It's Elementary:
Math for Grades One to a Zillion

Ethan D. Bolker

February 18, 2009

Contents

Preface

This is the proposal, gradually being edited into a preface

When I tell strangers I'm a mathematician (or they guess, because I wear an abacus as a belt buckle) their first response is often "I never could do mathematics." Readers of *It's Elementary* need no longer proudly proclaim that ignorance. They will discover that mathematics, like language, is not just something you study at school. The first two R's help you learn language, but you don't leave it behind at the schoolroom door. You invent language every day, speaking, writing, singing. You can take mathematics with you too, and meet the world in a new way.

For the last five years I've spent a morning a week at the Manning, a public elementary school in Boston's Jamaica Plain neighborhood, where I help kids, teachers and the principal think about math in a new way. *It's Elementary* describes what they and I learned in five years of those mornings. I tell stories using anecdotes, direct quotes and reconstructed dialogue culled from journal entries, email correspondence, and handouts to students and teachers. Many illustrations help make the book reader-friendly.

One of the most rewarding surprises in my Manning work has been how often elementary school mathematics resonates with much deeper ideas I deal with regularly in the other places where I practice my trade – teaching at the university, writing research papers, consulting about software development. *It's Elementary* is also about those resonances, and about becoming a mathematician.

Who might read this book?

Three kinds of readers:

- First, teachers. There are 1.5 million elementary school teachers in the United States. [1] They all teach mathematics, and could all enjoy *It's Elementary*. They don't need to discover it one by one: some of the principals and superintendents in the 100,000 elementary schools and 15,000 school districts[2] will want to recommend the book.

  As the subtitle suggests, *It's Elementary* will appeal as well to the half a million math teachers in middle and high schools[3] and to my 27,000 colleagues in the American Mathematical Society.

  About 200,000 students graduate each year from schools or programs in elementary education. Many are likely to have encountered *It's Elementary* as required or recommended reading during their training.

- Second, parents of the 36 million elementary school kids, or anyone concerned about schools or education, could learn from *It's Elementary*.[4]

[1] From the Bureau of Labor Statistics: http://stats.bls.gov/oco/ocos069.htm#emply.

[2] http://www.edreform.com/index.cfm?fuseAction=section&pSectionID=15&cSectionID=97

[3] Estimated from year 2000 data at http://www.ericdigests.org/2004-1/finding.htm

[4] Three quarters of the 20 million households with school age kids own more than 25 books (http://www.nationmaster.com/graph/edu_stu_fro_hou_wit_mor_tha_25_boo_age_13-than-25-books-age-13) – those are the ones where a copy of *It's Elementary* might find a home.

- Third, *It's Elementary* will appeal to curious general readers, particularly those who want to learn about mathematics. The success in the market of several books with significant mathematical content shows that there are many people who seek that kind of reading.

I don't know of any other book quite like this one. But there are some that resemble it in some ways. The list "Customers who bought this item also bought" on the Amazon.com web page for *It's Elementary* might contain:

- Amir Aczel's *Fermat's Last Theorem: Unlocking the Secret of an Ancient Mathematical Problem* (Four Walls Eight Windows, 1996). In this and his other books Aczel writes about the ideas behind serious mathematics, the history of those ideas, and the personalities of mathematicians. *It's Elementary* is more conversational – personal and present rather than descriptive and historical. It offers detail about everyday mathematics rather than sketches of the most advanced parts of the subject.

- Joseph Mazur's award winning memoir *Euclid in the Rainforest* (Pi Press, 2004). Here mathematical logic and personal history share center stage. *It's Elementary* is also part memoir, but focuses more on the kids and their teachers, and on the math we learned together.

- Lynne Truss' *Eats, Shoots & Leaves: The Zero Tolerance Approach to Punctuation* (Profile Books Ltd, 2003). That this grammar book earned a place on best seller lists throughout the English speaking world proves that the public will read about arcane intellectual matters when they are humorously and well explained.

- Carmen M. Latterell's *Math Wars: A Guide for Parents and Teachers* (Praeger, 2004) addresses the current controversy about elementary school mathematics. What should be taught? How should it be tested? How should teachers be recruited, trained, supported and rewarded? *It's Elementary* suggests answers to some of these complex questions.

- *I Am a Pencil: A Teacher, His Kids, and Their World of Stories* (Owl Books, reprint 2005). Sam Swope describes his yearlong adventure teaching a group of fifth graders to write outside their "regular" curriculum. Publishers Weekly named this a Best Book of the Year. I try to do for mathematics what Swope does for writing. My experiences are broader, perhaps less intense. They come from several years of observation as well as intervention. *It's Elementary* teaches some math too – Swope isn't trying to teach his reader to write.

- Jonathan Kozol's *The Shame of the Nation* (Three Rivers Press, reprint 2006), and Frank McCourt's *Teacher Man* (Scribner, 2006): Two of America's best known teachers reflect on their experiences. I can't really compare my five years spending one day a week in an elementary school to Kozol's passionate advocacy or McCourt's life's work. But readers who enjoyed their stories might appreciate in *It's Elementary* the commitment we share to kids and learning.

It's Elementary is inviting, fun, compelling, unthreatening, and unpolemical, even when it touches on contentious questions. It will not trigger a reader's latent fear of mathematics. It promises to change how teachers, parents, kids and concerned citizens think about the subject, and how it can be taught.

Chapter by chapter

1. Dragged Kicking and Screaming.

   This chapter introduces the school, and me, starting with why I said "yes" when a neighbor told me of a consulting opportunity helping teachers come to grips with a new math curriculum. I promised only to observe and comment, not to tell the teachers how or what to teach. The questions raised for me by what I saw were so compelling, and the atmosphere at the school so welcoming, that I stayed on. That first spring I built cardboard mathematical models with the third grade (I explain how here). Then I describe my growing involvement over the years, leading to what you can read about in the rest of the book.

2. Vocabulary. Talking about Numbers.

   Why pay attention to words in a math class? Isn't math about numbers and shapes? Yes, but you need words to talk about mathematics, and thinking about what the words mean, where they come from, and how they are understood or misunderstood, sheds light on both the mathematics and how to teach it. This chapter looks at words like "equals," "equation," "prime," "zero" and "zillion." Along the way it quotes Humpty Dumpty, Abraham Lincoln, Leopold Kronecker, the Book of Genesis and my grandson Solomon.

3. Geometry.

   Too much of the geometry in elementary school is taxonomy rather than mathematics – learning the *names* of things but not enough about them. My Manning experience suggests that kids are capable of more. They can understand the Pythagorean theorem when it's presented as a puzzle, and learn to think in three dimensions if given the chance. Watching the fifth graders study triangles leads to an interesting question: can you draw an equilateral triangle on graph paper if you must put the vertices at corners of the graph paper squares? This chapter starts to explain why you can't.

4. Explanations, Assumptions and Examinations.

   "Explain your thinking" is the current style in math teaching. This chapter explores the subtleties of asking for and interpreting explanations. When a stumbling student tries to explain how he solved a problem a teacher may learn why he stumbles. When a sharp student explains the same thing a teacher may learn how to solve that problem in a way that never occurred to her. Sometimes "I just know" is the best explanation, not an excuse for not thinking. How do you grade that answer if it's offered by a student taking a standardized test? Much of elementary school mathematics is driven by those tests. That can be good or bad – a debate I return to often. Here I write about hidden assumptions in tests. What might a teacher do when a student convincingly explains an answer but didn't use the method whose mastery the test is supposed to measure?

5. Christopher.

   One year I worked with a particularly gifted second grader. We wrote comments back and forth each week in his math notebook. This chapter talks about what he did and how I responded. Christopher is extraordinary, but not alone in needing challenges beyond the curriculum. Inventing them is hard for teachers who are both busy and tentative about their own mathematics. One of my self-appointed tasks in the classroom is to tailor the problems in the curriculum, making them a little harder or a little easier in order to give each student something he or she can do that's just at the edge of what they might think they can do. For example, one excellent second grade exercise asks kids to collect a dollar's worth of (fake) coins by rolling dice. A four and a three means add a nickel and two pennies to what they have so far. They trade in accumulated pennies for nickels and dimes along the way. I ask those who find this too easy to start with four quarters and roll the dice, spending money until they've nothing left. Trading in dimes and nickels for pennies is harder than the reverse. When they master that I ask them to do the computations in their heads: if there's a dime left and they roll a seven they should return the dime and take three pennies without first trading the dime for ten pennies in order to return seven. When the teachers see me invent variations on the curriculum themes they learn to do it too.

6. The Math Club.

   Every Wednesday morning before school begins I meet with half a dozen kids in the math club. Each year the club's popularity grows. It's been a satisfying challenge to find the time for all the students who want to join – whether or not they are the math whizzes. The principal and I agreed from the start that math club is for the kids who want to be there, not just for the talented ones, or the ones with ambitious parents. I deliberately keep the club topics separate from what the kids are learning in their classes, so it's not tutoring. We learn clock arithmetic, write numbers in different bases, count handshakes and build mathematical models.

7. The Math Wars.

   No book on mathematics in an elementary school would be complete without some observations on the current controversy about the curriculum. Should everyone learn to do long division the good old fashioned way? Does taking the time to let students invent their own algorithms help them learn, or just slow them down? Should we ban flashcards? Does today's "new math" dumb down the curriculum, or spice it up? The heated debate on questions like these in the professional literature, in the popular press, and in parent-teacher conferences is rarely illuminating. This chapter tells stories that prove that complex questions need complex answers.

8. What is the moral of the stories?

   I learned that elementary school teachers are dedicated professionals who work much harder than university professors. Given the chance most are eager and able to learn new mathematics, both to teach it better and for its own sake. I'm proud that I could help them do that, and pleased at how much new mathematics I, too, learned along the way. This chapter speculates on how what I've done and learned can help change the way mathematics is taught in more places than just the Manning. I start by considering how many elementary schools there are in the country. If I could recruit professional mathematicians willing to do what I've done would there be enough of us to make a significant difference? Probably not. What might teachers and parents be able to do for themselves once they know what's possible? I hope this book begins to answer that question.

9. Answers to Some Problems.

   Throughout the book I pose mathematical puzzles for the reader to play with. This chapter has answers to questions like these: What are the factors of zero? Which numbers have exactly three factors? What's a googol? What is the difference between googol and Google? How many ways are there to have a dozen pieces of fruit if you can have apples, bananas and cherries? Can you prove the Pythagorean theorem?

10. Not as Elementary.

    There's a lot less difference than you might think between the math in elementary school and the math that professional mathematicians do. From time to time my work at the Manning led to questions that aren't completely elementary. In order not to disrupt the flow of the narrative this chapter discusses some of those questions, both for innocent amateurs and professional colleagues.

Acknowledgments

Chapter 1

Dragged Kicking and Screaming

1.1 Magic

The Manning school's fifth grade math club crowds around the table. Ben cuts the face down small deck several times. Alejandro peeks at the top three cards and announces their colors: "red, black and black." A chorus sounds "me! me! me!" as the kids vie to name the cards. Lisa blurts out "eight of diamonds, ace of spades, three of clubs." She knows the cards when she's been told only the colors because we've spent the last hour studying the mathematics that makes this possible.

The night before I'd taken my UMass graduate class of prospective high school math teachers to hear Persi Diaconis talk at MIT about "Mathematics and Magic Tricks." He promised (and succeeded) in showing us first this trick and then that "the way a magic trick works is sometimes even more amazing than the trick itself." I woke the next morning with my common Wednesday morning Manning math club anxiety – I hadn't prepared and didn't quite know what we would do. I decided, hesitantly, to try to show the kids the trick and the mathematical secret that makes it work. The story unfolded successfully over the next few weeks. Persi says the secret is only for magicians and mathematicians (the kids in the math club qualify) until he finishes the book he is writing now, so I will leave the explanation as "magic." But I will tell many other compelling stories about the magic in teaching mathematics, and about the magic in mathematics itself.

1.2 How did I get here?

One day in February 2000 I walked down the hall from the principal's office to the second grade classroom and stood just inside the door. The kids were at their tables, writing (no desks in classrooms these days). The atmosphere was a little tense - not as quiet as "at their tables, writing" would suggest. Ms. Fitzgerald looked up from her desk (the teacher has a desk), interrupting her conference with one of the second graders to ask bluntly

"What can I do for you?"

I was a little surprised, and felt a little unwelcome. I only wanted to help (with math) and she was treating me as just another intruder in her busy day - which is in fact exactly what I was. I was lucky: I said

"What can I do for *you*?"

"Here - sit with Julio."

So I sat and listened while he told me the story he was having trouble writing down. I found out that his birthday was July 13. I said I'd send him a birthday card and marked the date in my calendar.

I don't know whether I helped Julio but I think I helped Susan. For a few minutes I dealt with a distraction that was keeping her from spending necessary time with the rest of the class. I may have started to convince her that I was not just another busybody who would try to tell her how to do her job. No mathematics happened, but perhaps the stage was set.

I was in Susan's class that day because a week or so earlier I had a phone call from Judy Manthei, a neighbor who's seriously committed to working in primary education (Ed.D. from Harvard, former elementary school principal and former professor of education, then staff development consultant in the Boston public schools, now teaching at Boston University. She may be retired by the time you read this, but she will be as active as ever). She asked if I might be interested in helping the principal, Casel Walker, help her teachers with the new mathematics curriculum recently introduced by the Boston Public School system.

My first instinct was to say "no." I knew I cared about elementary education, and had some opinions (call them prejudices, since they were literally "prejudgments" based on little evidence). But I thought I had no credibile help to offer. And this wasn't to be volunteer work: Casel had a few thousand dollars in her budget for a consultant. I didn't want to take that money under false pretenses.

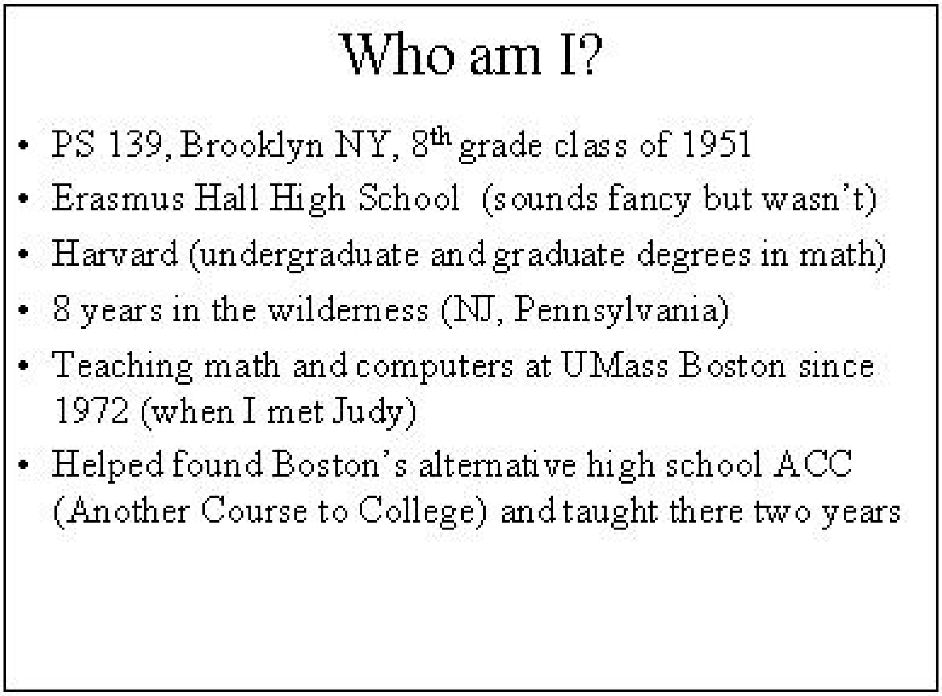
Fortunately, my wife, Joan, was there when when Judy called. I've learned from Joan that when asked to serve on a committee at UMass I am free to say "no" right away, but if I'm tempted to say "yes" I should say instead "let me think about it." In this case she offered the appropriate opposite advice: "I think you should do it, and I can tell you why if you want to know. So say 'yes' right now if you like, but don't say 'no' without thinking about it." First Joan, then Judy convinced me that I might in fact be useful, that I would surely enjoy the job, and that my scruples, while honorable, were extreme. "Why not at least meet with Casel?"

