Several Remarks on Non-Boolean Functions over Boolean Algebras

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Abstract

We investigate charactizations of n-argument Boolean functions in the class of functions defined on Boolean algebras and we extend our previous results centered around the approximation of non-Boolean functions by Boolean functions. We also generalize of the notions of upper and lower semi-Boolean functions to the case of n-variable functions.

1 Introduction

The purpose of this paper is to extend several results established for one-argument Boolean and non-Boolean functions over Boolean algebra (see [RS91, MR80]) to the case of functions of several arguments. This work is motivated by the substantial body of research involving the use of set-valued non-Boolean functions in circuit design [AKH89, KH88, Aok93, AH93, AKH92, AKH91] and [AKH90].

Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra, where B is a set, $\vee$ and $\cdot$ are binary operations on B called disjunction and conjunctions, respectively, $'$ is a unary operation, called the complementation operation, and $0, 1$ are two special elements of B, with $0 \neq 1$ such that the usual axioms of Boolean algebras are satisfied as given in [Rud74] or [Hál74].

Elements of $\mathcal{B}^n$, where $n > 1$ will be denoted by capital letters $X, Y, \ldots$, while elements of the algebra $\mathcal{B}$ will be denoted with small letters. Boolean functions will be denoted with small letters $f, g, \ldots$. Arbitrary functions will be denoted by capital letters: $F, G, \ldots$.

If $x \in B$ and $a \in \{0, 1\}$ we use the notation

$$x^a = \begin{cases} x & \text{if } a = 1 \\ x' & \text{if } a = 0. \end{cases}$$

Note that $a^a = 1$ for $a \in \{0, 1\}$. If $X = (x_1, \ldots, x_n) \in B^n$ and $A =$

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$(a_1, \dots, a_n) \in \{0,1\}^n$, define X^A as $x_1^{a_1} \cdots x_n^{a_n}$. We have

$$A^C = \begin{cases} 1 & \text{if } A = C \\ 0 & \text{if } A \neq C, \end{cases}$$

for every $A, C \in \{0,1\}^n$. Also, if $A \neq C$, we have $X^A X^C = 0$ for every $X \in B^n$.

The binary operation "+" is defined on B by $x + y = xy' \vee x'y$ for $x, y \in B$. An easy argument by induction on n shows that if $z_1, \dots, z_n \in B$ such that $z_i z_j = 0$ for $1 \leq i, j \leq n$ and $i \neq j$, then

$$\bigvee_{1 \leq i \leq n} z_i = \sum_{1 \leq i \leq n} z_i.$$

If $X = (x_1, \dots, x_n) \in B^n$, then we denote $x_1 \vee x_2 \vee \cdots \vee x_n$ by $X^{\vee}$.

Lemma 1.1 *Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra and let $X, Y \in B^n$. We have $X^A Y^A = 0$ for every $A \in \{0,1\}^n$ if and only if $X = Y'$.*

Proof. Choose i such that $1 \leq i \leq n$. For $Z \in B^n$ denote by $Z_i \in B^{n-1}$ the tuple $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$. The condition $X^A Y^A = 0$ for every $A \in \{0,1\}^n$ implies that $X_i^{A_i} Y_i^{A_i} x_i^{a_i} y_i^{a_i} = 0$. Consequently, $X_i^C Y_i^C x_i^{a_i} y_i^{a_i} = 0$ for every $C \in B^{n-1}$, so

$$\left(\bigvee_{C \in \{0,1\}^{n-1}} X_i^C Y_i^C \right) x_i^{a_i} y_i^{a_i} = 0$$

Thus, $x_i^{a_i} y_i^{a_i} = 0$ for every $a_i \in \{0,1\}$, that is, $x_i y_i = 0$ and $x_i' y_i' = 0$. The first inequality implies $x_i \leq y_i'$; the second implies $y_i' \leq (x_i')' = x_i$, so $x_i = y_i'$. Thus, $X = Y'$. The reverse implication is immediate. ∎

Another useful technical result is given next.

