DUALITY IN OPTIMIZATION

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UMB
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Primal and Dual Problems

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ and let

$\theta : \mathbb{R}^{m+\ell} \rightarrow \mathbb{R}$ given by:

$$\theta(u, v) = \inf_{x \in X} L(x, u, v),$$

where

$$L(x, u, v) = f(x) + u'g(x) + v'h(x)$$

is the Lagrangean function.

**Primal Problem:**

minimize $f(x)$

subject to: $g(x) \leq 0$

$h(x) = 0$

$x \in X$

**Dual Problem:**

maximize $\theta(u, v)$

subject to: $u \geq 0$
Theorem

Let \( x \in \mathbb{R}^n \) be a feasible point of the primal problem, that is,

\[
x \in X, \ g(x) \leq 0, \ h(x) = 0.
\]

Let \( \left( \begin{array}{c} u \\ v \end{array} \right) \) be a feasible solution to the dual problem, that is, \( u \geq 0 \). Then,

\[
\theta(u, v) \leq f(x).
\]

Proof: If \( u \geq 0 \), then \( f(x) + u'g(x) + v'h(x) \leq f(x) \) because \( g(x) \leq 0 \) and \( h(x) = 0 \). Therefore,

\[
\theta(u, v) = \inf_x (f(x) + u'g(x) + v'h(x)) \leq \inf_x f(x),
\]

which implies \( \theta(u, v) \leq f(x) \).
Corollaries

The weak duality theorem can be used to obtain lower bounds on optimal values of difficult problems.

A The dual optimum (max) is a lower bound of the primal optimum (min), that is,

$$\inf\{f(x) \mid x \in X, g(x) \leq 0, h(x) = 0\} \geq \sup\{\theta(u, v) \mid u \geq 0\}.$$ 

B If $f(\bar{x}) \leq \theta(\bar{u}, \bar{v})$, where $\bar{u} \geq 0$ and

$$\bar{x} \in \{x \in X, g(x) \leq 0, h(x) = 0\},$$
then $\bar{x}$ and $\left(\begin{array}{c} \bar{u} \\ \bar{v} \end{array}\right)$ solve the primal and dual problems, respectively.

C If $\inf\{f(x) \mid x \in X, g(x) \leq 0, h(x) = 0\} = -\infty$, then $\theta(u, v) = -\infty$. 
Recall

Let $S$ be a nonempty convex subset of $\mathbb{R}^n$ and $\bar{x} \notin S$. Then, there exists a $p \neq 0$ such that $p'(x - \bar{x}) \geq 0$ for every $x \in K(S)$.

In particular, if $0 \notin S$, there exists $p \neq 0$ such that $p'x \geq 0$ for $x \in K(S)$. 
Theorem

Let $X$ be nonempty and convex subset of $\mathbb{R}^n$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be convex and let $h : \mathbb{R}^n \to \mathbb{R}^\ell$ be given by $h(x) = Ax - b$.

If the system $(S1)$

$$\alpha(x) < 0, g(x) \leq 0, h(x) = 0$$

has no solution, then the system $(S2)$:

$$u_0 \alpha(x) + u'g(x) + v'h(x) \geq 0,$$

$$\begin{pmatrix} u_0 \\ u \\ v \end{pmatrix} \geq 0, \begin{pmatrix} u_0 \\ u \\ v \end{pmatrix} \neq 0,$$

has a solution. The converse holds when $u_0 > 0$. 
Proof

Suppose that (S1) has no solution and consider the set

\[
L = \left\{ \begin{pmatrix} p \\ q \\ r \end{pmatrix} \mid p > \alpha(x), q \geq g(x), r = h(x), x \in X \right\}
\]

The set \( L \) is convex. Since (S1) has no solution, \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin L \). By the result recalled earlier, there exists \( \begin{pmatrix} u_0 \\ u \\ v \end{pmatrix} \) such that \( u_0 p + u'q + v'r \geq 0 \) for each \( \begin{pmatrix} p \\ q \\ r \end{pmatrix} \in L \)
Fix $x \in X$. Since $p$ and $q$ can be made arbitrarily large, we have $u_0p + u'q + v'r \geq 0$ only if $u_0 \geq 0$ and $u \geq 0$. This shows that (S2) has a solution.

