CS/Math 320 Discrete Mathematics

Fall 2017

Solution to Test 3 5:30PM - 6:45PM

Each problem is worth 10 points.

Problem 1. Find the closed form of the summation below. Show your work.

$$\sum_{j=0}^{8} (2^{j+1} - 2^{j})$$

Answer:

$$\sum_{j=0}^{8}(2^{j+1} - 2^{j}) = \sum_{j=0}^{8}(2 \times 2^{j} - 2^{j}) = \sum_{j=0}^{8} 2^{j} = (2^{8+1} - 1)/(2-1) = 2^{9} - 1 = 511$$

Problem 2. Find $(99^2 \bmod 32)^3 \bmod 15$. Show your work.

Answer: Because 99 ≡ 3 (mode 32)

$(99^2 \bmod 32)^3 \bmod 15 = (3^2 \bmod 32)^3 \bmod 15 = (9 \bmod 32)^3 \bmod 15 = 9^3 \bmod 15 = 9 * 9 * 9 \bmod 15 = 81 * 9 \bmod 15 = (81 \bmod 15) * 9 \bmod 15 = 6*9 \bmod 15 = 54 \bmod 15 = 9$.

Problem 3. Find $123^{1001} \bmod 101$. Show your work.

Answer:

Little Fermat Theorem: $a^{p-1} \equiv 1 \pmod{p}$ for all prime p and a not a multiple of p.

Letting p=101, a=123, we have $123^{100} \equiv 1 \pmod{101}$ and so

$123^{1001} = 123^{10*100} . 123 \equiv 123 \equiv 22 \pmod{101}$

Problem 4. Solve the congruence $4x \equiv 5 \pmod{9}$ using the inverse modulo approach. Show your work, including the steps to find the inverse.

Answer:

Gcd(4,9)=1 and so inverse of 4 modulo 9 exists

9 = 4*2 + 1

1 = 9-4*2

Therefore (-2) is inverse of 4 modulo 9. Multiplying both sides of

$4x \equiv 5 \pmod{9}$

by (-2), we have

$x \equiv -10 \pmod{9} \equiv 8 \pmod{9}$

So any integer congruent to 8 modulo 9 is a solution.

Problem 5. Solve the system of congruence $x \equiv 3 \pmod{6}$ and $x \equiv 4 \pmod{7}$ using the method of back substitution (one of the two methods we discussed in the class, the other method being the Chinese Remainder Theorem). Show your work.

Answer:
Since $x \equiv 3 \pmod 6$, we can write $x = 6k+3$ for arbitrary k. Plugging this expression in $x \equiv 4 \pmod 7$ we have $6k+3 \equiv 4 \pmod 7$ or $6k \equiv 1 \pmod 7$. Therefore, k is an inverse of 6 modulo 7, which implies $k \equiv 6 \pmod 7$. We can express $k = 7t+6$ for some arbitrary t. Any integer of the following form will be a solution of the given congruence:
$x = 6k+3 = 6(7t+6)+3 = 42t + 39$.
Put another way, $x \equiv 39 \pmod{42}$.

Problem 6. How many bit strings of length seven either begin with two 0s or end with three 1s? No need to show your work. Just give the answer.
Answer: Using inclusion-exclusion principle:

A = set of 7-bit strings beginning with 00
B = set of 7-bit strings ending with 111
The answer is $|A \cup B| = |A|+|B|-|A \cap B| = 2^5+2^4-2^2 = 44$.

Problem 7. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? Explain.
Answer: Using pigeonhole principle, let us say the number of enrollment is n. These students belong to 50 states and so there is some state with at least $\lceil n/50 \rceil$ students. Setting $\lceil n/50 \rceil = 100$, we obtain the minimum $n = 50*100+1=5001$.

Problem 8. How many bit strings of length 10 contain at most four 1s? Just give the answer. No need to explain.
Answer:
Num of 10-bit strings with no 1: C(10,0)
Num of 10-bit strings with one 1: C(10,1)
Num of 10-bit strings with two 1s: C(10,2)
Num of 10-bit strings with three 1s: C(10,3)
Num of 10-bit strings with four 1s: C(10,4)
Answer = C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4).

Problem 9. Show that if n is a positive integer then $C(2n, 2) = 2C(n,2) + n^2$ using a combinatorial proof.
Answer: Consider the following counting problem: counting how many ways to choose 2 items from a set of 2n items. There are C(2n,2) ways to do so. Another way to count is as follows. Partition the set into 2 sets: set A with n items and set B with n items. There are 3 scenarios:
1) both items are selected from A: there are C(n,2) ways to choose them
2) both items are selected from B: there are C(n,2) ways to choose them
3) one item is from A and the other from B: there are n ways to choose the first

item and n ways to choose the second item, resulting in n^2 ways to choose them
The total count therefore $C(n,2)+C(n,2)+n^2$.
In conclusion, $C(2n,2)= C(n,2)+C(n,2)+n^2$

Problem 10. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it? Just give the answer. No need to explain.

Answer:
There are 5 types of coins. Use 4 divider lines to separate them
Pennies **|** nickels **|** dimes **|** quarters **|** half dollar

Choosing 20 coins can be represented by finding a way to interleave 20 coins with these 4 dividers, which is the same as choosing 20 position out of 20+4=24 positions. The total count is C(24, 20) or C(24,4).