = \psi(x_{p+1})$ implies $\phi(x_p) = \phi(x_{p+1})$, which, in turn, implies $\psi(x_{p-1}) = \psi(x_p)$. Repeating this process, we note that the inequalities (26) collapse into equalities. Since $\psi(x_0) = \psi(x_1)$ implies $\phi(x_0) = \phi(x_1)$ it follows that $\phi(x_0) = \psi(x_0)$, which is the desired equality.

If L is Noetherian, the dual argument works by replacing the functions ϕ, ψ by the functions μ, ν given by:

$$\begin{array}{rcl} \mu(x) & = & (a \vee x) \cdot (b \vee x) \\ \nu(x) & = & x \vee (a \cdot b). \end{array}$$

for $x \in L$. ∎

The previous theorem implies immediately the following result:

Corollary 3.2 *Let $(L, \vee, \cdot)$ be a finite lattice. If for every lattice function $f : L \longrightarrow L$ the set of values belongs to $\mathsf{INT}(L)$, then L is a distributive lattice.*

If L is an Artinian or Noetherian lattice that is not distributive, then it is always possible to find lattice functions whose range is not an interval. For example, suppose that L is the 5-element diamond lattice M_3 shown in Figure 1. It is well known that M_3 is a modular, but non-distributive lattice (cf. [Bir73]). For the function $\phi : M_3 \longrightarrow M_3$ given by $\phi(x) = (a \cdot x) \vee (b \cdot x)$ for $x \in M_3$ we have $\mathrm{Ran}(\phi) = \{0, a, b, 1\}$. Since $0 \leq c \leq 1$ and $c \notin \mathrm{Ran}(\phi)$ it follows that the range of ϕ is not an interval. However, this set is a sublattice of M_3.

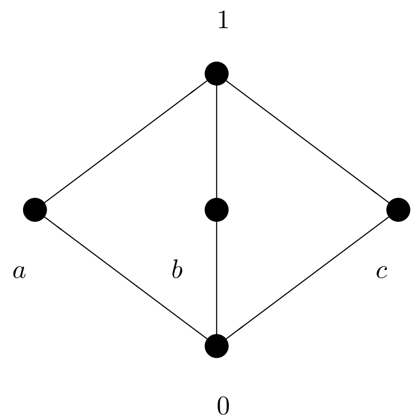

Figure 1: The modular and non-distributive lattice M_3

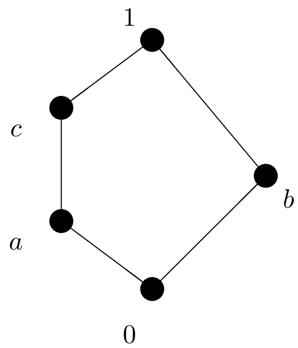

Figure 2: The non-modular lattice N_5

For the pentagon non-modular lattice N_5 given in Figure 2 consider the function $\phi : N_5 \longrightarrow N_5$ given by: $\phi(x) = (a \cdot x) \vee (b \cdot x)$. We have $\phi(0) = 0$, $\phi(a) = \phi(c) = a$, $\phi(b) = b$, and $\phi(1) = 1$. Thus, the set $\mathrm{Ran}(\phi) = \{0, a, b, 1\}$ is not an interval; it is however, a non-convex sublattice of N_5, because $\{a, 1\} \subseteq \mathrm{Ran}(\phi)$, $a \leq c \leq 1$, but $c \notin \mathrm{Ran}(\phi)$.

4 Open Issues

It would be interesting to examine ranges of algebraic functions of other lattice-based algebras (such as Post algebras, or more general subclasses of LM algebras) and obtain characterizations of those algebraic functions whose ranges are intervals. Using ranges of algebraic functions to characterize modular lattices in a manner similar to Theorem 3.1 is another possible research topic.

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