I did. She was very persuasive. As you read on in this book you'll find out how often she managed to get what she wanted for the Manning – which is fortunate for everyone. I'm not sure what she saw in me then but I'm glad she saw it. I hesitantly agreed to try to help. We decided that in a month or two I'd present something to the staff at a morning meeting. In order to prepare I'd look at the curricular materials, and visit some classes, fitting those visits into my teaching schedule at UMass.

Then I walked down the hall to Susan's room to start to figure out how I might help.

1.3 Introductions

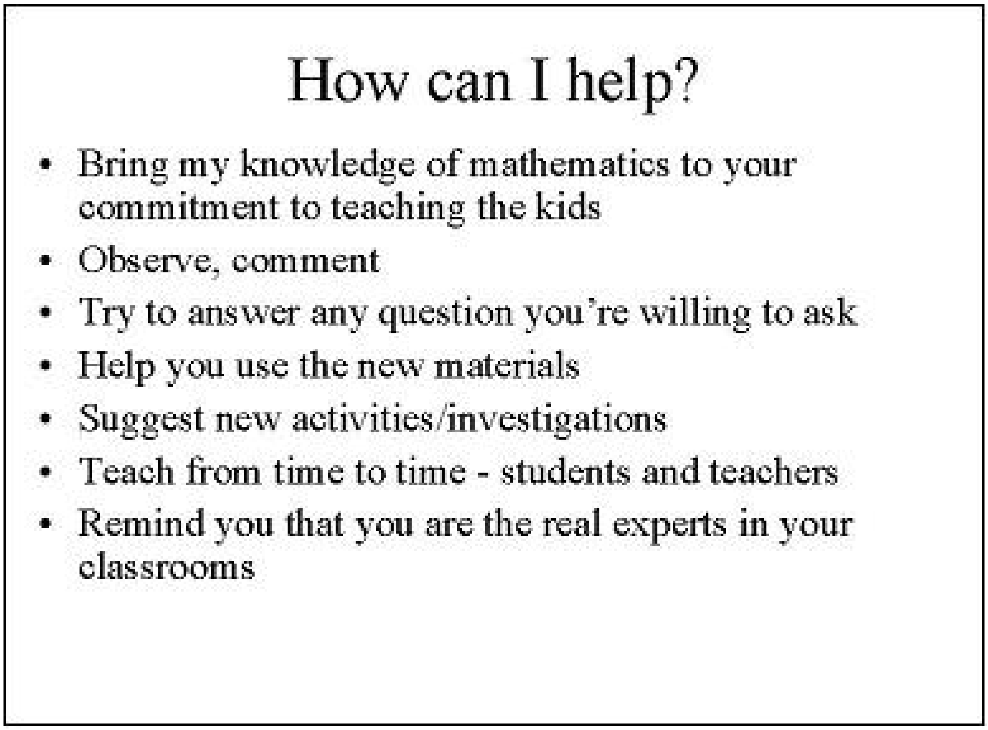
Six weeks later I began my presentation by introducing myself. Here's the first slide:



I reported on what I'd seen in the few weeks I'd been at the Manning, visiting in the second and fifth grades and reading the TERC[1] materials. I talked about the Math Wars, testing, solving problems by thinking about them. I won't reproduce the substance of the presentation here since it will all come out in the rest of this book.

I ended with this slide:

[1] TERC (formerly "Technical Education Research Centers" but now just "TERC") is the organization that authored the curriculum. Find out more about them at http://www.terc.edu/.



1.4 Stellated dodecahedra in the third grade

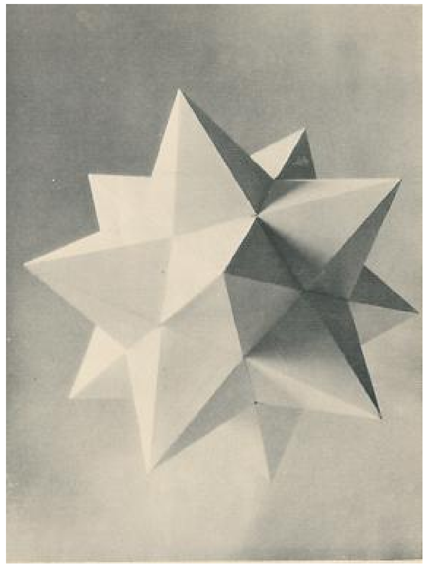
By then it was April. With just a few weeks left in the school year there was no time for more serious work. Yvonne, the third grade teacher, invited me to visit her class. Before I tell you what we did there I will tell you about the book that made me a mathematician.

In my freshman year in High School I devoured Hugo Steinhaus' *Mathematical Snapshots*. I can't remember how I came upon that book. My best guess is that it was a gift from my father, though now I can't imagine how he knew about it. The irony (if it was a gift from him) is that I have always looked on that book as the one that made me a mathematician – a choice he thought would be a disaster since he believed the only proper profession was medicine.

I can trace almost all my original mathematical thoughts to pictures I first saw in Steinhaus. Ideas there often seem elementary but in fact are deep – like much of the mathematics I found at the Manning. [2]

One of the first things I did when I read Steinhaus was to build myself cardboard models of some classical geometric figures. One of my favorites then and now is the stellated dodecahedron. Here's the picture from my 1950 Oxford edition.[3]

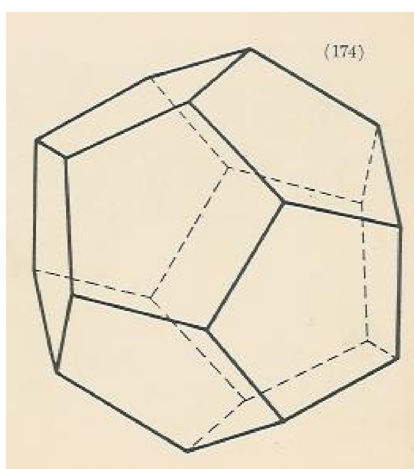
[2] Two other books nearly as influential were Polya's *Induction and Analogy in Mathematics* and Courant and Robbins' *What is Mathematics*, both of which I first read as a High School senior. They are not nearly as elementary.
[3] I need to get permission to reproduce this and the following figures from Steinhaus.



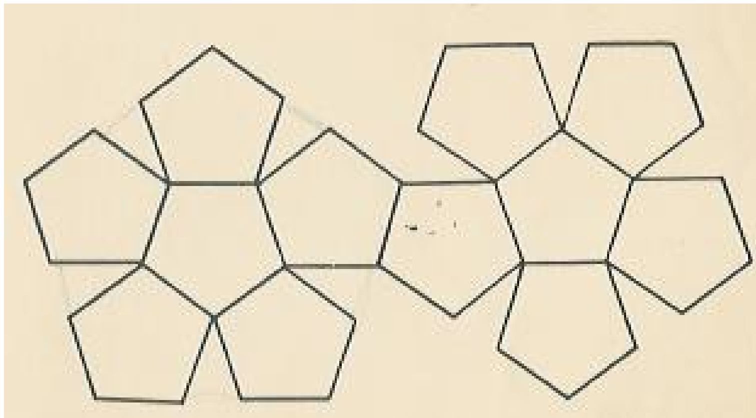
Maybe include a photo of the back of the shirt Joan appliquéd for me? Judy Manthei suggests that for the book jacket.

The stellated dodecahedron isn't part of any curriculum I know of. Nevertheless I rashly decided to build some with the third grade.

I started with the ordinary dodecahedron:



Steinhaus describes how to build one with cardboard and tape by folding this arrangement of five pentagons:



That's what we did the first week. Each kid had a copy of the pattern to cut, fold and tape.

Some were better than others - I hadn't realized how much hand eye coordination is required to use scissors accurately. And the taping is difficult because there's nothing to push against when trying to stick the tape down.

So I encountered yet another instance of something all elementary school teachers know:

> ... the gap between theory and practice in practice is much larger than the gap between theory and practice in theory
>
> Dr. Jeff Case

Reflecting on this experience (while writing, years later) I can understand some of the problems.

Part of the difficulty with scissors and tape is age specific. But some may have other causes. My grown daughter is as enamored of craft projects now as she was as a child herself. She does them with my grandchildren. It's clear watching them that they will be competent enough for this task by the time they reach the third grade. The kids who don't have a home environment like theirs can learn the craft skills in school, even if not as well – but only if its part of the curriculum. I have seen scissors practice in the first and second grades at the Manning, but perhaps not enough. Judy Manthei wonders somewhat ominously if less time is spent on these projects because more is devoted to test preparation.

Part of the problem was insufficient testing. My subsequent math club experience (which you can read about in that chapter) convinced me that these third graders would have done better if they'd had cardboard rather than paper, and tabs to glue rather than edges to tape.

But that was in the future. In spite of my blunders, most of the kids did succeed in building a dodecahedron, even though some were pretty lopsided. Once done they were surprisingly rigid.[4] Yvonne and I needed to restrain the boys from using them as footballs.

But my goal was not this dodecahedron but its stellated cousin. Given the struggle in that first week what I'd planned for the next week clearly wouldn't work. I needed literally to go back to the drawing board. Fortunately, I was able to figure out a design just doable enough so that each kid ended up with a model to take home. You can find the details in the chapter on answers to the problems.

I'm glad we did this project (so was Yvonne). Neither of us can say what the students *learned* in any measurable sense. But their enthusiasm and their ability to focus on a complex frustrating task for a full hour in each of three successive weeks made it worth the time.

I don't think I have the nerve to try it again in the third grade. But older kids could probably manage it.

[4] I shouldn't have been surprised. *Alexandrov's Theorem* on convex polyhedra predicts that rigidity.

1.5 The rest is history

My presentation to the staff and my third grade Wednesdays rounded out the consulting I was hired for that spring. But I'd had so much fun I wanted more. I decided to spend some of my sabbatical time the next fall at the Manning. The work continues to be play [5] so I have arranged to have it count as part of my teaching load at the university as a part of the UMass Boston urban mission.

In the years since then I've spent a morning a week at the Manning, visiting the first and second grades regularly, the third, fourth and fifth grades intermittently. I run the before school math club. Since the math club meets in the science room I spend some time talking with the science teacher about teaching science. Twice my software engineering students from UMass have written applications for the teachers: TWIST teaches kids keyboarding skills, KICS is a Kindergarten Information Control System.

This book about my experience was my next sabbatical project. I was not surprised to discover that writing about elementary mathematics and how to teach is harder than writing just about mathematics, since it must address both the mathematics and the audience (kids, teachers, readers of this book). In the event, as with the Manning work itself, I found it fun and rewarding. I hope you do too.

[5] "The work is play" is from Robert Frost's "Two Tramps in Mud Time." Maybe I can find a place in the book for the whole poem.

Chapter 2

Vocabulary – What's in a Name?

> 'I don't know what you mean by "glory,"' Alice said.
>
> Humpty Dumpty smiled contemptuously. 'Of course you don't – till I tell you. I meant "there's a nice knock-down argument for you!"'
>
> 'But "glory" doesn't mean "a nice knock-down argument,"' Alice objected.
>
> 'When *I* use a word,' Humpty Dumpty said in rather a scornful tone, 'it means just what I choose it to mean – neither more nor less.'
>
> 'The question is,' said Alice, 'whether you CAN make words mean so many different things.'
>
> 'The question is,' said Humpty Dumpty, 'which is to be master – that's all.'
>
> Lewis Carroll, *Through the Looking Glass*

2.1 Equals

In the first grade the equals sign in the expression

$$3 + 7 = 10$$

means "the answer is." The plus sign means "put the 3 and the 7 together."

This makes perfect sense. It's also just the sequence of keys you'd press on a calculator to make it compute the 10 for you. (Of course the first graders don't use a calculator.) But there's more to the equals story.

One day I asked Cynthia, the first grade teacher, "Would it be OK to write $10 = 3 + 7$?"

"I don't know. Why would you want to?"

"Well for a mathematician, 'the answer is' isn't the only meaning for 'equals.' The first thing I think when I see the word or the equals sign is 'these things are the same.' So for me $10 = 3 + 7$ is just as correct as $3 + 7 = 10$."

"I never thought of it like that. It makes sense. But I don't think I want to use = that way with my first graders."[1]

Despite her reply, she and the other teachers seem to welcome my sometimes pedantic dissections of everyday vocabulary. It helps them see the mathematics clearly for themselves; they can then decide when and how to use that understanding in their teaching. So one service I offer is simply raising vocabulary awareness.

In fact, thinking about the vocabulary of mathematics is important for me as a mathematician. Just recently (January 2008) while reading a review of William Byers' *How Mathematicians Think* (Princeton

[1] My friend and occasional consultant Judy Manthei, reading this part of the manuscript, noted that I'd stumbled on an issue Piaget addresses. The ability to see two problems or expressions as essentially the same is in part developmental. Kids can't do it until they're ready, no matter how well they're taught.

University Press, 2007) in the December 2007 *Notices of the American Mathematical Society* I realized that at least part of the ambiguity in the meaning of "=" is already present in the meaning of "+": does "$1+1$" mean "add 1 to itself" or does in name the number that results from that operation?

Why pay so much attention to the words in a math class? Isn't math about numbers and shapes? Well, yes, but we need the words to talk about the mathematics (even though sometimes a picture can wordlessly convey 1000 times more, which is why there are many in this book). With everyday words like "equals" we rely on the everyday meanings that everyone assumes are perfectly clear. But sometimes they're not. That's an opportunity for thought. [2]

2.2 Equations

Our examination of "equals" showed that in mathematics, as in ordinary English, a word can have several useful meanings. Sometimes there's one clear meaning, but several words for it. One of the first things I noticed in the new math curriculum is that equations are always called "number sentences." That's good and bad; let's look first at the bad.

The phrase "number sentence" is clumsy. Why replace one word with two? A second dictionary definition of "neologism" should make us pause:

> neologism
> 1 : a new word, usage, or expression
> 2 : a meaningless word coined by a psychotic
>
> Merriam-Webster Online Dictionary

The curriculum designers who invented "number sentence" aren't psychotic. I think they are trying to honor an unnecessary and unhealthy wish for precision, based on an abstract view of mathematics reminiscent of the "new math" of the sixties. Even if the words were better, it's a mistake to propagate them this way. Kids should learn the words the world uses every day.

"Number sentence" sows confusion. If you search at the teacher-to-teacher web site `mathforum.org/t2t/` you can join a discussion that began years ago. It starts with this query:

```
From: Melissa
To: Teacher2Teacher Service
Date: Nov 13, 2001 at 17:53:08
Subject: Difference between "equation" and "word sentence"
   I am having a difficult time trying to understand the
difference between an equation and a number sentence.
```

No one at t2t has offered a satisfactory answer.

What's good about "number sentence"?

Synonyms enrich language. Using "number sentence" for "equation" reminds us that "$3 + 7 = 10$" really is a sentence; "equals" (meaning "is the same as") is its verb.

When school kids learn to write they learn that sentences should end with periods. That's true even when they are number sentences, written with mathematical symbols. Every teacher I've suggested this to agrees, and tries – for a while – to implement the idea in class, putting periods at the end of number sentences she writes on the board, and asking the kids to do the same. [3]

Finally, "number sentence" and "equation" aren't exactly the same. When the curriculum turns to inequalities there is a difference between them.

$$3 + 7 = 10$$

[2] For still more thought about "equals" see the *Not as Elementary* chapter.

[3] Then they stop. They have more important things to remember, and I don't want to wear out my welcome by constantly reminding them.

is both an equation and a number sentence while

$$3 + 6 < 10$$

is a sentence about numbers, but it doesn't assert that two numbers are equal.