Conversely, suppose that (S2) has a solution \[
\begin{pmatrix}
u_0 \\ u \\ v
\end{pmatrix}
\] such that

\[u_0 > 0 \text{ and } u \geq 0\]

that satisfies

\[u_0\alpha(x) + u'g(x) + v'h(x) \geq 0,\]

for $x \in X$.

Let $x$ be such that $g(x) \leq 0$ and $h(x) = 0$. Since $u \geq 0$, it follows that $u_0\alpha(x) \geq 0$. Since $u_0 > 0$, it follows that $\alpha(x) \geq 0$, so (S1) has no solution.
Framework for Strong Duality

Let $X \subseteq \mathbb{R}^n$ be a nonempty convex subset.

- $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be convex functions and let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $h(x) = Ax - b$.
- there exists $\hat{x}$ such that $g(\hat{x}) \leq 0$ and $h(\hat{x}) = 0$;
- $0 \in h(X) = \{h(x) \mid x \in X\}$.

Strong Duality:

$$\sup \{ \theta(u, v) \mid u \geq 0 \} = \inf \{ f(x) \mid x \in X, g(x) \leq 0, h(x) = 0 \}$$
Strong Duality Theorem

Theorem

Let $X$ be a nonempty convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a convex function and let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function given by $h(x) = Ax - b$.

Suppose that the following constraint qualification holds:
- there exists $\hat{x}$ such that $g(\hat{x}) \leq 0$ and $h(\hat{x}) = 0$;
- $0 \in I(h(X))$, where $h(X) = \{h(x) \mid x \in X\}$.

Then,

$$\inf\{f(x) \mid x \in X, g(x) \leq 0, h(x) = 0\} = \sup\{\theta(u, v) \mid u \geq 0\}$$
Proof

Let

$$\gamma = \inf \{ f(x) \mid x \in X, g(x) \leq 0, h(x) = 0 \}.$$  

If $\gamma = -\infty$, then by Corollary (C), $\sup \{ \theta(u,v) \mid u \geq 0 \} = -\infty$ and the equality holds.

Suppose $\gamma$ is finite and consider the system:

$$f(x) - \gamma < 0, g(x) \leq 0, h(x) = 0, x \in X,$$

which has so solution. Therefore, by the Theorem on slide 7, there exists

$$\begin{pmatrix} u_0 \\ u \\ v \end{pmatrix}$$

such that

$$u_0(f(x) - \gamma) + u'g(x) + v'h(x) \geq 0, \quad (1)$$

for $x \in X$.  

Proof (cont’d)

We claim that $u_0 > 0$. Suppose that this is not the case, that is, $u_0 = 0$. By the constraint qualifications there exists $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$. The inequality $u_0(f(\hat{x}) - \gamma) + u'g(\hat{x}) + v'h(\hat{x}) \geq 0$, reduces to $u'g(\hat{x}) \geq 0$. Since $g(\hat{x}) < 0$ and $u \geq 0$, $u'g(\hat{x}) \geq 0$ is possible only if $u = 0$.

Again, from $u_0(f(x) - \gamma) + u'g(x) + v'h(x) \geq 0$, $u_0 = 0$ and $\hat{u} = 0$ implies $v'h(x) \geq 0$ for $x \in X$.

Since $0 \in I(h(X))$, we can move from $0$ in the direction $-\lambda v$ and still remain in $h(X)$, that is, there exists $x \in X$ such that $h(x) = -\lambda v$, where $\lambda > 0$. Therefore,

$$0 \leq v'h(x) = -\lambda \|v\|^2,$$

which implies $v = 0$. We have shown that $u_0 = 0$ implies \begin{pmatrix} u_0 \\ u \\ v \end{pmatrix} = 0,

which is impossible. Hence $u_0 > 0$. 
Proof (cont’d)

Dividing Inequality (1) by $u_0$ we get

$$f(x) + \bar{u}'g(x) + \bar{v}'h(x) \geq \gamma$$

(2)

for $x \in X$, where $\bar{u} = \frac{1}{u_0} u$ and $\bar{v} = \frac{1}{u_0} v$. This shows that

$$\theta(\bar{u}, \bar{v}) = \inf_{x \in X} f(x) + \bar{u}'g(x) + \bar{v}'h(x) \geq \gamma.$$  