Lemma 1.2 *Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra. If $X, Y \in B^n$, then $1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A = (X + Y)^{\vee}$.*

Proof. For $A \in \{0,1\}^n$ we write $A = (a_1, \dots, a_n)$, where $a_i \in \{0,1\}$ for $1 \leq i \leq n$. Then,

$$\begin{aligned}
\bigvee_{A \in \{0,1\}^n} X^A Y^A &= \sum_{A \in \{0,1\}^n} X^A Y^A \\
&= \sum_{a_1} \cdots \sum_{a_n} \prod_{i=1}^{n} (x_i y_i)^{a_i} \\
&= \prod_{i=1}^{n} \sum_{a_1} (x_1 y_1)^{a_1} \cdots \sum_{a_n} (x_n y_n)^{a_n} \\
&= \prod_{i=1}^{n} (x_1 y_1 + x_1' y_1') \cdots (x_n y_n + x_n' y_n') \\
&= \prod_{i=1}^{n} (1 + x_1 + y_1) \cdots (1 + x_n + y_n).
\end{aligned}$$

Therefore, an application of the DeMorgan law yields the equality:

$$1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A = \bigvee_{i=1}^{n} (x_i + y_i) = (X+Y)^{\vee}.$$

∎

2 A Characterization of n-argument Boolean Functions

Characterization of Boolean functions in the class of all functions defined on Boolean algebra has been investigated in various contexts. One of the earliest such characterization belongs to McColl [McC77, McC78, McC79] who proved that an n-argument function $F : B^n \longrightarrow B$ is Boolean if and only if $X^A f(X) = X^A f(A)$ for every $A \in \{0,1\}^n$. An equivalent condition that is useful in the current context is given next:

Theorem 2.1 *An n-argument function $F : B^n \longrightarrow B$ is Boolean if and only if*

$$F(X) + F(Y) \leq 1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A \tag{1}$$

for every $X, Y \in B^n$.

Proof. The condition is sufficient. Indeed, we have

$$\begin{aligned} X^A (F(X) + F(A)) &\leq X^A \left(1 + \bigvee_{C\in\{0,1\}^n} X^C A^C\right) \\ &\leq X^A + X^A = 0. \end{aligned}$$

Thus, $X^A (F(X) + F(A)) = 0$; equivalently, we obtain $X^A F(X) = X^A F(A)$ for every $A \in \{0,1\}^n$ which is McColl's characterization of Boolean functions.

Conversely, to prove the necessity, suppose that F is Boolean and thus, the McColl's conditions hold. We can write:

$$\begin{aligned} &(F(X) + F(Y)) \left(1 + \bigvee_{A\in\{0,1\}^n} X^A Y^A\right) \\ &= F(X) + F(Y) + \bigvee_{A\in\{0,1\}^n} F(X) X^A Y^A + \bigvee_{A\in\{0,1\}^n} F(Y) X^A Y^A \\ &= F(X) + F(Y) + \bigvee_{A\in\{0,1\}^n} F(A) X^A Y^A + \bigvee_{A\in\{0,1\}^n} F(A) X^A Y^A \\ &= F(X) + F(Y), \end{aligned}$$

which shows that

$$F(X) + F(Y) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A.$$

∎

Lemma ¡1.2 shows that an equivalent condition to the Equality (1) is:

$$F(X) + F(Y) \leq (X + Y)^{\vee} \tag{2}$$

for every $X, Y \in B^n$.

The necessity of the condition (2) was established by McKinsey in [McK36]. Our result shows that this condition is also sufficient and this, it is actually a characterization of Boolean functions.

Other useful equivalent forms of condition (1) are:

$$\bigvee_{A \in \{0,1\}^n} X^A Y^A \leq 1 + F(X) + F(Y) \tag{3}$$

and

$$(F(X) + F(Y)) \bigvee_{A \in \{0,1\}^n} X^A Y^A = 0 \tag{4}$$

for every $X, Y \in B^n$.

3 Local Approximation of Non-Boolean Functions

Non-Boolean functions can be locally approximated by Boolean functions. When the number of such Boolean approximants is small this can lead to cheaper implementations of non-Boolean functions.