Fortunately, in the time I've spent at the Manning the teachers have begun using both expressions. That can lead to the best of both worlds.

2.3 Zero

Where do the numbers start? Bonnie, the special ed teacher, is working with the first grade. [4]

"Do you think zero is a number?"

Chorus of "yes" answers.

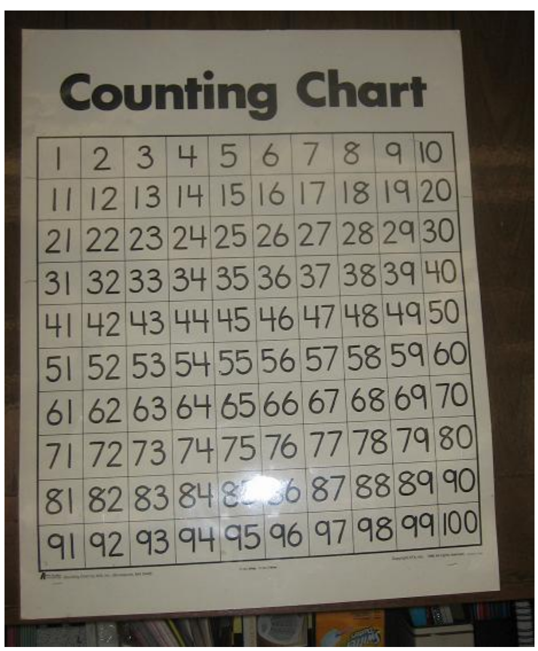
"Please try to remember that in first grade we raise our hands when we have something to say. Curtis?"

"Zero kids aren't here today!"

"Excellent! Zero is a number, and a very useful one too. Look at the number line above the whiteboard. It starts with zero before the one. Some of you know about negative numbers. We need the zero to separate the positive numbers from the negatives."

I was delighted to hear about 0 so early in the curriculum. The kids regularly asked whether they could use it when asked to find combinations that sum to 10. It's one of the values on the playing cards they use for arithmetic games. Bonnie's casual reference to the negative numbers surprised me. They aren't on the posted number line. But the kids didn't seem to mind. I think some understood and appreciated it while others (who didn't yet know about negative numbers) properly ignored it. After class I told Bonnie that it's tricky to figure out how many years separate an AD from a BC date because there's no year 0 in our calendar. Year 1 AD comes right after 1 BC.

Many classrooms display a "hundred's chart" which looks like this



[4] The Manning is the designated school in its district for kids who need special help. So there are two classes in each grade, one for most kids, another, much smaller, for those who need extra attention. Whenever possible - and that means usually for math - the classes meet together.

Since the kids know about zero, I suggested to the teachers that the hundred's chart should start there. Unfortunately, changing how suppliers of classroom materials print them isn't an option. Fortunately, some come with pockets to put the numbers in. Cynthia changed hers right away:

Here's why I recommend the change.

First, it never hurts to reinforce the presence and place for zero. It's on the number line. This puts it on the hundred's chart too.

Second, the second through last rows model better the way kids count: "$20, 21, 22, \ldots$" rather than "$21, 22, 23, \ldots$."

Third, all the numbers in a row have the same digit in the tens place – the thirty-somethings are all together. That's useful when teaching place value.

For more on starting the hundred's chart at 0, see the discussion in the chapter on technical matters.

There's not really much of a problem - in the twenty first century - accepting 0 as a number. Mathematicians and other people who, like the kids at the Manning, choose to think about the question pretty much agree.

Sometimes it's hard to remember how strange and difficult a new concept can be before it's seen as obvious and inevitable. People did serious arithmetic for centuries before some Indian mathematician realized how useful a number nothing could be. The names for some kinds of numbers – "negative," "irrational," "imaginary" – hint at their controversial history.

2.4 Prime

Sometimes clarifying a definition points a way to deep and interesting mathematics.

In math club we were thinking about primes. Celin's list began

$$1, 2, 3, 5, 7, \ldots$$

"Why is 1 in the list?"

"Because the only things that go into it are 1 and itself."

That's how most people define "prime," and that definition includes 1. But mathematicians would rather not call 1 prime. Celin and her math club buddies deserve to know why. They're entitled to something more than "our teacher told us so."

Explaining my reason requires a rather long digression. The fifth grade spends time thinking about the *factors* of a number: the numbers that go into it evenly. The factors of 24 are $1, 2, 3, 4, 6, 8, 12$ and 24 itself.

What are the prime factors of 24? Celin would say 1, 2 and 3, or, if she believed me, just 2 and 3. I think the answer should be "$2, 2, 2$ and 3," omitting the 1 but listing the 2 three times.

To see why let's explore "factor" as a verb as well as a noun. To factor 24 means to write it as a product of other numbers. You might begin this way

$$24 = 4 \times 6 \ .$$

Then continue, factoring each factor as long as you can:

$$\begin{aligned} 24 &= 4 \times 6 \\ &= (2 \times 2) \times (2 \times 3) \\ &= 2 \times 2 \times 2 \times 3 \ . \end{aligned}$$

Your work stops when all the factors are primes. And you can see that you have three 2's, one 3, and no 1. [5]

You might have started differently:

$$\begin{aligned} 24 &= 3 \times 8 \\ &= 3 \times (2 \times 4) \\ &= 3 \times 2 \times 2 \times 2 \ . \end{aligned}$$

Whatever path you follow, you will finish with the same list of primes, perhaps in a different order. So 24 (or any other number) is simply the product of all its prime factors (for 24 those are $2, 2, 2$ and 3) – as long as you allow repetitions in the list.

That's an important fact about arithmetic. Mathematicians give it an important name, the "Fundamental Theorem of Arithmetic. " It says:

> *Every integer greater than* 1 *can be written in just one way as a product of prime numbers. The only freedom you have is in the order of the factors.*

Mathematicians prefer not to count 1 as a prime because it plays no role in the fundamental theorem. It never comes up when you factor a number into primes.

I wonder if people are surprised when they discover (as we just did with 24) that when you factor a number all the way to primes you always get the same list (perhaps in a different order). Were you? Perhaps it seems obvious. It's not. The ancient Greeks proved it, although they are better known for their geometry than for their arithmetic.

The math club tolerated my excursion, but I know they didn't appreciate the elegance of the fundamental theorem of arithmetic. [6] And it didn't really satisfy Nathan's wish for a definition of "prime" that excluded 1 in a natural way. He did like "a number with exactly two factors." He could see that 1 has exactly one factor, primes have two and all the other numbers have at least three. [7]

This vocabulary story continues in the fifth grade, when a student teacher asks the class "What is the definition of a prime number?"

Alejandro, who's in the math club, volunteers "A number that only 1 and itself go into, except that 1 is not a prime."

"Why isn't 1 a prime?"

"Because Dr. Bolker says so."

So much for my explanation of *why* 1 should be excluded, and my new definition that excludes it, and my wish that people own their mathematics rather than see it as handed down like the tablets on Sinai. But

[5] This chain of equations makes sense only when you think of "=" as "is the same as" rather than "the answer is."

[6] A student's definition of mathematical elegance: "something that's short, difficult to understand, and that the teacher seems particularly fond of."

[7] Exercise for the reader: The factors of 4 are 1, 2 and 4. Describe all the other numbers with exactly three factors. You can find the answer in the back of the book.

perhaps I shouldn't be surprised. It takes time for all of us – not just fifth graders – to absorb new ideas. While we're doing that it's natural to fall back on an appeal to authority.

Yvonne, the fifth grade teacher, was intrigued when I did explain my reasons to her, after class. I don't think it had occurred to her that definitions were not the same as mathematical truths. They are just decisions you make about what to call things - but what you call them doesn't change their nature.

So in a sense we are free to define words as we please, as long as we honor our own definitions when we use those words to discuss the "facts" of the world. [8] That's the essence of what Humpty Dumpty was saying in the quotation that starts this chapter. But some definitions are better than others. Abraham Lincoln knew that when he asked

> If you call a tail a leg, how many legs has a dog? Five? No, calling a tail a leg don't make it a leg.

The only way calling a tail a leg could make it a leg would be to change the definition of "leg" so that it no longer meant just "appendages designed for locomotion." [9] Changing the definition that dramatically would be almost as bad as Humpty Dumpty's redefinition of "glory."

I've strayed from vocabulary lessons to discuss the fundamental theorem of arithmetic and the arbitraryness of definitions. But that's part of the point I hope to make: in order to choose good names for things we need to understand first what they are and how they behave. My students at UMass are often surprised when I tell them that a mathematician can only define a new concept *after* she understands it, not before. Unfortunately, mathematical exposition in both textbooks and research monographs often *begins* with definitions. That may be logical, but it's neither historical nor psychological nor a good way to teach concepts.

2.5 Etymology

Knowing where a word comes from can help unravel its meaning. "Forty" is just a quick way to say "four tens." "Fifty" is "five tens." Kids in first and second grade know the words and know the representations "40" and "50". Telling them the connection (which they find cool) lets them use what they know about single digit arithmetic to compute with bigger numbers

$$\begin{aligned} 40 + 50 &= 4 \text{ tens} + 5 \text{ tens} \\ &= 9 \text{ tens} \\ &= \text{ninety} \\ &= 90\ . \end{aligned}$$

Some kids come to first grade knowing this. But for most, who don't yet, it's a way to teach them. Here's a conversation with Jack, struggling with just this problem.

"If you have four apples and I give you five more apples how many apples do you have now?"

"9"

"Nine what"

"Apples"

"If you have four tens and I give you five more tens how many do you have now?"

"Nine"

"Nine what"

"Tens"

"If you say 'nine tens' fast what number does it sound like?"

"Ninety."

[8] Nathan really likes Enzensberger's *The Number Devil*. I can understand why. But I wish the author hadn't gratuitously renamed all the lovely mathematical ideas he discusses. For him a prime is a "prima-donna." Granted, he knows what he's done, and provides a kind of out (p 254). And 1 is not a prima-donna, although he doesn't say why he wants to exclude it.

[9] Forgive the use of the word "designed" – that's a controversy for another book, on biology ...

"Can you write that as a number?"
"Sure."
"Can you write the number sentence you just figured out?"
Jack writes

$$40 + 50 = 90 \ .$$

Hidden here is an application of the arithmetic fact mathematicians know of as the distributive law:

$$(4 + 5) \times 10 = 4 \times 10 + 5 \times 10 \ .$$

You may remember that law from algebra as "multiply out the parenthesis." If you turn the equality around

$$4 \times 10 + 5 \times 10 = (4 + 5) \times 10$$

you can see how Jack used it (implicitly) to find $40 + 50$ when he knew $4 + 5$. [10]

Etymology helps too with the word "times" for multiplication: "six times seven" is really "six, seven times". [11]

2.6 Fractions

Pluggers, Boston Globe, March 29, 2007
permission to reproduce pending

By third grade (often by first grade) kids know that four quarters make a dollar, and that each quarter is 25 cents and that a dollar is 100 cents. But some have trouble with the question "how much is a quarter of 100?" when it comes up in the classroom in a unit on fractions. I'm not surprised when kids can't connect math in school with everyday vocabulary. Would it help if we made explicit the fact that it's no accident that we use the word "quarter" in two different ways: the quarter coin is a quarter of a dollar? I don't know. Only teachers trying it out can answer that question.

But there are more subtle questions about fractions that thinking about meanings can clarify. One reason fractions are hard is that an expression like "3/5" has at least five meanings – related, but distinct.

[10] Written backwards this way the distributive law tells that 10 is a *factor* of 40 + 50.

[11] If you happen to be familiar with reverse polish notation (which is unlikely unless you are a computer scientist or used to own an HP scientific calculator) you will appreciate the value of writing the operator after the operands – "6 7 ×" – instead of the usual "6 × 7". If you're not, skip this confusing footnote.

The simplest is that it's a shorthand description of what we get when we divide something into five parts and take three of them. But some time around the third grade it acquires a second meaning. Suddenly it's a *number* in its own right. Since the kids have implicitly defined "number" (for themselves) as "whole number" it's hard for them to imagine numbers between zero and one, except perhaps for a half. Number lines would help here, but that visual aid seems to vanish from classrooms after the first and second grades.

Familiarity with calculators suggest a third meaning – not a number, but instructions to find a number by dividing three by five. The answer appears as a decimal, of course. Since calculators are ubiquitous perhaps decimals should be taught sooner than they are. Unfortuntately, finding the decimal fraction 0.6 on a number line marked as a ruler is even harder than finding 3/5, unless it happens to be a *metric* ruler.

The fourth meaning is related to the second. "3/5" is a number expressed in a particular form: one integer "over" another. Mathematicians call those *rational* numbers. [12] The emphasis is on the ability to think of the top and the bottom of a rational number separately (the top and the bottom are the numerator and the denominatory in a vocabulary lesson). To manipulate fractions correctly you must learn some new, complex rules for computing tops and bottoms.

It's not too hard to explain the rule for adding fractions using the cut up a pie metaphor. If you have 1/3 of a pie and 1/2 of a pie then you can draw a picture to convince yourself that you have 5/6 all together. [13] It's harder to explain the rule for dividing fractions without moving toward a fifth interpretation of "3/5" as the answer to the question "what do you multiply 5 by to get 3?"

Finally, this is my chance to tell the world that I think the person who invented our notation for fractions got it upside down. The first meaning of "3/5" is "cut into five parts and take three of them" but when we read from left to right and top to bottom we see the three before the five. Conceptually the five should come first – the number of pieces into which the pie is cut is knowledge you need before you can make sense of how many of them you get. The problem recurs with the rule for adding fractions, where it's counterintuitive to remember that you have to calculate the bottom *before* the top. [14]

2.7 Zillions of numbers

Kids like big numbers.

In math club Nathan wants to know how the sequence "thousand, million, billion, trillion, ..." continues. However far I go he wants to know the next. I try to teach them the more useful prefixes that come with the metric system ("kilo, mega, giga, tera, peta, ...") but they aren't interested. They want to write lots of zeroes. Some of them even know about a googol. [15] But they all know about a zillion. It's a number with *lots* of zeroes. It must be really big, because its name begins with the last letter of the alphabet.

There's even a definition on the internet:

> Zillion
> A generic word for a very large number. The term has no well defined mathematical meaning.
>
> `http://mathworld.wolfram.com/Zillion.html`

I wondered whether other languages had an equivalent, so I asked a friend in Florence. Here's his answer.

```
From: "Batterman, Henry" <batterman@gonzaga.edu>\index{Batterman, Henry}
To: <eb@cs.umb.edu>
Subject: R: zillions
Date: Sun, 1 Jun 2008 06:29:16 -0700
```

[12] That there are numbers that are not rational – *i.e.* not fractions – is not elementary. It is disturbing. That's why those other numbers are called *irrational*. Read more about irrational numbers in the chapter on technical matters.

[13] In Chapter 5 we'll see one stunning second grader figure that out for himself.

[14] Later on when you learn calculus you learn that in "dy/dx" the numerator dy is the change in y caused by the change dx in x. The fraction is upside ddown there too – reading from top to bottom you see the effect, dy, *before* the cause, dx.

[15] If you don't, you can google it on the internet, or look it up in the back of the book.

```
Yes there is, though I hadn't heard of it until you asked.  I asked
Giulio, a big fan of Topolino, or the Mickey Mouse comic books and he
remembered from his reading the word "fantasatiliardo"  which  is a
combination of    FANTASTI(CO) and (MI)LIARDO      Miliardo is the
Italian for billion.  The other is  "fantastilione" , union of
fantastico and milione.  The dictionary says the word was coined by
translators of Mickey Mouse.  Unlike English however, the word seems to
refer almost exclusively to enormous sums of money. \index{Mickey Mouse}

Henry
```

But the last words belong to my then four year old grandson Solomon, who asked me

"How big is a zillion?"