Therefore, $\theta(\bar{u}, \bar{v}) = \gamma$, so $\bar{u}$ and $\bar{v}$ solve the dual problem.
Suppose that \( \bar{x} \) is an optimal solution to the primal problem, that is, \( \bar{x} \in X, \ g(\bar{x}) \leq 0, \ h(\bar{x}) = 0 \) and \( f(\bar{x}) = \gamma \). Choosing \( x = \bar{x} \) in Inequality (2):

\[
f(x) + \bar{u}'g(x) + \bar{v}'h(x) \geq \gamma
\]

we get \( \bar{u}'g(\bar{x}) \geq 0 \).

Since \( \bar{u} \geq 0 \) and \( g(\bar{x}) \leq 0 \), we have \( \bar{u}'g(\bar{x}) = 0 \).
Example (non-convex $X$)

Consider the primal problem:

- minimize $-2x_1 + x_2$;
- subject to $x_1 + x_2 - 3 = 0$, $x \in X$,

where

$$X = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 4 \\ 4 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

The optimal solution to the primal is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with objective value $-3$.

$(2, 1)$ is the optimal solution of the primal; the dual objective function is

$$\theta(v) = \min \{-2x_1 + x_2 + v(x_1 + x_2 - 3) \mid x \in X\}.$$
Example (cont’d)

\[
\begin{pmatrix}
0 \\ 0 \\ -3v \\
0 \\ 4 + v \\
4 \\ 4 + 5v \\
4 \\ -8 + v \\
0 \\ 0 \\
1 \\ 2 \\
2 \\ 1 \\
-3
\end{pmatrix}
\]

This implies

\[
\theta(v) = \begin{cases}
-4 + 5v & \text{if } v \leq -1 \\
-8 + v & \text{if } -1 \leq v \leq 2 \\
-3v & \text{if } v \geq 2
\end{cases}
\]
Example (non-convex $X$)

Let $A \in \mathbb{R}^{n \times n}$ be a matrix. Consider the problem

- minimize $x'Ax$;
- subject to $x_i^2 = 1$, that is, $x_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

The feasible set contains $2^n$ points.

The Lagrangean function is

$$L(x, v) = x'Ax + \sum_{j=1}^{n} v_j(x_j^2 - 1)$$

$$= x'(A + \text{diag}(v_1, \ldots, v_n))x - 1'v.$$

The dual function is

$$\theta(v) = \inf_{x} L(x, v) = \begin{cases} 
-1'v & \text{if } A + \text{diag}(v_1, \ldots, v_n) \succeq 0 \\
-\infty & \text{otherwise,}
\end{cases}$$

provides lower bounds on the optimal value of the primal problem. For example, we can take $v = -\lambda_{min}1$, where $\lambda_{min}$ is the smallest eigenvalue of $A$. This is dually feasible because $A + \text{diag}(v) = A - \lambda_{min}I_n \succeq 0$, so the optimal value of the primal $p^*$ satisfies

$$p^* \succeq -1'v = n\lambda_{min}.$$
Example

Consider the problem

- minimize $\| x \|$ for $x \in \mathbb{R}^n$;
- subjected to $Ax = b$, where $A \in \mathbb{R}^{m \times n}$.

Minimizing $\| x \|$ is tantamount to minimizing $x'x$ and this is the function we are using.

The Lagrangean is

$$ L(x, v) = x'x + v'(Ax - b) $$

and is a quadratic convex function in $x$. The objective function of the dual is $\theta(v) = \inf_x L(x, v)$, where $v \in \mathbb{R}^m$.