Let $F : B^n \longrightarrow B$ be a function. Define the set $B_X^F \subset B^n$ by:

$$B_X^F = \{U \in B^n \mid F(X) + F(U) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A U^A\}.$$

The binary relation $\sim_F$ is given by $X \sim_F Y$ if $B_X^F = B_Y^F$.

It is easy to verify that the relation $\sim_F$ is an equivalence on B^n. The equivalence class of an element $X \in B^n$ will be denoted by $\langle X \rangle_F$. Note that if $Y \in \langle X \rangle_F$, then $Y \in B_X^F$.

For $n = 1$, $X = (x), W = (w)$, and we have $1 + \bigvee_{A \in \{0,1\}} X^A W^A = 1 + x^0 w^0 + x^1 w^1 = 1 + (1 + x)(1 + w) + xw = x + w$. Thus, the current definition reduces to the definition of the equivalence $\sim_F$ that we introduced in [RS91] for the case $n = 1$.

Observe that if $X \sim_F Y$, then, by taking $W = Y$ we have $F(X) + F(Y) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A Y^A$. In other words, the equivalence class of X is included in the set B_X for every $X \in B^n$.

Let $\mathcal{C}$ be an equivalence class of the relation $\sim_F$. Define the Boolean function $f_{\mathcal{C}} : \mathcal{B}^n \longrightarrow \mathcal{B}$ by

$$f_{\mathcal{C}}(U) = \bigvee_{Y \in \mathcal{C}} F(Y) \cdot \bigvee_{A \in \{0,1\}^n} U^A Y^A$$

for $U \in B^n$. Equivalently, the function $f_{\mathcal{C}}$ can be written as

$$f_{\mathcal{C}}(U) = \bigvee_{A \in \{0,1\}^n} \left(\bigvee_{Y \in \mathcal{C}} F(Y) \cdot Y^A \right) U^A$$

for $U \in B^n$. Then, $f_{\mathcal{C}}$ is also given by:

$$f_{\mathcal{C}}(A) = \bigvee_{Y \in \mathcal{C}} F(Y) \cdot Y^A \tag{5}$$

for $A \in \{0,1\}^n$.

Theorem 3.1 *Let $F : B^n \longrightarrow B$ be a function on the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$. We have $F(U) = f_{\langle X \rangle}(U)$ for every $U \in \langle X \rangle_F$ for every $X \in B^n$.*

Proof. Let $U \in \langle X \rangle_F$. If $Y \in \langle X \rangle_F$, then $U \sim_F Y$, and therefore $F(U) + F(Y) \leq 1 + \bigvee_{A \in \{0,1\}^n} U^A Y^A$, or equivalently, $\bigvee_{A \in \{0,1\}^n} U^A Y^A \leq 1 + F(U) + F(Y)$. Consequently,

$$F(Y) \cdot \bigvee_{A \in \{0,1\}^n} U^A Y^A \leq F(Y)(1 + F(U) + F(Y)) = F(Y)F(U). \tag{6}$$

Note that we can write

$$\begin{aligned} f_{\langle X \rangle_F}(U) &= F(U) \left(\bigvee_{A \in \{0,1\}^n} U^A \right) \vee \\ &\quad \bigvee_{Y \in \langle X \rangle - \{U\}} F(Y) \cdot \bigvee_{A \in \{0,1\}^n} U^A Y^A \\ &= F(U) \vee \bigvee_{Y \in \langle X \rangle_F - \{U\}} F(Y) \cdot \bigvee_{A \in \{0,1\}^n} U^A Y^A \\ &\quad \text{(because } \textstyle\bigvee_{A \in \{0,1\}^n} = 1) \\ &= F(U) \\ &\quad \text{(because the inequality (6)).} \end{aligned}$$

4 A-Boolean Functions

By relaxing the condition from Theorem 2.1 that characterizes Boolean functions one obtains interesting classes of non-Boolean functions that extend to the case of functions of n arguments the classes of upper and lower semi-Boolean functions of one-variable introduced in [MR80].