"A zillion really isn't any particular number. It's just a word we use for a really big number."

"But how do you know that if you keep counting you won't come to a zillion? If there are no people in the world some day will there still be more numbers, even without people to count them?"

. . . which leaves me speechless. He knows the numbers go on forever, and he's innocently raised serious philosophical questions about distinguishing between numbers and their names and about whether mathematics is invented or discovered.

Those questions are not at all elementary. You could begin to ponder them with these quotes.

> And out of the ground the LORD God formed every beast of the field, and every fowl of the air; and brought them unto Adam to see what he would call them: and whatsoever Adam called every living creature, that was the name thereof.
>
> Genesis 2, verse 19
> (King James bible translation)

> God made the integers; all else is the work of man
>
> Leopold Kronecker (December 7, 1823 - December 29, 1891)
> quoted in E. T. Bell's *Men of Mathematics*, 1986 (reprint)
> p. 477)

Chapter 3

Geometry

Quote for this chapter needed. Kronecker or Genesis? Find something geometrical. Maybe a picture? Pythagorean Theorem?

3.1 Taxonomy

A large part of the TERC math curriculum in the early grades is about classification. Lots of attention paid to attributes and categories, Venn diagrams.

Taxonomy is important. Sorting things into categories is a useful skill. There's subtlety involved in understanding, teaching, finding words for the distinction between the number of categories and the number of items in each particular category.

The taxonomy of polygons is a favorite example.

Pattern blocks

Sample exercises:

vocabulary

what's my rule?

I think these are good exercises. My complaint is that they are labelled as teaching geometry. But there are no theorems to go with the names.

I think that's a missed opportunity.

In the first or second grade the theorem that the diagonals of a rectangle are equal (and perhaps the failure of the converse, since they know about trapezoids) is within reach. Fifth grade should be able to do more.

3.2 Polygons – first grade

```
November 19, 2008. Second or third class with \Marissa, the new first
grade teacher (\Cynthia\ has taken a leave of absence). I watch the
kids filling in outlined shapes with pattern blocks.

They are pretty good by now at the terminology.

\Marissa\ agrees with me that the long white rhombus is out of place
in the set. In fact, the orange square is too. They allow for
interesting pictures, but they get in the way of building shapes
based on the tesselation of the plane into equilateral triangles.

After class we discussed the fact that some kids are good at tracing
```

```
the patterns they've constructed - they know that the pattern really
has vertices and edges. They can accurately trace the shapes. Others
draw pictures that are more or less random (include an example here,
real or made up - with shapes sprinkled through the outline). What's
interesting is that some kids can improve their work when you
``teach'' them, but for others instruction goes right by them. There's
a mix of intellectual level and developmental level that is very hard
to untangle - particularly by one teacher in a room of 20-25 diverse
six and seven year olds. This rather makes a hash of frameworks and
goals and assessments based on standardized tests!

With one particulary competent girl I suggested trying to build a
second pattern on top of the first -- one that covered all the
internal edges of the first - impossible with the white diamonds used
for the dog's ears and tail.
```

3.3 Polygons – fifth grade

The fifth grade class is restive, unruly, unfocussed at the start of the day's one hour math lesson.

Yvonne firmly reminds them about proper behavior.

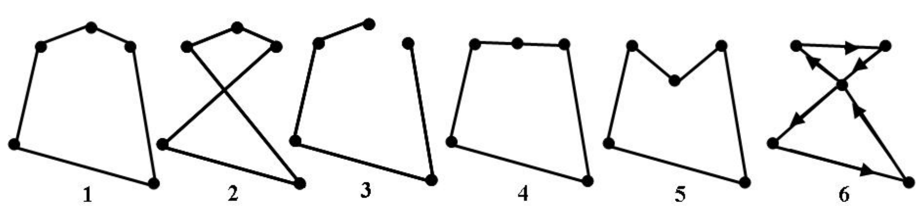
Terel is jumpy: "Sometimes raising your hand is not the thing to do because your hand gets tired from having it raised all the time when you're not called on so you just have to shout out." [1]

What's going on, besides no mathematics? I don't doubt Yvonne's ability to manage a classroom, given what I watched her do with third graders building dodecahedra a few years ago. Maybe it's fifth grade hormones, maybe it's because fifth graders are seniors at the Manning. Whatever the reason, it reminds me of how much of an elementary school teacher's job is teaching the kids how to work together, at whatever age appropriate task they're set, and how hard a job that is. Much harder than mine, teaching at the university.

Finally, order restored, she resumes her promised "five minute review" of polygon vocabulary. She asks for definitions of

- polygon
- regular
- vertex/vertices
- angle
- coordinate grid
- ordered pair [2]
- internal line

Which of the following figures is a polygon?



[1] Sure enough, later on in the class he isn't seen or heard when he wants to say that he has no more empty pages in his math journal to write in.

[2] This first appears on the whiteboard as "order pair". Yvonne thanks me when I correct it.


```
Casel noted that the whole fourth grade
stumbled on a "rhombus" question on the mcas last Spring - she/we will
investigate whether it's a naming question or something with
mathematical content.
```

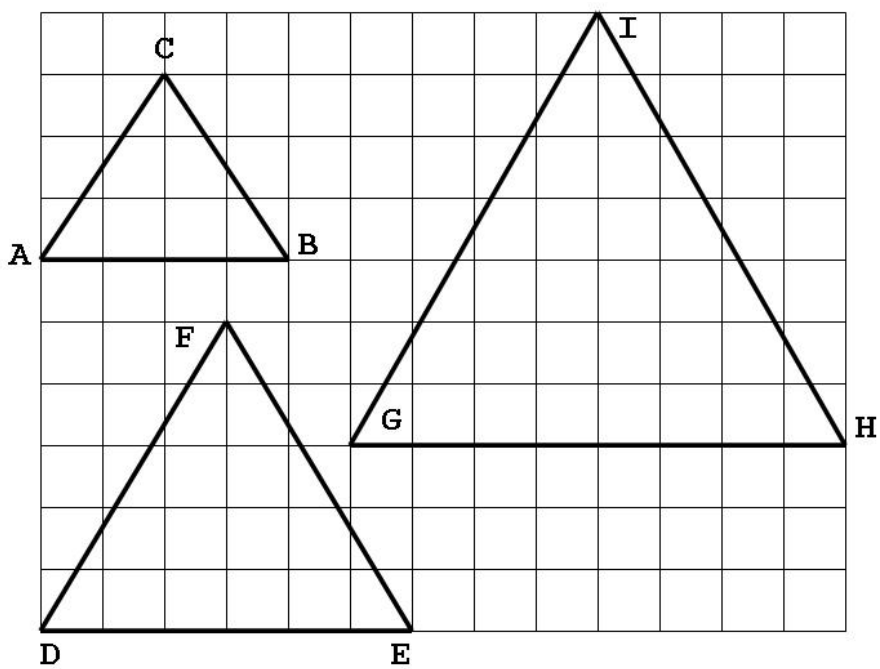
Now there's just twenty minutes left for the new material, on equilateral triangles.

3.4 Equilateral Triangles

The students are given small paper cutouts of polygons to classify. To decide whether one of them is an equilateral triangle they measure the sides with a ruler. When the measurements seem to agree they declare success. They show no sense of error or error analysis.

In my woodshop I try to let things measure themselves when that's possible. If I need two boards the same size I first cut one. Then I lay it on the second and mark where it ends as the place to cut next. Working with Davis I suggested a similar strategy for testing her triangle. Rather than measuring and recording lengths to compare, try folding the triangle and comparing the actual sides. This makes sense to her. It does leave unaddressed the interesting questions about mathematical proof versus "proof by measurement."

The kids have sheets of coordinate paper to play on. The integral lattice points are prominent. They draw polygons and then classify them. Looking for equilaterals they quickly find



(I have added the letter labels to the vertices.)

These look "pretty good" but I know they're not really equilateral. We measure them. With one inch squares the measurements are

$$
\begin{aligned}
AB &= 4 \text{ inches} \\
DE &= 6 \text{ inches} \\
GH &= 8 \text{ inches}
\end{aligned}
$$

(these are exact and easy) and

$$\begin{aligned} AC &= BC &= 3\frac{5}{8} \text{ inches} \\ DE &= EF &= 5\frac{7}{8} \text{ inches} \\ GI &= HI &= 8\frac{1}{16} \text{ inches} \end{aligned}$$

Then it's off to the computer with the usual suspects - the fifth graders who happen to be in the math club: Alejandro, Davis, Nathan, ...

We put each of these figures into Geo-Logo and ask for the distances between the points - and see values rounded to integers! So these triangles which our measurements tell us are only almost equilateral seem to be exactly so in software.

I'm angry at the software, but that's not very useful. I try to turn it into a good lesson in not trusting the computer (too much). I tell the kids how to compute the distance between two points in the coordinate plane. They don't know the Pythagorean theorem but they cheerfully believe me when I tell them the formula. They are comfortable finding square roots with their calculator.

(Further exploration after class lets me set the number of decimal places.)

We didn't carry this any further in class. But I did some more work on the problem – you can see it in the answers chapter.

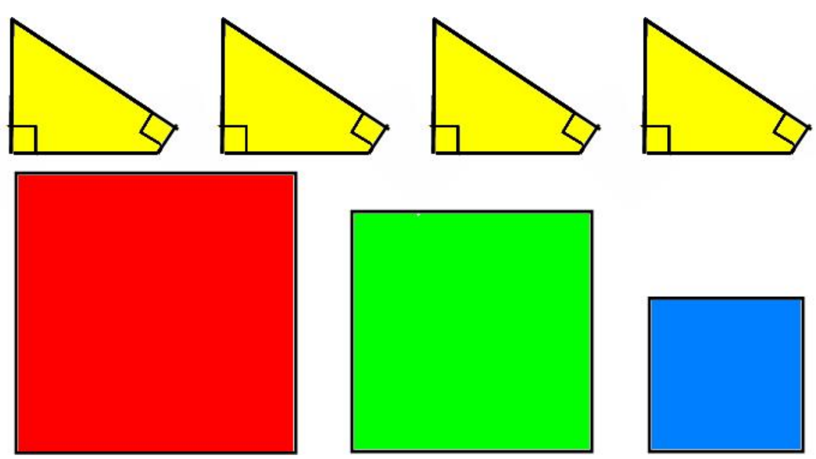
What lessons should we draw from what happened?

- Too much reliance on measurement?
- Why don't they know/learn the Pythagorean theorem?

Perhaps this is a good place to discuss the down side of concrete manipulatives. They can prevent the kids/teachers from forming abstractions appropriate to the developmental level. Mathematics isn't really about the manipulatives. It's about the ideas - the Platonic entities the manipulatives stand for. But it may be too easy to think that the manipulatives *are* the mathematics.

3.5 Pythagoras

By fifth grade I'm sure that the Pythagorean theorem is within reach. And I mean geometrically. I gave this puzzle to Christopher:



Cut out these pieces. First arrange the four yellow quadrilaterals and the blue square so that they just cover the red sqare. Then arrange the four yellow quadrilaterals so that they just cover the green square.

From the geometry it's not hard to get the algebra. Square roots aren't really a problem for kids with calculators, and that's a really good thing. No need to rely on integral Pythagorean triples.

This is a set of manipulatives that actually serves to prove an abstract theorem.[3]

I used this puzzle in math club too. This should really be the math club discussion. Which chapter does it go in?

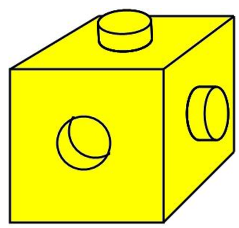
3.6 Symmetry

Second grade. Symmetry with pattern blocks. Ask Daniel about two, three, four and sixfold rotational symmetry (everyone does bilateral).

The diamond and the square are red herrings. Good for making cool pictures but bad/frustrating for teaching geometry.

3.7 Snap cubes

One of the most frequently appearing manipulatives in the first grade are the *snap cubes*.

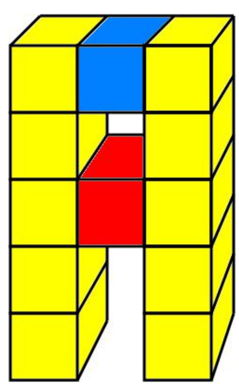


Check that this is really what they look like - how many male and female faces are there in each of the two kinds? See footnote in math wars chapter where I discuss the two kinds. Perhaps combine that discussion with one here?

The kids use them when they need things to count (they make sticks of 10) and for patterns (see later). (One problem is that they are so much fun to build with that counting lessons often deteriorate/morph into building - and in fact girls build houses while boys build robots or space ships.)

I suggested to Cynthia that she have the kids build letters from snap cubes.

I pointed out that the 'A' would look funny since you couldn't easily build diagonal lines:



She said that would be fine since the kids knew about fonts. Interesting what the computer age has made possible/common.

[3] You do need to do some old fashioned geometry to know that the puzzle really does fit together as it seems to.

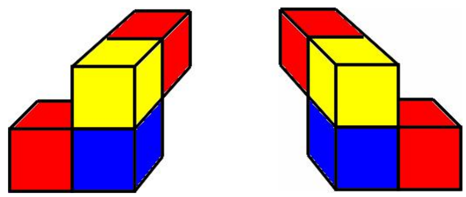
3.8 Three dimensions

There's not really much 3D geometry in the standard curriculum.

One day I showed first/second graders how to draw a cube, then an arbitrary prism, so that it looked three dimensional.

Details go here

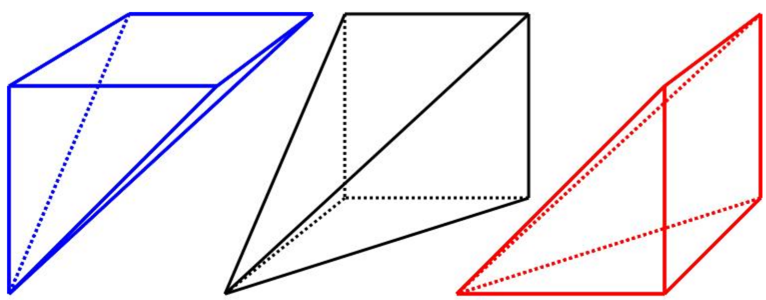
Three dimensional mirror symmetry is difficult/interesting. Asking kids if these are the same provokes useful discussion:



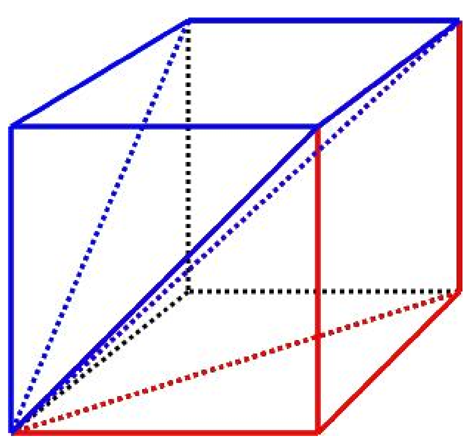
3.9 Dissection

The three dimensional geoblocks have the wonderful dissection of a cube into three congruent pyramids. That gives the formula for the volume of the pyramid analogous to the formula for the area of a triangle, and hints at calculus/archimedes.

Here are three copies of the same geoblock:



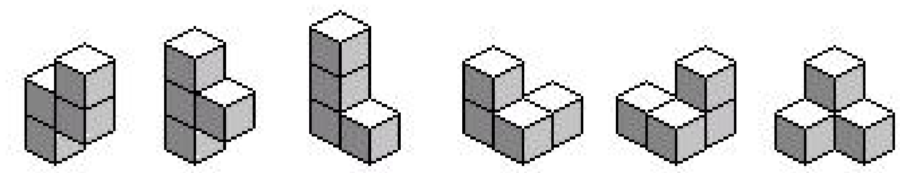
They fit together to make a cube!



3.10 Soma cubes

Move to math club chapter?