The minimum of $L$ is determined from

$$ \nabla_x L(x, v) = 2x + v'A = 0, $$

so $x = -\frac{1}{2}v'A$. Therefore,

$$ \theta(v) = -\frac{1}{4}v'AA'v - b'v, $$

which is a concave quadratic function with domain $\mathbb{R}^m$. The weak duality theorem states that

$$ -\frac{1}{4}v'AA'v - b'v \leq \inf \{ x'x \mid Ax = b \}. $$
Saddle Point Theorem

**Theorem**

Let $X \neq \emptyset$ be a subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$. Suppose that there exists $\bar{x} \in X$ and $\bar{u}, \bar{v}$ with $\bar{u} \geq 0$ such that

$$\Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v}) \leq \Phi(x, u, v)$$  \hspace{1cm} (3)

for all $x \in X$ and all $u, v$ with $u \geq 0$, where

$$\Phi(x, u, v) = f(x) + u'g(x) + v'h(x).$$

Then $\bar{x}$ solves the primal problem, and $u, v$ solve the dual problem.

Conversely, suppose that $X, f, g$ are convex, $h(x) = Ax - b$, $0 \in I(h(X))$, there exists $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$. If $\bar{x}$ is an optimal solution to the primal problem, then there exist $\bar{u}, \bar{v}$ with $\bar{u} \geq 0$ so that the inequalities (3) hold.
Suppose that there exists $\bar{x} \in X$ and $\bar{u}, \bar{v}$ with $\bar{u} \geq 0$ such that

$$
\Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v}).
$$

Since

$$
f(\bar{x}) + u'g(\bar{x}) + v'h(\bar{x}) = \Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v}) = f(\bar{x}) + \bar{u}'g(\bar{x}) + \bar{v}'h(\bar{x}),
$$

for all $u \geq 0$ and all $v \in \mathbb{R}^n$ it follows that

$$(u - \bar{u})'g(\bar{x}) \leq (\bar{v} - v)'h(\bar{x}),$$

for all $u \geq 0$ and all $v \in \mathbb{R}^n$, which implies $g(\bar{v}) \leq 0$ and $h(\bar{x}) = 0$. Thus, $\bar{x}$ is a feasible solution of $P$.

By taking $u = 0$, it follows that $\bar{u}'g(\bar{x}) \geq 0$. Since $\bar{u} \geq 0$ and $g(\bar{x}) \leq 0$, we have $\bar{u}'g(\bar{x}) = 0$. 

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By the inequalities of the hypothesis we get

\[
f(\bar{x}) = f(\bar{x}) + \bar{u}'g(\bar{x}) + \bar{v}'h(\bar{x}) = \Phi(\bar{x}, \bar{u}, \bar{v}) \\
\leq \Phi(x, \bar{u}, \bar{v}) = f(x) + \bar{u}'g(x) + \bar{v}'h(x)
\]

for every \( x \in X \). This implies \( f(\bar{x}) \leq \theta(u, v) \). Since \( \bar{x} \) is feasible for the primal problem and \( \bar{u} \succeq 0 \), it follow that \( \bar{x} \) and \( \bar{u}, \bar{v} \) are optimal for the primal and the dual problem, respectively.
Conversely, suppose that $\bar{x}$ is optimal for the primal problem. By the Strong Duality Theorem, there exist $\bar{u}, \bar{v}$ with $\bar{u} \leq 0$ such that $f(\bar{x}) = \theta(\bar{u}, \bar{v})$ and $\bar{u}'g(\bar{x}) = 0$. so

$$f(\bar{x}) = \theta(\bar{u}, \bar{v}) \leq f(x) + \bar{u}'g(x) + \bar{v}'h(x) = \Phi(x, \bar{u}, \bar{v})$$

for $x \in X$. Since $\bar{u}'g(\bar{x}) = \bar{v}'h(\bar{x}) = 0$,

$$\Phi(\bar{x}, \bar{u}, \bar{v}) = f(\bar{x}) + \bar{u}'g(\bar{x}) + \bar{v}'h(\bar{x}) \leq \Phi(x, \bar{u}, \bar{v})$$

for $x \in X$ (the second inequality of 3). The first inequality holds because $\bar{u}'g(\bar{x}) = 0$, $h(\bar{x}) = 0$, $g(\bar{x}) \leq 0$ and $u \geq 0$. 
KT Conditions and Saddle Points

**Theorem**

Let \( S = \{ x \in X \mid g(x) \leq 0, h(x) = 0 \} \) and consider the primal problem \( \mathcal{O}(f, g, h) \). Suppose that \( \bar{x} \in S \) satisfies the KTC, that is, there is \( \bar{u} \geq 0 \) and \( \bar{v} \) such that

\[
(\nabla f)(\bar{x}) + \bar{u}'(\nabla g)(\bar{x}) + \bar{v}'(\nabla h)(\bar{x}) = 0 \quad (4)
\]

\[
\bar{u}'g(\bar{x}) = 0. \quad (5)
\]