Definition 4.1 Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra and let $F : B^n \longrightarrow B$ be a function. For $A \in \{0,1\}^n$, F is an A-Boolean function if $F(X) + F(Y) \leq 1 + X^A Y^A$ for every $X, Y \in B^n$. □

Note that if F is A-Boolean for every $A \in \{0,1\}^n$, then

$$\begin{aligned} F(X) + F(Y) &\leq \prod_{A \in \{0,1\}^n} (1 + X^A Y^A) \\ &= \left(\sum_{A \in \{0,1\}^n} X^A Y^A \right)' \\ &= 1 + \sum_{A \in \{0,1\}^n} X^A Y^A, \end{aligned}$$

which is precisely the condition of Theorem 2.1. Thus, a function that is an A-Boolean function for every $A \in \{0,1\}^n$ is a Boolean function. Note that when $n = 1$, every 1-Boolean function is an upper semi-Boolean function and every 0-Boolean functions is a lower semi-Boolean function.

Theorem 4.2 *A function $F : B^n \longrightarrow B$ is A-Boolean if and only if it satisfies the McColl's condition for A, namely $X^A F(X) = X^A F(A)$ for every $X \in B^n$.*

Proof. Suppose that F is A-Boolean, that is $F(X) + F(Y) \leq 1 + X^A Y^A$ for every $X, Y \in B^n$. Choose $Y = A$. This implies $F(X) + F(A) \leq 1 + X^A$ for every X, hence $X^A F(X) + X^A F(A) \leq 0$, which implies the McColl's condition $X^A F(X) = X^A F(A)$.

Conversely, suppose that the McColl's condition $X^A F(X) = X^A F(A)$ holds for every X. Then, $Y^A F(Y) = Y^A F(A)$ and, therefore, we have both

$$X^A Y^A F(X) = X^A Y^A F(Y) = X^A Y^A F(A)$$

for every $X, Y \in B^n$. Thus, $X^A Y^A (F(X) + F(Y)) = 0$, which implies $F(X) + F(Y) \leq 1 + X^A Y^A$ for every $X, Y \in B^n$. ∎

Note that for any function $F : B^n \longrightarrow B$ the function $G : B^n \longrightarrow B$ defined by $G(X) = X^A F(X)$ for $X \in B^n$ is an C-Boolean function for every $C \neq A$ because $X^C G(X) = X^C X^A F(X) = 0$ and $X^C G(C) = C^A F(C) = 0$.

Note that if $F : B^n \longrightarrow B$ is an A-Boolean function, then we have

$$F(X) = X^A F(A) + F(X) \sum_{D \in \{0,1\}^n - \{A\}} X^D \tag{7}$$

for every $X \in B^n$ because $\sum_{D \in \{0,1\}^n} X^D = 1$ and $X^A F(X) = X^A F(A)$. Therefore, if F is A-Boolean, then $G(X) = X^A F(X)$ is a Boolean function. To prove this fact note that $G(A) = F(A)$, $G(C) = 0$ if $C \neq A$, and

$$X^C G(X) = X^C X^A F(X) = \begin{cases} 0 & \text{if } C \neq A \\ X^A F(X) & \text{if } C = A \end{cases}.$$

Therefore, $X^C G(X) = X^C G(C)$ for every $C \in \{0,1\}^n$, which means that G is indeed Boolean.

Theorem 4.3 *A function $F : B^n \longrightarrow B$ is A-Boolean if and only if there is an element $k \in B$ and a function $K : B^n \longrightarrow B$ such that*

$$F(X) = X^A k + K(X) \sum_{D \in \{0,1\}^n - \{A\}} X^D$$

for $X \in B^n$. Moreover, F is a Boolean function if and only if K is a D-Boolean function for every $D \neq A$.

Proof. The necessity of the condition follows immediately from Equality (7). The condition is sufficient because it implies $X^A F(X) = X^A k$ and $F(A) = k$. Thus, $X^A F(X) = X^A F(A)$, which shows that F is indeed A-Boolean.