The dodecahedron I wrote about in Chapter 1 was only one of many things I learned from Steinhaus' *Mathematical Snapshots*.. I built the *soma cube* puzzle too:



That's not the picture from Steinhaus. I found it on the web at web.inter.nl.net/users/C.Eggermont/Puzzels/Soma/, where I learned more: these six pieces are exactly the nonconvex (three dimensional) 4-tetrominos and the single nonconvex tromino[4] The wonderful surprise is that they can be assembled into a cube!

At the time I'm writing this I haven't tried the soma cube puzzle at the Manning. *It's Elementary* has a mathematical life of its own (something that happened to me and is still happening with Steinhaus, and that I hope will happen to my readers with this book.

How might I introduce soma cubes at the Manning? Building the pieces from the first grade snap cubes wouldn't work.

January 2008: I did soma cubes with the fourth grade math club. Prepared the ground last week with plane pentominoes . Had them try to find them all (using snap cubes. They are in fact good for the exploration, but not good for assembling as a puzzle.

Building a rectangle from the snap cube pentominoes is hard for two reasons. First, the protuberances get in the way. I had prepared some cutouts for the kids. But even with good models the problem is hard. So I showed them a solution and asked them to think about it, draw it.

If I'd prepared better I could have done more in advance about which rectangles were buildable.

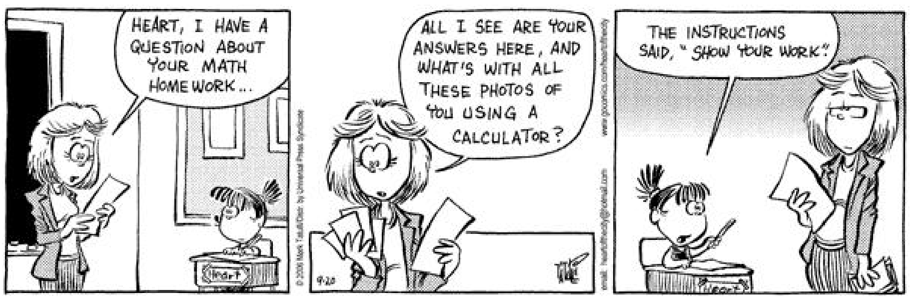
The following week we did the same with soma cubes. I brought some in. Valentina was good at drawing them! She preferred that to the cube assembly puzzle.

picture here

[4]Note that two of them are the two mirror images pictured above.

Chapter 4

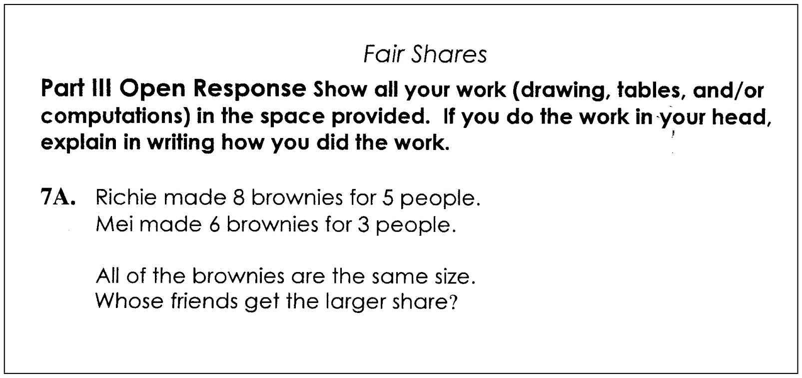
Explanations, Assumptions and Examinations



Heart of the City
Boston Globe, September 20, 2006
(Permission to reproduce pending)

4.1 Following the scoring rubric

From the third grade End-Of-Unit Assessment for the *Fair Shares* unit:



Fair Shares

Part III Open Response **Show all your work (drawing, tables, and/or computations) in the space provided. If you do the work in your head, explain in writing how you did the work.**

7A. Richie made 8 brownies for 5 people.
Mei made 6 brownies for 3 people.

All of the brownies are the same size.
Whose friends get the larger share?

When I came into the third grade class that day in early June Maria, the teacher, asked if I would go over that problem with her since she was unsure about how to interpret the scoring rubric provided.

I asked her first how she solved it for herself. She said

"Twice 3 is 6, so each of Mei's friends gets two brownies. There's no way Richie's friends can get two each."

That's essentially how I thought too. It's what any mathematician is likely think of first – some of the kids at school, my wife, when I asked her, probably you too. We're all mathematicians.

Now let's look at the scoring rubric:

Task # 7A,7B, Sharing brownies…

4 **Exceeds the standard**	3 **Meets the standard**
Answers to both parts must be correct with work or written explanation. A. Mei's friends with supporting work B. 10 brownies with supporting work	Answers to both parts must be correct with work or written explanation. A. Mei's friends with supporting work B. 10 brownies with supporting work
Response shows **complete** understanding of fraction sense, the relationships of fractions to each other and to the whole; how to construct fractions; the importance of equal parts; the significance of the size of the wholes being compared.	Response shows **substantial** understanding of fraction sense, the relationships of fractions to each other and to the whole; how to construct fractions; the importance of equal parts; the significance of the size of the wholes being compared.
Solution in pictures, numbers, or words gives evidence of an **efficient and refined mathematical process.** **For example:** A. Student response is purely numerical… I know that $\frac{8}{5}$ equals $1\frac{3}{5}$ and $\frac{6}{3}$ equals 2, so Mei's friends get the larger share.	Solution in pictures, numbers, or words gives evidence of an **efficient mathematical process.** **For example:** A. Draws "brownie" shares for Richie's friends and Mei's friends. Shows that Richie's friends' share is $1\frac{3}{5}$ and shows that Mei's friends' share is 2. Drawing, model, or labeling shows that, given wholes of the same size, 2 is larger than $1\frac{3}{5}$

Oh my. Now we see Maria's problem. Strictly speaking, our common sense solution seems worth neither a 4 (Exceeds the standard) nor even a 3 (Meets the standard) because it does not mention fractions at all! We reasoned with multiplication instead of division, comparing the relations

$$2 \times 3 = 6$$

and

$$2 \times 5 > 8$$

to solve the problem. As mathematicians, we know it's a better solution than the ones the rubric suggests. It truly "gives evidence of an efficient and refined mathematical process." Does it deserve a 4?

Maria is, and should be, cautious about substituting her own opinions for the rubric's guidelines. All teachers want their students to do well, so it's tempting when grading to give each answer the benefit of every doubt. And these days there are financial and political benefits when test scores go up, whether or not those scores measure anything of educational significance. So Maria asked for my help with her quandary. Fortunately, asking me was an acceptable strategy. The instructions accompanying the rubric say explicitly that it's

> ... a guide. Not all student responses will be represented there. Based on each student's written work on this task, use your best judgment, conversation with your colleagues, the rubric, the ideas in "About the Mathematics in this Unit," "About Assessment in this Unit," the checklist in the Assessment Sourcebook and conversations with your colleagues to determine a score ...

Since part of my job at the Manning is to invoke my credentials as a mathematician in order to bless ordinary common sense Maria awarded this solution a 4.

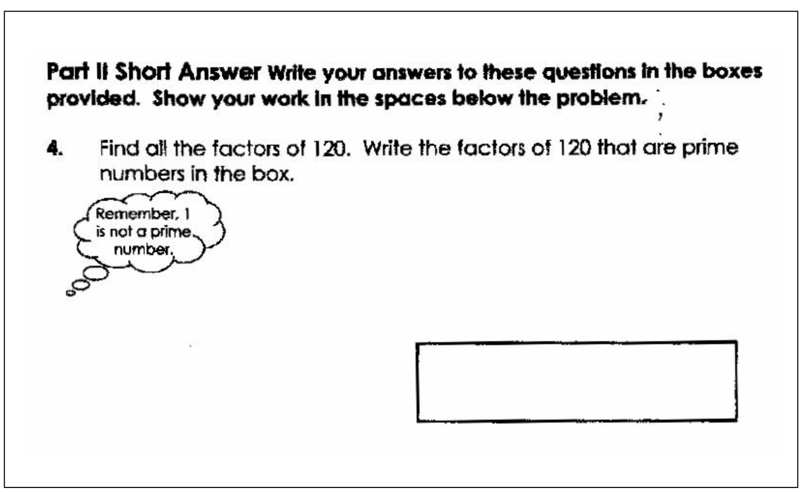
Why did the problem occur in the first place? The curriculum designers want to teach about fractions by embedding the arithmetic in everyday problems. That's a reasonable goal, but leads to the kind of skewed scoring we've just seen. Once you ask a real question (rather than one designed to test a particular skill) you are compelled (or should be) to accept any reasonable answer.

In fact, it's quite possible that kids who could do this problem easily out of school get it wrong here because their goal and their teacher's is not to solve the problem, it's to learn how to do well on an exam by guessing what the examiner wants. [1]

Perhaps faulty as it is this kind of problem is a better way to teach the arithmetic of fractions than a stack of flash cards, but I'm not sure. I am sure that the curriculum errs in preferring so called "mixed numbers" like $1\frac{3}{5}$ to "improper fractions" like 8/5 where the numerator is larger than the denominator. As a kid in school, as an adult mathematician and when reading recipes I've always found the latter easier to work with than the former.

4.2 Factoring redux

Yvonne, with whose third grade class I once built dodecahedra, is teaching fifth grade now. One day I visited her there and watched her student teacher work on this problem with the class:



Part II Short Answer Write your answers to these questions in the boxes provided. Show your work in the spaces below the problem.

4. Find all the factors of 120. Write the factors of 120 that are prime numbers in the box.

Remember, 1 is not a prime number.

The class knows that the factors of a number are the numbers that go into it evenly. [2] The problem asks

[1] That's a learnable skill. I find myself unconsciously teaching it myself: when I ask a question in class at the University and call on a student to answer I sometimes notice that he is not thinking about the question, he's guessing answers and watching me to see when I nod my head. As educators we should remember Clever Hans, the horse who could point to the correct solution to a small addition problem – because he could detect his trainer's excitement when he was at the right number.

[2] Since we've agreed that 0 is a number, we can ask what *its* factors are. After you've thought about the question you can look up the answer in the back of the book.

about 120 since it has lots. The kids rapidly volunteer 1, 2, 3, 4, 5, 6, 10, 12, 20, 30, 40, 60 and 120. The student teacher consults the list she'd prepared and says one is missing. [3]

After a while Davis says "24 should be there."

"How do you know?"

"I can't explain it."

Jacob raises his hand. When called on he said "I can do the problem Dr. Bolker's way."

He starts writing on the whiteboard:

$$\begin{aligned} 120 &= 30 \times 4 \\ &= 15 \times 2 \times 2 \times 2 \end{aligned}$$

Alejandro interrupts: "15 isn't prime."

So Jacob continues with

$$120 = 3 \times 5 \times 2 \times 2 \times 2\ .$$

Then he starts multiplying together groups of the prime factors to find the other factors. I'd shown that method to five of the kids in the class the previous week, when Yvonne asked me to engage them with some material more challenging than what she was reviewing with the class as a whole that day.

But time in math class and space on the whiteboard are both running out. Yvonne intervenes to move on to the pretest to see how much they remember about polygons. She copies this outstanding problem to the "parking space" to come back to.

I don't know whether they ever finished it.

After class I talked with Yvonne about what I saw as the waste of time writing down all the factors of 120 in order to list the prime factors, which Jacob found much more rationally. She agreed. I asked if this answer to the original problem would be correct:

> $120 = 2 \times 2 \times 2 \times 3 \times 5$ so 2, 3 and 5 go in the box.

I have the right answer and I showed how I got it. Would I get full credit even though I didn't find all factors of 120?

Her answer: "no." We agreed that that was outrageous. She and other teachers are now teaching not only that you need to be able to answer the question and explain your work, but that you need to provide the particular explanation that the examiner is looking for. So the exciting idea that the new curriculum is about "investigations" is lost when test time comes. It's still about the algorithms. They aren't the same algorithms I was taught. They are a little more interesting although often a lot less efficient.

I have talked about this at length with Casel, the principal. But there's nothing to be done about it. The assessment test is MCAS[4] practice. The MCAS isn't basically a bad test but the grading policy (and the stakes) are turning it into one. Sharp kids like Nathan and Celin both did poorly on it, which proves that it's a poor measure of mathematical knowledge or talent. Celin was off the charts on the general purpose Stanford 9 exam, while Leah, a "good girl," shows up as advanced in math. I suppose it's important to remember that the MCAS is a better exam than the old ones that were pure drill, and to work at the school to make incremental changes where possible. But teaching to a test – even to a good test – serves neither students nor teachers well.

4.3 I just know ...

When and why is "explain how you know" not a good thing?

The first grade surveyed their tastes in ice cream and found that 17 kids liked vanilla, 7 chocolate. (Those were the only choices. If your favorite was butter pecan you were out of luck.) Bonnie asked

[3] In fact, two are missing. What are they?

[4] MCAS is an acronym for "Massachusetts Comprehensive Assessment System."

"How many more kids like chocolate?"
since questions just like this will come up on the explicit assessment they will have to take soon.

It's only January; the first grade hasn't "studied" subtraction. They can count accurately at least to thirty – most to 100 – and have worked out lots of arithmetic problems like this one that can be solved by counting. But some of them know more. In this case Lee said

"The answer is ten."

"How do you know?"

"I was thinking about the survey and the math just popped into my head."

When Bonnie asked for more information Lee couldn't find good words. When he tried to explain he became more and more confused. And his confusion didn't help those of his classmates who didn't "just know."

Here's the email I wrote to Cynthia and Bonnie after that class:

```
   I was thrilled (as were you) when Lee said he was thinking about the survey and
the math suddenly popped into his head:  10+7=17.  I think it's really hard to push
him to tell you where he got the 10 from - his "I just knew" is a respectable answer.
(If the TERC folks don't think so I think they're wrong.  Kids should be able to
explain ideas that are new to them, but once they've internalized them they shouldn't
have to introspect that much.  No one asks you how you know 5 x 7 = 35.  You just
know.)  So maybe when the first question elicited the "I just knew" Bonnie could
have explained to the rest of the class that what Lee knew was that 17 was ten and
seven, maybe observing 11 is 10 and 1, 12 is 10 and 2, so that the pattern is clear.
Maybe some day do the etymology of the word "seventeen" which really has both the
"seven" and the "te(e)n" in it so it tells you its meaning.
   What fun!
```

On another day in another year Cynthia's first grade class is reviewing answers to another subtraction question:

> Miguel's class went to the zoo. He drew 15 pictures in all. 8 were pictures of zebras. The rest were pictures of tigers. How many tigers did Miguel draw? [5]

When they took the practice test the day before most of the class got the problem right. That's reasonable for May. What Cynthia focussed on today was the form their answers took, since the grading rubric required explanations. Simply writing

$$8 + 7 = 15$$

or

$$15 - 7 = 8$$

earns a "3" but not a "4".

But the class has practiced for this. Here are some of the responses.

Maja: "I drew 15 boxes. I crossed off 8. I counted the rest. There were 7."

Ethan: "8 is 2 from 10. 10 and 5 is 15. 5 and 2 is 7."

Declan: "Twice 8 is 16. I only need to get to 15 so take one away from the 8 to get 7."

What's the moral of this story? If the kids had learned subtraction with flash cards Maja wouldn't have had to to draw all those boxes and count them, risking an error. But we might never have discovered Ethan's and Declan's lovely mental gymnastics.

[5] The second sentence should start "Eight were" rather than "8 were" – but for first graders the numeral is easier to understood than the word.