Suppose that \( f, g_i \) for \( i \in I_X \) are convex, \( v_j \neq 0 \), and \( h(x) = Ax - b \). Then \( \bar{x}, \bar{u}, \bar{v} \) satisfy the saddle point conditions

\[
\Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v})
\]

for all \( x \in X \) and all \( u, v \) with \( u \geq 0 \), where

\[
\Phi(x, u, v) = f(x) + u'g(x) + v'h(x).
\]
Proof

Suppose that $\bar{x} \in S$ satisfies the KTC, that is, there is $\bar{u} \geq 0$ and $\bar{v}$ such that

$$
(\nabla f)(\bar{x}) + \bar{u}'(\nabla g)(\bar{x}) + \bar{v}'(\nabla h)(\bar{x}) = 0 \quad (6)
$$

$$
\bar{u}'g(\bar{x}) = 0. \quad (7)
$$

Since $f$ and $g_i$ are convex, we have

$$
f(x) \geq f(\bar{x}) + (\nabla f)'(x - \bar{x}),
$$

$$
g_i(x) \geq g_i(\bar{x}) + (\nabla g_i)'(x - \bar{x}) \quad \text{for } i \in I_x
$$

$$
h_j(x) = h_j(\bar{x}) + (\nabla h_j)(\bar{x})'(x - \bar{x}) \quad \text{for } 1 \leq j \leq \ell, (\bar{v})_j \neq 0,
$$

for $x \in X$. 

Proof

Multiplying and adding the equalities:

\[ f(x) \geq f(\bar{x}) + (\nabla f)'(x - \bar{x}), \]

\[ g_i(x) \geq g_i(\bar{x}) + (\nabla g_i)'(x - \bar{x}) \quad \text{for } i \in I_x \]

\[ h_j(x) = h_j(\bar{x}) + (\nabla h_j)(\bar{x})'(x - \bar{x}) \quad \text{for } 1 \leq j \leq \ell, (\bar{v})_j \neq 0, \quad \text{by } (\bar{v})_j, \]

and taking into account \( \bar{u}'g(\bar{x}) = 0 \), it follows that \( \Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v}) \).

Since \( g(\bar{x}) \leq 0, g(\bar{x}) = 0 \), and \( \bar{u}'g(\bar{x}) = 0 \), it follows that

\( \Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v}) \).

Thus, \( \bar{x}, \bar{u}, \bar{v} \) satisfy the saddle point conditions.
Reciprocal Theorem

**Theorem**

If \( \bar{x}, \bar{u}, \bar{v} \) with \( \bar{x} \in I(X) \) and \( u \geq 0 \) satisfy the saddle point conditions, then \( \bar{x} \) is feasible to the primal problem, and \( \bar{x}, \bar{u}, \bar{v} \) satisfies the conditions 6 and 7.
Proof

Suppose that $\bar{x}, \bar{u}, \bar{v}$ with $\bar{x} \in I(X)$ and $\bar{u} \geq 0$ satisfy

$$\Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v})$$

Since $\Phi(\bar{x}, u, v) \leq \Phi(\bar{x}, \bar{u}, \bar{v})$ for all $u \geq 0$ and all $v$, it follows that

$$g(\bar{x}) \leq 0, h(\bar{x}) = 0, \bar{u}'g(\bar{x}) = 0.$$ 

Thus, $\bar{x}$ is feasible to the primal. Since

$$\Phi(\bar{x}, \bar{u}, \bar{v}) \leq \Phi(x, \bar{u}, \bar{v}),$$

it follows that $\bar{x}$ minimizes $\Phi(x, \bar{u}, \bar{v})$ subject to $x \in X$. Since $\bar{x} \in I(X), \nabla_x \Phi(\bar{x}, \bar{u}, \bar{v}) = 0$, that is $(\nabla f)(\bar{x}) + (\nabla g)(\bar{x})\bar{u} + (\nabla h)(\bar{x})\bar{v} = 0.$