Suppose that F is a Boolean function. Then, $X^D F(X) = X^D F(D)$ for every $D \in \{0,1\}^n$. Let D be different from A. We have $F(D) = K(D)$, and $X^D F(X) = X^D K(X)$, which implies $X^D K(X) = X^D K^D$. Thus, K is a D-Boolean function for every $D \neq A$. ∎

5 Chain-Valued Functions

Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra. A *p-chain-valued function* is a function $F : B^n \longrightarrow B$, where $F(B^n) \subseteq \{c_0, \ldots, c_p\}$, and $0 = c_0 < c_1 < \cdots < c_{p-1} < c_p = 1$. The number p is the length of the chain $(c_0, c_1, \ldots, c_p)$.

Initially, we focus our attention on binary functions on Boolean algebras, that is, on chain-valued functions $F : B^n \longrightarrow B$, where the chain is of length 2. In other words, $F : B^n \longrightarrow B$ is a binary function if $f(B^n) \subseteq \{0,1\}$. For such functions, the sets B^F_X have a special, simple form.

Lemma 5.1 *Let $F : B^n \longrightarrow B$ be a binary function on the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$. If $X \in B^n$, then $B^F_X - \{X'\} = F^{-1}(F(X))$.*

Proof. Suppose that $U \in B^F_X$. Then, we have $F(X) + F(U) \leq 1 + \bigvee_{A \in \{0,1\}^n} X^A U^A$.

If $U \neq X'$, then by Lemma 1.1, $\bigvee_{A \in \{0,1\}^n} X^A U^A \neq 0$, so

$$1 + \bigvee_{A \in \{0,1\}^n} X^A U^A < 1,$$

which implies $F(X) + F(U) = 0$ because F is a binary function. This means that $F(X) = F(U)$, so $U \in F^{-1}(F(X))$. Thus, $B^F_X - \{X'\} \subseteq F^{-1}(F(X))$.

Conversely, if $U \in F^{-1}(F(X))$, then $F(U) = F(X)$, so $F(U) + F(X) = 0$, which implies $U \in B^F_X$. ∎

Theorem 5.2 *Let $F : B^n \longrightarrow B$ be a binary function on the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$. We have*

$$B_X^F = F^{-1}(F(X)) \cup \{X'\}.$$

for every $X \in B^n$.

Proof. Suppose initially, that $F(X') = F(X)$. Then, $X' \in F^{-1}(F(X))$, so $B_X^F = F^{-1}(F(X))$ by Lemma 5.1. Suppose now that $F(X') \neq F(X)$. In this case, we have $B_X^F = F^{-1}(F(X)) \cup \{X'\}$, by the same lemma. In either case the above equality is verified. ∎

Note that $F^{-1}(F(X))$ is the equivalence class $[X]_F$ of X relative to $\mathbf{ker}F$. If F is not constant then there are two such classes, namely $F^{-1}(0)$ and $F^{-1}(1)$. Thus, the statement of the theorem can also be written as

$$B_X^F = \begin{cases} [X]_F & \text{if } X' \in [X]_F \\ [X]_F \cup \{X'\} & \text{otherwise.} \end{cases}$$

This allows us to conclude that $X \sim_F Y$ if and only if

$$[X]_F \cup \{X'\} = [Y]_F \cup \{Y'\}. \tag{8}$$

Suppose that $X \sim_F Y$.

If $F(X) \neq F(Y)$, the classes $[X]_F$ and $[Y]_F$ are disjoint. Since $[X]_F \cup [Y]_F = B^n$ the Equality (8) is possible only if $[X]_F = \{Y'\}$ and $[Y]_F = \{X'\}$, which means that $B^n = \{X', Y'\}$. This is possible only if $B = \{0, 1\}$ and $n = 1$.

If $F(X) = F(Y)$ and $X \neq Y$, then we must have $X', Y' \in [X]_F = [Y]_F$, that is, $F(X') = F(Y') = F(X) = F(Y)$. Thus, if $n \geq 1$ we have $X \sim_F Y$ if and only if $F(X) = F(Y) = F(X') = F(Y')$.