4.4 I just know ... (part two)

In *Blink* Malcolm Gladwell quotes Sibley on birding:

> Most of bird identification is based on a sort of subjective impression – the way a bird moves and little instantaneous appearances at different angles and sequences of different appearances, and as it turns its head and as it flies and as it turns around, you see sequences of different shapes and angles...
>
> All that combines to create a unique impression of a bird that can't really be taken apart and described in words. When it comes down to being in the field and looking at a bird, you don't take the time to analyze it and say it shows this, this, and this; therefore it must be this species. It's more natural and instinctive. After a lot of practice, you look at the bird, and it triggers little switches in your brain. It looks right. You know what it is at a glance. [6]

Sometimes a kid makes a leap of understanding that outstrips his own knowledge of what he's done. A solution comes in a rush. A student – usually but not always one of the stronger ones – will blurt out the answer without even raising her hand. The teacher then praises the student (for the answer, not for the interruption) and asks for an explanation – which is usually cryptic at best. I can sometimes know what method the kid invented even when she can't explain it. But the teacher, who's much more experienced with second graders but knows less mathematics than I do can't follow it. And after more prompting, the student sometimes can no longer solve the problem, let alone explain how she did it the first time. In this instance it's potentially useful to find a way to make the method explicit. That's how insights can become things that more kids reliably "just know" so they free themselves to have insights about the next set of problems.

When I reflect on mathematics with my professional mathematician's hat on (rather than just as an observer in first and second grade) I'm struck by how the essence of the subject is often to figure something out (somehow) and then find the words that make the insight rigorous so that I can write it down in a form that my colleagues can follow step by step. But, that form often obscures the excitement and wonder at the insight that led to the initial discovery. Even in my profession things were sometimes better in the old days. Here's Edward Sandifer commenting on Leonhard Euler's mathematical style:

> Modern readers may complain that he did it "wrong" in view of what we now know about series and convergence. This may be true, but Euler's results are correct if we make appropriate translations into the language of limits. Also, these proofs have a beauty and a charm that rigorously correct modern proofs of the same theorems lack.
>
> C. Edward Sandifer
> The Early Mathematics of Leonhard Euler

4.5 Everyone knows that ...

Every test comes in an implicit context. The person making up the questions just assumes everyone starts in the same place. In these instances it's the questioner, not the answerer, who hasn't explained enough.

During my first year in graduate school I needed to take a special exam to determine whether I would be allowed to extend my student exemption from the draft.

This was one of the questions, reconstructed from memory:

[6] This is giss birding: "giss" stands for "general impression of size and structure."

Station	Departs	Station	Arrives
Loop	7:12 AM	Oak Park	7:40 AM
Oak Park	7:45 AM	Loop	8:17 AM
Loop	5:23 PM	Oak Park	5:51 PM
Oak Park	5:56 PM	Loop	6:21 PM

How much time does the daily commute take?

This is nominally a question about reading a table, twice finding the difference between two times, and adding the results. The answer is exactly one hour – but to know that you need to know that the Loop is downtown and Oak Park is a suburb, and that Dick and Jane's father takes the train to work in the Big City for his nine to five job. So this isn't just an exercise in reading a table, it's a test of culture.

John Norman observes in his *The Design of Everyday Things* that everyday knowledge resides in the world as well as in the head. It's those things "everyone (supposedly) knows" that can skew the results of tests when not everyone knows.

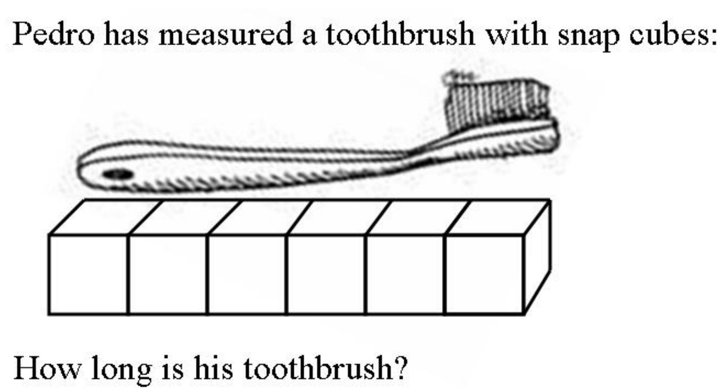
see similar discussion in math wars chapter, or combine the two in one of the two places

Sometimes, in fact, what everyone knows happens to be wrong. Andy's thoughts about Euclid and Eudoxus ...

4.6 Making life more interesting

This might turn into a discussion of measurement. Then it belongs somewhere else - maybe in the geometry chapter.

The first grade has worked on this question:



All the kids know the answer:

"6 cubes. I know because I counted them."

Morris says "I know because $3 + 3 = 6$."

Lisa says "I counted them differently." She goes to the whiteboard and counts them for the class this way:

first, last, second, fifth, third, fourth

What do I think is going on? Morris and Lisa are bored with trivial problems, and intentionally make them harder to make them more interesting. Like doing easy crossword puzzles in ink.

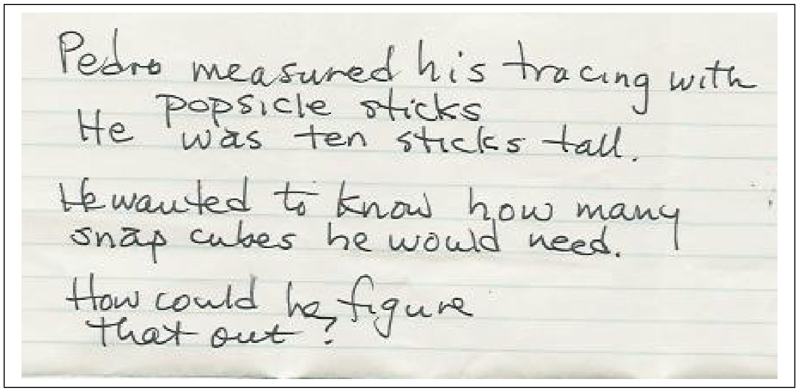
But why is the toothbrush being measured with snap cubes at all? I'd have thought it more useful to learn about inches.

In the first and second grades the TERC curriculum seems to want to measure things in anything but standard units. Snap cubes are a favorite. And they are really annoying for large numbers.

February 2006. Cynthia reviews a problem about kids who traced each other's outlines onto paper and then measure heights, with snap cubes. One of the kids had no snap cubes and measured himself with popsicle sticks. I asked her if she thought any of the kids in her class could figure out the height in snap cubes without remeasuring, given the height in popsicle stick. She gives me some popsicle sticks and a box of snap cubes.

"Try it with Allison "

I wrote out



"Can you solve this problem? I'll read it to you, since my handwriting isn't too clear."

I watch silently. She thinks for a while, takes out a handful of snap cubes and arranges five of them in a row next to a popsicle stick – just the way Pedro measured his toothbrush.

"How many snap cubes for 10 popsicle sticks?"

"5, 10, 15, 20, 25 ..."

Several such attempts fail. She can't quite keep track of how many fives she's counted.

"How about trying five tens instead of ten fives?"

Success. Perhaps I should have remained silent – but math time was ending. And I'm not sure the success was fairly achieved. It's true that $5 \times 10 = 10 \times 5$ but not at all clear why you can solve this problem with 5 groups of 10. The units don't match up.

The class goes on to measure things with a ruler. Cynthia demonstrates a bad way to measure by failing to start the ruler at the edge of the object. She talks about "starting at 1". I raise may hand; she calls on me. I note that what she means is really "start at 0" and that the ruler should have the zero marked – just as does the posted number line, and the posted hundred's chart..

Most of the items are less than a foot long, so addition doesn't enter the problems. After class I finish the day with Cynthia suggesting that the class should really have a supply of ten inch rulers (with zero marked at the left end). That would make it easy to get a count of inches, and reinforce place value notation and counting by tens.

4.7 Last words

> I rarely think in words at all. A thought comes, and I may try to express it in words afterward.
>
> Albert Einstein
> Quoted in H. Eves *Mathematical Circles Adieu*
> Boston 1977

Ramanujan, having solved a problem posed verbally: “Immediately I heard the problem it was clear that the solution should obviously be a continued fraction. I then thought, Which continued fraction? And the answer came to my mind.”

Robert Kanigal
The Man Who Knew Infinity, p 215

and then quoting Hardy

“All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.”

ibid, p 216
(find the place where Hardy said it)

Chapter 5

Christopher

Last week I decided wanted to learn the world of fractions.

Christopher

5.1 One half plus one third

One September 29 Ms. Fitzgerald asked her new second grader Christopher to show me how he constructed yesterday's number of the day.

He said "13 3/4 + 13 3/4 + 1/2."

"How did you figure that out?"

"Last week I decided wanted to learn the world of fractions."

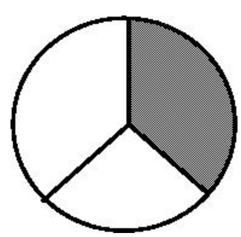
"Did you ask someone to teach you, or did you read something?"

"I just thought about it."

"Do you ever write down the things you think about?"

"Sometimes, but mostly I just think about it."

Since he knew about halves and quarters I asked him about thirds. They were new to him. I drew a three piece pie



and he understood immediately.

"What's 3/3?"

"1."

Then I wrote

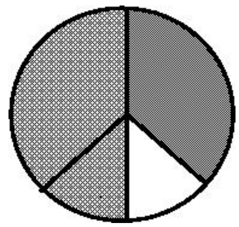
$$1/2 + 1/3$$

on the board and asked him to think about it while I worked with some other kids. A few minutes later he called me over and said he thought the answer was three fourths. I was a little surprised, and a little disappointed. I pointed out that he knew $1/2+1/4$ was 3/4, so that couldn't be the right answer. He agreed. I started to suggest how he might do the problem but he told me he didn't want any help: "let me think about this some more."

Five minutes later he had figured out 5/6.

"How do you know?"

He took my picture of 1/3, extended the vertical radius to a diameter, shaded a half and a third



and said "there's one sixth left so a half plus a third is five sixths."

I wondered how far he could go. He'd just heard about thirds and generalized to sixths on his own. So I asked him to try $1/2+1/5$ at home, and to write down any thoughts he had about that or other mathematics he was interested in. Ms. Fitzgerald was delighted, and told me his parents would be thrilled, since neither she nor they had any idea what to do with him in math. Then she found a blank notebook for him to use.

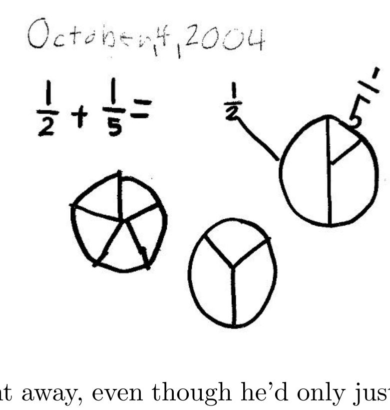
Judy Manthei says I need more frame here. Why Christopher? Partly it's my pleasure at working with)just) the good students. But also it's a chance to learn from him about what might work for more - note that some math club stuff started here, before the math club. And I do talk later in the chapter about enriching for all, or at least more kids

5.2 The notebook

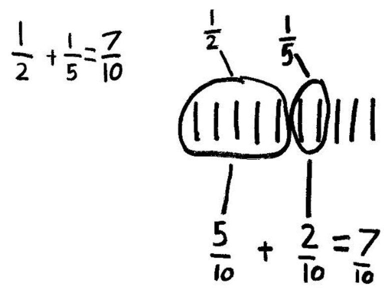
Christopher was clearly a special case as a second grade mathematician, but he was just one of 25 starting second graders. I had to restrain my impulse to single him out from his classmates. It wouldn't be fair to spend all or even much of my second grade classroom time with him, tempted though I was. So I decided that each week I would write out some problems for him to work on at home. When we met at school I'd read and comment on what he'd written and pose appropriate new problems for the next week. We wouldn't talk much about the math, and I wouldn't explicitly try to *teach* him anything. Just see what he could learn.

He's graciously given me a copy of his notebook, and permission to discuss it here. I hope his head doesn't swell (too much) when he reads what I have to say.

This was his response that first week, with my written comments, and my analysis. Remember that the analysis is mine, not ours. I didn't discuss his work with him this way.

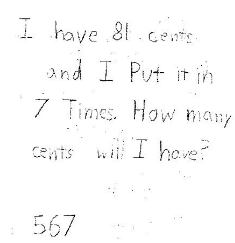


He clearly understood fifths right away, even though he'd only just learned about thirds. So he's grasped the principle of arbitrary denominators. The first circle suggests that he started the problem I'd posed by trying the same method he used for $1/2+1/3$. But he could't quite see how to use the half to bisect one of the fifths, as he'd done with the thirds in class. He drew a five part pie and then redrew a three part pie to see if he could get any inspiration from it. His pictures suggest that nothing helped. So he turned the page and began again:



The kids learn about tally marks explicitly in first grade, which may explain why they occurred to him as potentially useful here. He did realize that he needed them all vertical, so did not use the diagonal slash for every fifth one. I don't know how he knew that tenths would matter. He's figured out for himself that $1/2 = 5/10$ and that $1/5 = 2/10$. His pictures are particularly clear and his handwriting particularly legible - few second graders (few second grade boys) write this well.

In keeping with the silent plan I'd made I did not quiz him on his thinking. Insofar as possible we just communicate by passing the notebook back and forth. I turned the page to find the next blank one and found instead first of many problems Christopher made up - sometimes for himself, sometimes to challenge me:

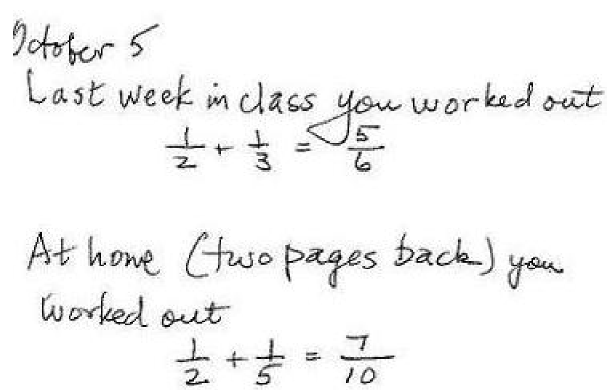


Since I couldn't see how he'd solved it, I asked.

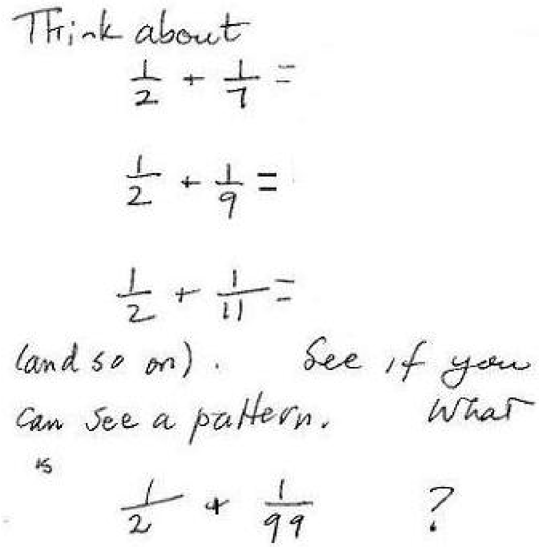
"I used a calculator."

"That's fine. Next time you do a problem that way just write that down."

Then I wrote out some questions for the next week[1]

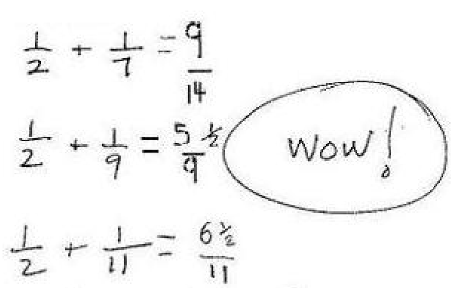


[1] His handwriting is neater than mine - I discovered that I had to pay extra attention to make it legible.

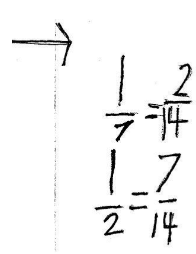


I was pretty sure he couldn't do the last question with either tally marks or cut up pies. To answer it he'd have to see a pattern, find a general method.

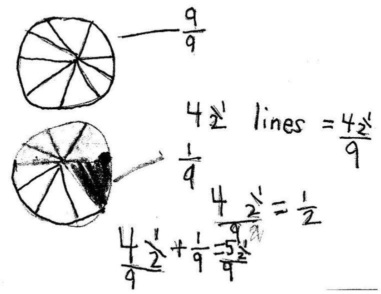
A week later he offered these answers. The comment is mine.

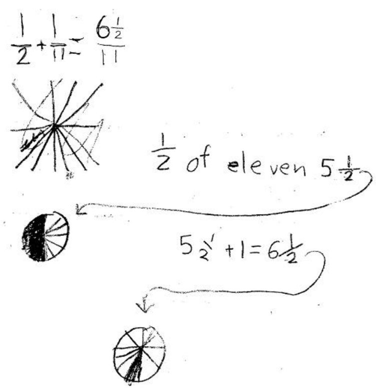


In the next three pages he showed what he was thinking. Here's the work for the first problem, clearly mirroring what he discovered for $1/2 + 1/5$ using tally marks.



Then for some reason he develops a different strategy, returning to the pies but this time making them work:





He's clearly mastered a method. What struck me when I thought about his wonderful unexpected audacity writing $5\frac{1}{2}/9$ was the contrast between the two ways he tackled the problem. To compute $1/2+1/3$ and $1/2+1/5$ he found the least common denominator (6 and 10 respectively) but for $1/2+1/9$ and $1/2+1/11$ he deals entirely in ninths and elevenths. That prompted my next week's query.

OCTOBER 20

A few pages ago you said

$\frac{1}{2} + \frac{1}{7} = \frac{9}{14}$ (it's correct)

ALSO – just now

$\frac{1}{2} + \frac{1}{7} = \frac{4\frac{1}{2}}{7}$ (also correct)

See if you can figure out
<u>why</u> $\frac{9}{14}$ and $\frac{4\frac{1}{2}}{7}$

are really the same.

Then see if you can rewrite
$\frac{5\frac{1}{2}}{9}$ in a different way
(without the $\frac{1}{2}$)

Then do any other problems
(fractions or not) that
you find interesting.

and his answer:

$\frac{9}{14} = \frac{4\frac{1}{2}}{7}$

This is beacaure
in fractions 14ths half of
7ths. →  $\frac{1}{2}$ of 9 =
$4\frac{1}{2}$. so they are the same

5.3 Is there a curriculum?

I am thrilled each time I look at my copy of Christopher's notebook. We worked lots on fractions, some with subtraction, some with fractions in the denominator.

I tried to interest him in patterns and limits, suggesting he work out

$$\begin{aligned} \frac{1}{2}+\frac{1}{4} &= ? \\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= ? \\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= ? \end{aligned}$$

which would lead to the sum of the infinite geometric series. But he made an arithmetic error, thought the last sum was 13/16. [2] So he couldn't see the emerging pattern $3/4, 7/8, 15/16 \ldots$. I suggested he go back and try it again. The next week there were two notes in his book. First, from him:

> Dear Doctor Bolker,
>
> I worked on this for an hour and I had some help from my dad..

and this from his dad:

> He seems to be a little overwhelmed by this – and is getting a little frustrated with fractions.
>
> Thank you
> David
> (Dad)

P.S.: Perhaps some logic-oriented problems?

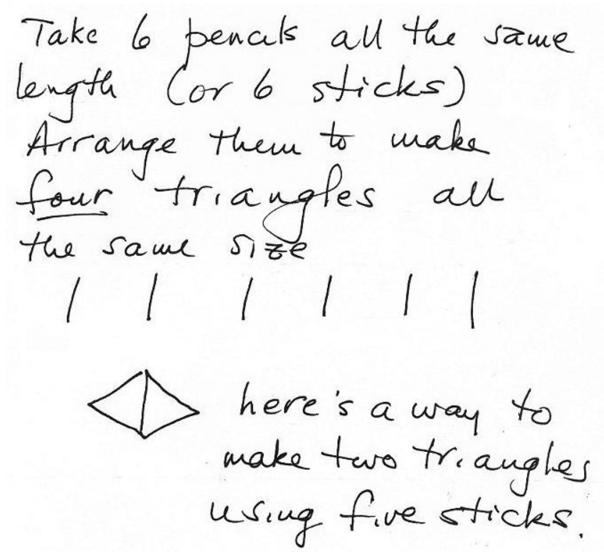
I answered:

> We (Chris and I) need to think about this. I wouldn't mind some problems like this (arrow to "logic problems") but Chris has said he's really interested in thinking about fractions. I've been trying to (text not available in my xerox copy of the notebook)

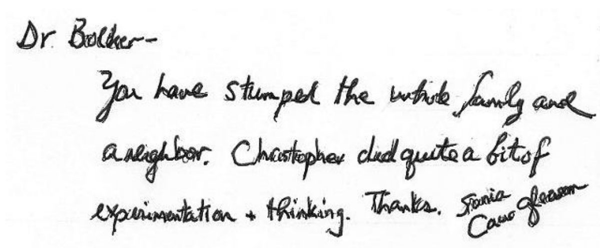
A few weeks later, at a parents' night, Chris' dad asked me if I thought he should be helping his son with the problems I posed. I told him I thought not, even though there would surely sometimes be questions that were too hard – or ones that turned out to be inappropriate. The point was not to teach him more stuff, but to see where he went with my suggestions. But of course work with him whenever he wants to do that – just don't worry about getting the answers. He understood.

Christopher and I went on to explore lots of mathematics – more fractions, magic squares, how to draw prisms and pyramids and other three dimensional figures, Euler's formula, shortest paths in grids, the Pythagorean puzzle discussed in Chapter xx, ... Here's another puzzle I gave him.

[2] In similar situations I've heard people say "I got the mathematics wrong." I often take that opportunity to reply "No. You got the mathematics right. You just got the arithmetic wrong."



Try it. If you can't do it you won't be the first to fail. Here's the note from Christopher's mother at the bottom of the page:



Lest you lose sleep, you will find the answer at the back of the book. But if you're willing to lose some sleep you might find in the morning that the answer has come to you in the middle of the night.

There aren't many Christophers in a math teacher's life. The only other one in mine is my friend Paul Mason, who was a UMass freshman when I first met him in 1972. I sometimes wonder if I was anyone's Christopher.

5.4 Mentoring

Christopher is exceptional. But he's not the only kid who can benefit from stretches that go beyond the curriculum. In some ways he wasn't even the strongest student in his second grade class. Often Daavi was faster and more accurate. So he's not the only one who can take advantage of extra attention.

Nathan Dickler discussion is all here, for now. But it's evolving. It may well turn into more of a comparison with Casel and less about intervention with good students. So some of it - perhaps all of it - will move to other places, in time.

Extra attention matters. It can be memorable. When I was a senior in High School my college application asked for a description of an educational experience that mattered to me. [3] I knew that the essay needed to be distinctive. And it needed to be funny too. I chose to tell stories about Dr. Nathan Dickler, the principal of P.S. 139 in Brooklyn – who happened to have been my father's high school math teacher at Boys' High many years early. I started at P.S. 139 when I was six. When we visited the school Dr. Dickler interviewed me. The conversation went something like this.

[3] I just looked in my files for the copy of the essay I thought I'd saved, but couldn't find it. I will keep looking, but in the meanwhile I can reconstruct it from memory – the ideas if not the words.

"How old are you, Ethan?"
"Six."
"How much is four and two?"
"Six."
"How old will you be next year?"
"Seven."
"How much will four and two be next year?"
Perhaps I paused before saying "six."

"How come if you are six now and will be seven next year, four and two which is six now won't be seven next year?"

I know I stuck to my guns, but can't remember what reason I gave.

Several years later Dr. Dickler visited my fourth grade class. He asked "If a horse weighs 2000 pounds when it stands on all four legs, how much will it weigh when it stands on two?" We had a heated debate, which was settled only when two kids were sent to the nurse's office to weigh themselves, first standing on two feet, then on one.

Sometimes I think I may be channeling Dr. Dickler when I ask provoking questions.

/marginpar The next thoughts are were triggered by a remark from Joan

Everyone I tell these stories to laughs. I tell them that way. I wrote them that way to make the Harvard admissions committee laugh. But further thoughts about these stories are more disturbing. In each of these stories (but particularly in the first one) there's an undercurrent of superiority: "I know something you don't and you are ignorant." Neither of these questions is really about teaching mathematics. The first is a kind of linguistic trick, which can certainly make the kid who's asked it very uncomfortable – if it made me uncomfortable I have forgotten that fact. The second is really physics - it's about a conservation law.

What kind of teacher/principal was Nathan Dickler? My mother in law despised him (strong word) – she had a run in with him about Bob (says Joan). (I should ask Bob whether he remembers anything about this.) He was probably an elitist – good for math geeks like me but what about for all the other kids, and the teachers. I have always wondered (at the back of my mind) how/why a high school math teacher bacame an elementary school principal. I'd be surprised if he was as good as Casel.

His nominally sound plan to base grades on expectations – so my A didn't mean anything because it's just what I was supposed to get.

But I can try to channel the better parts of Dr. Dickler. So one of the things I do in the classes I sit in on is to ask provoking questions. Sometimes that's when the class is assembled as a group, sometimes it's while I am wandering about while the kids are working, looking looking over shoulders. I invent problems on the fly that are just a little bit more challenging (or, on occasion, less challenging) than the ones the teacher has just assigned, that the stronger (or weaker) mathematicians can work on.

I can do this when the teachers can't, for several reasons. First, I am more of a mathematician than they - although they get better at it as they become more comfortable with the new curriculum, as they watch me, as they think of themselves as mathematicians. Second (what did I have in mind when I started this paragraph?)

I want now to describe some of those interventions – often but not always for the kids who need more challenge. I'll show you more of Christopher's notebook, too, transcribed rather than reproduced now that you have the flavor of the original.

I thought of making this stuff another chapter, called "Excursion", but for now think I will leave it here. Maybe the title of this chapter includes both "Excursions" and "Christopher".

5.5 Christopher isn't the only one

Examples (turn these into dialogues)

One of the best exercises in the second grade uses coins. Kids get plastic pennies, nickels, dimes and quarters. Two take turns rolling a die, taking the resulting number of coins from the central pot and adding them to their accumulating pile. The game ends when someone reaches 50 cents. They're encouraged to trade low denominations for high when they can, and most do. Nickels for five pennies and dimes for two nickels (or ten pennies) are easy. Quarters are harder. Most kids don't bother, so end up winning with five dimes.

It's easy to make this game somewhat harder by using two dice, making the goal a dollar, and requiring an exact finish. But those changes don't introduce any new principles.

This game is more interesting if you encourage the kids to do some mental arithmetic first and deal only with the net result. For example, the boring straightforward way to add four cents to a pile with a nickel and three pennies is to take four pennies, note that you now have more than five pennies, so turn in five for a nickel. Then see that you have two nickels and trade them for a dime.

I've tried asking the kids to figure out – after they roll the die but before they touch any coins – what they will end up with. In this case that would be a dime and two pennies. They find this difficult. I don't push. (That's a principle I try to honor. When I suggest something that isn't working I assume the problem is mine, not the kids', and move on to something else. I do this at all levels - you can see it from time to time in Christopher's notebook when we return to it.)

But what's even more interesting, and does seem to work, is to ask a pair of kids to start out with two quarters each, roll a die (just one is fine) and return the indicated amount to the pot. There the very first move requires a trade!

Oct 4 2006. Cynthia asks me to work with Morrisand Lisa on my first day at the Manning. They're the sharpest (at the end of this chapter it's Morris who does the big doubles). They are among the few using two dice right away, collecting 25 cents. After one round in which they use almost all pennies I ask if they want a challenge. They agree. So I leave them with lots of dimes and nickels but just four pennies.

Too hard. "It's impossible"

I don't really think so. But there wasn't enough time. I want to return to this later. For sure it's an idea for the second grade.

Second grade. When kids are asked to draw rectangles with given areas I ask them to find all the ways. And then to see which numbers have only one way.

Second grade. The magic doubling pot. I suggest recursive use. How many uses until answer exceeds x? (logarithms to the base 2).

First grade. More elaborate patterns (already discussed).

First grade. Double compare. Do it without having to add, when a1 ¿ b1 and a2 ¿ b2.

Second grade. Billy: simultaneous equations.

Chapter 6

The Math Club

> You should really do the math club. It's fun, not like math.
>
> One girl from last year's math club, to a friend.

6.1 In the beginning

The math club didn't start with the magic that starts Chapter One.

Planning for my third year at the Manning I was looking for a way to see some of the older kids I'd met in first and second grade and the principal Casel was trying to solve a political problem. She wanted to show parents visible evidence that she was paying attention to the most motivated students, without abandoning her committment to leave no child behind (her philosophy long before it became a popular phrase in Washington). We looked for a way to make my presence more visible, beyond what I was doing quietly for Christopher and other adventurous kids.

A math club met her needs and my wishes. The Manning is a "late school." To allow shool busses to make two trips, early schools start at seven and late ones at ten. So the math club could meet at nine for an hour.

I knew the best mathematicians in the third grade, and suggested asking the fourth and fifth grade teachers to recommend their best students. I'd pick eight from that group. Casel would have none of it.

"You and I went to school when tracking was the rule of the day. Since we were in the top track we got the education we needed – but at the expense of many of the other kids. We don't have 'bluebirds' here at the Manning. Sure, some kids are smarter than others, and some work harder (not necessarily the same some). So let's figure out a way to have all kinds of kids in the math club."

At the University no one is my boss. But in an elementary school – particularly in Casel's school – the principal is the boss. Since I was just a volunteer I could choose not to do the math club on her terms, but I couldn't just do it my way. And I knew Casel well enough by this time to know that she would have good reasons for her way.

I found her reasoning convincing, even though I suspected (rightly, as it happened) that opening up the club that way would make it harder for me to manage, would shut out some of the kids who would most enjoy it and might make it less interesting for the stronger students.

We planned an application form that asked kids why they wanted to join the club and posed a few problems from an on line source (about which more later). Even though part of the purpose of the club was to convince ambitious parents that we cared about their kids (who were, by their unbiased estimate, the smartest and most deserving) we wanted only those kids who themselves wanted to be in the club. So we handed out applications to be returned in school that day.

Here's what we asked them. You might try filling it out yourself.

scan application here

Discuss some responses.

In evaluating responses we looked for enthusiasm as much as for talent. Some kids got no problems right and were invited anyway. And we needed gender, race and upstairs/downstairs balance. So it's not just about the math – that would be easy.

6.2 Math League problem sets

Although I knew much more about the kids than I did when I began, this was a new experience for me. I'd be fully responsible for the eight kids for an hour, not just an observer. And mindful of My experience with the rhombic dodecahedra in the third grade and my short attempts at curricular design in the second grade, I wanted to avoid overpreparing overambitious material. And I didn't have much time in my schedule to prepare. I hoped the web would help, and it did. A short search found mathleague.com, an organization that prepares mathematics contest materials for grades 4-12. The sample problems on the questionnaire are typical - this isn't Olympiad level material. It's not just for the bluebirds. It's intended to be inclusive rather than selective. Just right. I made copies of some of the sample problems on the web site to use in the club until the books I sent away for arrived.

We met in a small resource room adjacent to the cafeteria/auditorium where some of the kids (both in the math club and not) had before school breakfast. Not a good venue.

I planned to study the problems one at a time, first having them work individually or in whatever small groups formed naturally, then pointing out to the group what they or I found interesting. Here'a the first problem from the 1996-1997 fourth grade contest. It's probably not the contest I started with (I have no notes from those weeks to refer back to) but the ideas are there.