Example 5.3 Let $a \in B - \{0, 1\}$, where we assume that $|B| > 2$, and let $F : B^2 \longrightarrow B$ be the function defined by:

$$F(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + x_2 \leq a \\ 0 & \text{otherwise} \end{cases}$$

for $x_1, x_2 \in B$. The function F is not Boolean. Indeed, if F were Boolean we could write:

$$F(x_1, x_2) = p + qx_1 + rx_2 + sx_1x_2$$

for every $x_1, x_2 \in B$. If x_1, x_2 range in the set $\{0, 1\}$ we obtain immediately that $F(x_1, x_2) = 1 + x_1 + x_2$. Now, by choosing $x_1 = a$ and $x_2 = 1$, we obtain $F(a, 1) = 0$ because $a + 1 \not\leq a$. On another hand, $1 + a + 1 = a$, so $F(a, 1) = 0 \neq a$. This shows that F is not Boolean.

Note that we always have $F(X) = F(X')$ for $X \in B^2$ because if $X = (x_1, x_2)$, then $x_1 + x_2 = x_1' + x_2'$. Thus, the relation $\sim_F$ coincides with $\mathbf{ker}f$ and we can approximate F using two Boolean functions: $f_{F^{-1}(1)}$ and $f_{F^{-1}(0)}$.

For the function $f_{F^{-1}(1)}$ we can write:

$$\begin{aligned} f_{F^{-1}(1)}(U) &= \bigvee_A U^A \left(\bigvee_{Y \in F^{-1}(U)} Y \right) \\ &= u_1 u_2 \vee u_1 u_2' a \vee u_1' u_2 a \vee u_1' u_2' \\ &= (1 + u_1 + u_2) \vee a(u_1 + u_2) \\ &= (1 + u_1 + u_2) + a(u_1 + u_2) \\ & \quad \text{(because } (1 + u_1 + u_2) \cdot a(u_1 + u_2) = 0) \\ &= 1 + a'(u_1 + u_2). \end{aligned}$$

It is easy to see that $f_{F^{-1}(0)}$ is the constant function 0. ∎

Next, we consider an example involving a 3-chain-valued function.

Example 5.4 Consider the 4-element Boolean algebra Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$, where $B = \{0, a, a', 1\}$, and the function $F : B^2 \longrightarrow B$, defined by the following table:

	$F(x_1, x_2)$	x_2: 0	a	a'	1
x_1	0	0	1	0	a
	a	1	0	a	0
	a'	0	a	0	1
	1	a	0	1	0

Note that this function is a 3-chain-valued function, because its range is the chain $0 \leq a \leq 1$. This function is not Boolean because $F(0, a) + F(a, a) = 1 + 0 = 1$, $((0, a) + (a, a))^{\vee} = (a, 0)^{\vee} = a$ and $1 \not\leq a$. The function is clearly symmetric and, in addition, $F(X) = F(X')$ for every $X \in B^2$.

A direct but tedious computation shows that:

$B_{(0,a')} = B_{(0,1)} = B_{(a,a')} = B_{(a,1)} = B_{(a',0)} = B_{(a',a)} = B_{(1,0)} = B_{(1,a)} = B^2$
$B_{(0,0)} = B_{(a,a)} = B^2 - \{(0, a), (a, 0)\}$
$B_{(a',a')} = B_{(1,1)} = B^2 - \{(1, a'), (a', 1)\}$
$B_{(a,0)} = B_{(0,a)} = B^2 - \{(0, 0), (a, a)\}$
$B_{(1,a')} = B_{(a',1)} = B^2 - \{(1, 1), (a', a')\}$.

Thus, there are five equivalence classes of $\sim_F$:

$$\begin{aligned} \mathcal{C}_1 &= \{(0, a'), (0, 1), (a, a'), (a, 1), (a', 0), (a', a), (1, 0), (1, a)\}, \\ \mathcal{C}_2 &= \{(0, 0), (a, a)\}, \\ \mathcal{C}_3 &= \{(a', a'), (1, 1)\}, \\ \mathcal{C}_4 &= \{(a, 0), (0, a)\}, \\ \mathcal{C}_5 &= \{(1, a'), (a', 1)\}. \end{aligned}$$

The corresponding Boolean functions that approximate F can be obtained from formula (5):

$$\begin{aligned} f_{\mathcal{C}_1}(A) &= a[(0,1)^A \vee (a,a')^A \vee (a',a)^A \vee (1,0)^A], \\ f_{\mathcal{C}_2}(A) &= f_{\mathcal{C}_3}(A) = 0, \\ f_{\mathcal{C}_4}(A) &= (a,0)^A \vee (0,a)^A, \\ f_{\mathcal{C}_5}(A) &= (1,a')^A \vee (a',1)^A, \end{aligned}$$

for every $A \in \{0,1\}^2$. □

The study on non-binary chain-valued functions is significantly more complicated. We are presenting here several initial results involving such functions.

Let $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$ be a Boolean algebra and let $a \in B$. Define the relation $X \equiv_a Y$ if $(X+Y)^{\vee} \leq a$. It is immediate that $\equiv_a$ is reflexive and symmetric. To show that it is transitive suppose that $X \equiv_a Y$ and $Y \equiv_a Z$, where $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, and $Z = (z_1, \ldots, z_n)$. This means that $x_i + y_i \leq a$ and $y_i + z_i \leq a$ for $1 \leq i \leq n$, that is $x_i + y_i = x_i a + y_i a$ and $y_i + z_i = y_i a + z_i a$. Adding these equalities we obtain $x_i + z_i = (x_i + z_i)a$, that is $x_i + z_i \leq a$ for $1 \leq i \leq n$, which implies $(X+Z)^{\vee} \leq a$. Thus, $\equiv_a$ is an equivalence for every $a \in B$. Moreover, by Lemma 1.2, $\equiv_a$ is a congruence on B^n for every $a \in B$. Note that $\equiv_1$ equals the set $B^n \times B^n$.

Theorem 5.5 *Let $Q = \{c_0, \ldots, c_p\}$ be a chain of the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$. The mapping $d_Q : B^n \times B^n \longrightarrow \mathbb{R}$ defined by*

$$d_Q(X,Y) = k \text{ if } k \text{ is the least number such that } (X+Y)^{\vee} \leq c_k,$$

for $X, Y \in B^n$ is an ultrametric on B^n.

Proof. We have $X \equiv_0 X$; therefore, $d_Q(X,X) = 0$ for every $X \in B^n$. Conversely, if $d_Q(X,Y) = 0$ we have $(X+Y)^{\vee} = 0$, and, therefore $X = Y$.

Suppose now that $d_Q(X,Y) = k$ and $d_Q(Z,Y) = h$. We have $x_i + z_i \leq c_k$ and $z_i + y_i \leq c_h$ for $1 \leq i \leq n$. This means that $x_i + z_i = x_i c_k + z_i c_k$ and $z_i + y_i = z_i c_h + y_i c_h$ for $1 \leq i \leq n$, which, in turn implies $x_i + y_i = x_i c_k + z_i(c_h + c_k) + y_i c_h$. Furthermore, suppose that $h \leq k$, which means that $c_h = c_h c_k$. This allows us to write $x_i + y_i = x_i c_k + z_i(c_h c_k + c_k) + y_i c_h c_k = (x_i + z_i(c_h + 1) + y_i c_h) c_k \leq c_k$, which shows that $d_Q(X,Y) \leq \max\{d_Q(X,Z), d_Q(Z,Y)\}$. Thus, d_Q is an ultrametric on B^n. ■

The sphere centered in $X \in B^n$ with radius k in the ultrametric space (B^n, d_Q) will be denoted by

$$S_X^k = \{W \in B^n \mid d_Q(X,W) < k\}.$$

Note that $S_X^0 = \emptyset$.

Lemma 5.6 *Let $Q = \{c_0, \ldots, c_p\}$, be a chain in the Boolean algebra $\mathcal{B} = (B, \vee, \cdot, ', 0, 1)$, where $0 = c_0 < c_1 < \cdots < c_{p-1} < c_p = 1$. If $c_i, c_j, c_k \in Q$ are such that $c_i + c_j \leq c_k$, then either $c_i = c_j$, or we have both $c_i \leq c_k$ and $c_j \leq c_k$.*

Proof. Observe first that $c_i \leq c_k$ if and only if $c_j \leq c_k$. Indeed, the inequality $c_i + c_j \leq c_k$ implies $c_i + c_j = c_i c_k + c_j c_k$. Thus, $c_i \leq c_k$ is equivalent to $c_i = c_i c_k$, and this, in turn, is equivalent to $c_j = c_j c_k$, which means $c_j \leq c_k$.

Suppose that $c_i \neq c_j$ and that $c_k < c_i$, and therefore, $c_k < c_j$. This implies $c_k = c_k c_i = c_k c_j$, so $c_i + c_j = 0$, that is, $c_i = c_j$. This contradicts the assumption that c_i, c_j are distinct. So, the single possible conclusion is that we have both $c_i \leq c_k$ and $c_j \leq c_k$. ∎

The next two results offer the possibility to simplify the computations of the sets B_X^F for chain-valued functions.

Theorem 5.7 *For any chain-valued function $F : B^n \longrightarrow B$, where $F(B^n) \subseteq \{c_0, \ldots, c_p\}$, and $0 = c_0 < c_1 < \cdots < c_{p-1} < c_p = 1$ if $Y \in B_X$, then for every j, $0 \leq j \leq p$, $d_Q(X, Y) = k$ implies either $F(X) = F(Y)$ or both $F(X) \leq c_k$ and $F(Y) \leq c_k$.*

Proof. Since $Y \in B_X$ we have $F(X) + F(Y) \leq (X+Y)^{\vee}$. On the other hand, $d_Q(X, Y) = k$ means that $(X + Y)^{\vee} \leq c_k$, which implies $F(X) + F(Y) \leq c_k$. Since we have $F(X), F(Y) \in Q$, by Lemma 5.6, we have either $F(X) = F(Y)$ or both $F(X) \leq c_k$ and $F(Y) \leq c_k$. ∎

Theorem 5.8 *Let $F : B^n \longrightarrow B$ be a chain-valued function , where $F(B^n) \subseteq \{c_0, \ldots, c_p\}$, and $0 = c_0 < c_1 < \cdots < c_{p-1} < c_p = 1$. For any $X \in B^n$ such that $F(X) = c_i$ we have $(B_X - F^{-1}(F(X)) \cap S_X^k = \emptyset$ if $k < i$ and*

$$\big(B_X - F^{-1}(F(X))\big) \cap S_X^k \subseteq \bigcup_{j \leq k} F^{-1}(c_j).$$

if $i \leq k$.

Proof. Let $W \in \big(B_X - F^{-1}(F(X))\big) \cap S_X^k$. We have $F(W) \neq F(X)$ and $F(X) + F(W) \leq (X+W)^{\vee}$ by the definition of B_X. Since $W \in S_X^k$ we also have $d_Q(X, W) \leq k$, which means that $(X + W)^{\vee} \leq c_k$. Thus, $F(X) + F(W) \leq c_k$. By Lemma 5.6, this implies $c_i = F(X) \leq c_k$ and also, $c_j = F(W) = c_j \leq c_k$. Thus, $W \in \bigcup_{j \leq k} F^{-1}(c_j)$. ∎

6 Conclusions

We extended our previous results centered around the approximation of non-Boolean functions by Boolean functions over Boolean algebras. This extension involved a new characterization of Boolean functions (obtained by proving the sufficiency of McKinsey inequality) as well as a generalization of the notions of upper and lower semi-Boolean functions (introduced by Melter and Rudeanu in [MR80] to the case of n-variable functions.

It would be interesting to investigate further the Boolean approximability of various classes of non-Bollean functions, such as the A-Boolean functions introduced here, or the generalized Boolean functions introduced in [Ţân81, Ţân82, Ţân84].

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