1. $(2 + 8) + 10 = 2 \times ?$

A) 8 B) 10 C) 12 D) 18

Learn enough TEXto get alignment right, or, perhaps better, scan.

Most of the kids needed to know what the "?" stood for. So I explained that they were to discover what number made the equation true. In later years this explanation became less and less necessary as the regular curriculum did more with blank spaces in equations - a kind of proto algebra. (I can't remember whether I needed to make explicit the second meaning of "equals" - see the Vocabulary chapter.) Then all had no trouble figuring out that the left side was 20 so that the unknown number must be 10.

A few questions further along in the same contest we encountered

6. $444 + 444 + 444 = (3 \times 400) + (3 \times ?)$

A) 38 B) 40 C) 42 D) 44

I watched the kids start computing $444 + 444 + 444$, silently wincing, waiting for the teaching moment. Some got that part right. Then they were puzzled. One or two knew (at least in principle) that they should subtract 400 times 3 from the result and divide by 3 to find the answer.

In everyday conversation most nonmathematicians I've met would call this "doing the math." When I hear that I'm tempted to correct them: all that's left in the problem is "doing the arithmetic." The "math" is the thinking, and that part's been done. Sometimes, if it doesn't seem impolite, I offer a correction, phrasing it as a compliment: "No, that's just arithmetic. You've *done* the math."

We could have spent lots of math club time doing that arithmetic. But I saw what I hoped was an opportunity. I pointed out that the contest had 30 problems and they had half an hour to do them all. So any problem that needed lots of arithmetic would need lots of time, so working on it would probably lower their score a lot even if they got it right. The secret, I said, was to look for "a way to do the problem without really doing it."

In this particular case the way to cheat is to see that three “four hundred forty fours” is really just three “four hundreds” and three “forty fours.” So the answer is D: 44.

The kids agreed that that was “a cool way to cheat.”

If you now revisit the first problem with an eye toward cheating by not doing it you can see (and so could the kids) that you never need to know that both sides of the equation represent 20. It’s clear at a glance that both sides represent two tens.

The deep mathematics that underlies this trick in both the problems is known as the distributive law, which I discuss in more detail elsewhere.

From then on when working the Math League problems this kids always looked for a shortcut. From time to time we find a problem where none of us (including me) could see a solution other than doing the arithmetic. And we are all disappointed.

using UMass professional development money to buy materials

6.3 Math club reproduces

2005/6:

Met with Casel to set up math club. We’ll do every other week for teams 3 and 45. There were 10 kids in Grade 3 who responded immediately. Christopher, of course, and Daavi, and Rafi, and a new kid Michael who is said to be strong. Annelisa(?). But also Terel, who is sweet but not up for math club. How I will handle this I don’t yet know. Should I get 10 notebooks and do 10 tutorials, maybe in small groups. About to write Maria to find out what they are doing now so that I can do something orthogonal to it.

I expect 10 more from grades 4 and 5.

The next year the new principal has even stronger feelings about including everyone. And there’s even greater demand since the math club has become known and popular. The parents know about it and the kids ask about it as soon as I show up for the first time in September. (This helps my political position with the new principal.)

Describe how Genteen and I come up with this year’s four math clubs.

The number of math clubs seems to be growing exponentially (1, 2, 4). Clearly that can’t continue. I wonder what will happen next year?

By the end of the first year I was finding the Math League problems boring, and so were the kids. And I was feeling much more confident that I could bring in new material and teach it (or, better, offer it) to the kids. So I gradually did. I’ve now been at this long enough to have repeated some topics. In the discussion that follows I’ll collect all my thoughts on each one, rather than providing a chronological account. (But I will try to indicate what didn’t work the first time and how I changed it and might change it again the next time.)

6.4 Bases 12, 8 and 2

When I was in high school (1951-1955) Tom Lehrer (mathematician and satirist) was all the rage in my set. I interviewed him for *Papyrus*, the school’s mathematics magazine, when I visited Harvard for an interview in my senior year. I don’t have the date for this song, but the decade is probably right. (Maybe the new math really came along ten years later, since I can remember some of the controversy when I’d already more or less become a mathematician – check the date.)

```
Some of you who have small children may have perhaps been put in the embarrassing position
of being unable to do your child's arithmetic homework because of the current revolution in
mathematics teaching known as the New Math.  So as a public service here tonight I thought
I would offer a brief lesson in the New Math.  Tonight we're going to cover subtraction.  This
is the first room I've worked for a while that didn't have a blackboard so we will have to
make due with more primitive visual aids, as they say in the "ed biz." Consider the following
subtraction problem, which I will put up here:  342 - 173.
```

Now remember how we used to do that. three from two is nine; carry the one, and if you're under 35 or went to a private school you say seven from three is six, but if you're over 35 and went to a public school you say eight from four is six; carry the one so we have 169, but in the new approach, as you know, the important thing is to understand what you're doing rather than to get the right answer. Here's how they do it now.

You can't take three from two,
Two is less than three,
So you look at the four in the tens place.
Now that's really four tens,
So you make it three tens,
Regroup, and you change a ten to ten ones,
And you add them to the two and get twelve,
And you take away three, that's nine.
Is that clear?

Now instead of four in the tens place
You've got three,
'Cause you added one,
That is to say, ten, to the two,
But you can't take seven from three,
So you look in the hundreds place.

From the three you then use one
To make ten ones...
(And you know why four plus minus one
Plus ten is fourteen minus one?
'Cause addition is commutative, right.)
And so you have thirteen tens,
And you take away seven,
And that leaves five...

Well, six actually.
But the idea is the important thing.

Now go back to the hundreds place,
And you're left with two.
And you take away one from two,
And that leaves...?

Everybody get one?
Not bad for the first day!

Hooray for new math,
New-hoo-hoo-math,
It won't do you a bit of good to review math.
It's so simple,
So very simple,
That only a child can do it!

Now that actually is not the answer that I had in mind, because the book that I got this problem out of wants you to do it in base eight. But don't panic. Base eight is just like base ten really - if you're missing two fingers. Shall we have a go at it? Hang on.

You can't take three from two,
Two is less than three,
So you look at the four in the eights place.
Now that's really four eights,
So you make it three eights,
Regroup, and you change an eight to eight ones,
And you add them to the two,
and you get one-two base eight,
Which is ten base ten,
And you take away three, that's seven.

Now instead of four in the eights place
You've got three,
'Cause you added one,
That is to say, eight, to the two,
But you can't take seven from three,
So you look at the sixty-fours.

"Sixty-four? How did sixty-four get into it?" I hear you cry. Well, sixty-four is eight squared, don't you see? (Well, you ask a silly question, and you get a silly answer.)

From the three you then use one
To make eight ones,
And you add those ones to the three,
And you get one-three base eight,
Or, in other words,
In base ten you have eleven,
And you take away seven,
And seven from eleven is four.
Now go back to the sixty-fours,
And you're left with two,
And you take away one from two,
And that leaves...?

Now, let's not always see the same hands.
One, that's right!
Whoever got one can stay after the show and clean the erasers.

Hooray for new math,
New-hoo-hoo-math,
It won't do you a bit of good to review math.
It's so simple,
So very simple,
That only a child can do it!

Come back tomorrow night. We're gonna do fractions. Now I've often thought I'd like to write a mathematics text book someday because I have a title that I know will sell a million copies. I'm gonna call it Tropic Of Calculus.

Tom Lehrer
The New Math
www.casualhacker.net/tom.lehrer/the_year.html#math

www.sing365.com/music/lyric.nsf/SongUnid/EE27EF26A4F581BE48256A7D002575E1
(need permission if I want to use this)

Several times I've spent some math club time on work in other bases. What follows is a synthesis of my attempts with third, fourth and fifth graders. Each time I do it I get a little better at adjusting the pace and the questions to the kids' ages and abilities. If you try it you'll have to do the same.

The essence of the idea is to begin the discussion of other number bases with a seemingly entirely different question – counting words.[1]

I start with the question

"How many letters are there in the alphabet?"

All: "Twenty six."

"How many one letter words can you make?"

Rory: " `a` and `i`"

"I meant words of any kind. They don't need to make sense, or be pronouncable."

All: "Twenty six."

"Suppose you had an alphabet with only two letters. Then how many one letter words could you make?"

"What are the letters?"

"Which ones do you want to use?"

"How about `a` and `d`"?[2]

"OK. How many one letter words?"

"Just one: `a`."

"Remember, they don't need to be real words, or pronouncable words."

"Then there are two"

"What are they?"

"'a' and 'd'."

"How many two letter words do you think you could make from alphabet?"

"Two: 'ad' and 'da'."[3]

I'm pleased to see that each of the kids has remembered to bring his or her math notebook to math club, and opened it to the first blank page. So I say "write down those two."

Then I ask "Can you use the same letter twice in a word?"

They're dubious, but willing to accept my hypothetical. They come up with

```
ad da aa dd
```

They have fun trying to pronounce "dd"; I need to corral them back to thinking about the task at hand.

"How many three letter words?"

"Six" is the most common guess.

"Write down as many as you can."

Looking over shoulders I see a typical random list

```
aaa ada add dad ddd daa
```

Since they expect the answer is six they tend to stop after they've found six. Most of them find the ones they can pronounce, and `ddd`. But usually the group generates more than one list, and often someone has seven words. So I ask them to collate what they have and come up with a common list with all the words anyone has found.

Confusion ensues.

I ask each kid to write his or her list in alphabetical order in his or her notebook.[4]

[1] This idea occurred to me as a strategy because as a mathematician I am by inclination a combinatorist. My research is often about how to count things. That proclivity combined with the years I've spent teaching computer science prompted by decision to count strings on alphabets as a prelude to introducing number bases.

[2] I'd have chosen `a` and `b`, of course. But the whole exercise works better if the kids choose the alphabet. They are more likely to own the work.

[3] It's an interesting accident that both of these are almost real words, and actually pronouncable.

[4] How can I avoid the frequent ugly "his or her" and still be grammatically and politically correct?

"How many four letter words?"

Everyone guesses 16.

"How could you know there were 16 without either just believing the pattern or writing them all down and counting? Why don't you start writing them down."

They all begin with `0000, 0001`. Some say `0011` comes next, but Caswellknows, and insists, that it should be `0010`. When they have worked their way correctly through the first 10 numbers

```
0000
0001
0010
0011
0101 (corrected after some kids suggested 0111)
0110
0101 (Caswell to the rescue)
0110
0111
1000
1001
```

Rory says "it's just the rest of the list of three letter words, with a one in front instead of a zero."

From there it's a small step to understanding why the word count doubles each time you allow an extra letter.

Then I ask the same set of questions for a ten letter alphabet.

"What should the letters be?"

All start with "`1, 2, 3 ...`".

When they get to `10` they realize something is wrong. They don't have a letter to go there.

Daniel says "Use the numbers from `0` to `9`!"

I agree, ask him what led him to that thought. I expected him to say it was the analogy with the two letters `0` and `1` we used for the two letter alphabets but he surprised me:

"I remember in the first grade thinking that the number line should begin with `0`, not with `1`."

I start talking about the hundred's chart in the first grade classroom, which had been rearranged to start at `0`, and then starting to tell them that when some day they learn to program a computer they will see that it's often better to start counting at `0`. But I sense the kids drifting off, losing interest, not following what I say, so I kill the digression and move on, making sure first to praise Daniel for his suggestion. So let's move on.

The kids clearly know that there are 10 one letter words. Rory guesses that there will be 20 two letter words. That's a common kind of error – addition is more primitive than multiplication; it's often a first guess. I suggest that they try starting to write out the two letter words. They clearly see that the sequence continues `11, 12, ..., 19`. Then Caswell (again) realizes that they are just counting to 100. After a little discussion about whether there are 99 words or 100 they settle on 100 since the zero must be counted. [5]

"Look at the list you've started to write. You were thinking about this as a list of two letter words. If you think of it instead as a list of the two digit numbers it's easy to count them!"

They can then immediately fill in the table

table here up to seven digit numbers

Then I point out that the words with letters `0` and `1` can represent numbers in base 2, which I explain.

I ask whether anyone knows why there are 10 digits (counting the zero) before we start the next column. No one knows. I say it's because you can't count higher on your 10 fingers.

Interesting to make the transition from base 2 words to base 10 words to base 10 numbers and then back to base 2 numbers.

[5] I resist digressing to talk about fence post problems..

I talk about abacus (a.k.a. counting frame), show them my belt buckle, then draw pictures of a base 2 abacus.

I promise to continue this in two weeks at the last Red team third grade math club of the academic year. I'll start with

> In the binary system we count on our fists instead of on our fingers.
>
> Author unknown

The vocabulary issues are fun – a fact that Tom Lehrer takes advantage of. What do you call "100" when it's a number in base 8 – "one hundred" or "sixty four"?

6.5 Clock arithmetic

The second graders learn to tell time. It's hard. I remember that is was hard for my daughter – I think because she really hated the tyranny of the clock. It's hard for everyone for several reasons.

- Kids nowadays are more familiar with digital than with analog clocks.
- The circular dial on the clock represents two scales simultaneously - hours and minutes. (The minutes scale may or may not be visible.)
- Fractions appear: halves and quarters.
- On a real clock, the hour hand points directly at a number only on the hour. So reading the hour value at any other time is tricky. It's not the *nearest* hour.

Teachers (and parents) will probably be more effective if they're conscious of the reasons.

None of that came up in math club. Kids there are third, fourth and fifth graders. For them telling time is a skill I can rely on to build other skills. [6]

In particular: what time is it five hours after it's eight o'clock? They know it's three. So they can in fact do modular arithmetic.

The only change we need to make in the clock is to use a 0 instead of a 12. [7]

Then it's easy to introduce addition mod 12.

eight, two and three hour clocks, addition and multiplication tables. Zero divisors. Can we get to the theorem that says Z_n is a field just when n is prime? That probably goes in the answers chapter.

Note the similarities/differences between arithmetic in base n and arithmetic mod n.

6.6 Platonic Polyhedra

Over the years I iterated through ways to study the Platonic polyhedra. . The first time I brought models.

Outline:

tetrahedron. Show model, count faces vertices and edges. "edges" was a spelling word that day. It's important to count first.

Two drawings, one the square with diagonals, one the Schlegel diagram.

Then same for octahedron. The two drawings are the square with internal stuff (vertex on view), the other is the top down view showing reciprocal opposite triangles. Turns out that one is better. So I try the perspective view too, the one I always draw.

Important for the kids to get a chance to build them.

[6] Another instance of the fact that after you understand something it's useful to "just know it" rather than having to understand it all over again when you need it. Save the processing power for the *current* problem rather than for one already mastered.

[7] Yet another place where starting at 0 is better than starting at 1. (The other places - the hundreds chart and in computer programs.)

```
October 5.

First day of the Team3 math club, day after Rosh Hashonah and I am not
prepared. Joan says bring your models. I think about the plastic
triangles and squares (from "But it moves") but wonder if I can find
them. Then early in the morning they are in the first place I
accidentally look - beshert. So I sprint to Staples for notebooks (I
don't want them to use scrap paper this year), grab them and head to
school.

I ask them to unfold a regular tetrahedron as many ways as they can,
and to find configurations of four triangles that won't fold, and to
record what they discover.
(Turns
out there are two configurations that fold and one that doesn't.)
Wildly successful. Some kids trace, some can actually draw the
configurations relatively accurately freehand. I show them how to draw
a perspective view of a tetrahedron (essentially as K_4 with the four
points on the exterior of their convex hull and one of the diagonals
visibly behind the other). Of the 10 or so third graders all started
out focussed. When they'd exhausted the tetrahedron I gave them a
square base and asked them to do the same for the square pyramid. More
variety here. At this point some kids (Daavi in particular) just
wanted to play with the shapes, building cool stuff. I couldn't really
stop them, nor would that have been a good idea. I did prompt them to
record whatever they did, but to little avail.

Christopher is now in third grade. He came in late because he hadn't
been told (or had forgotten) that math club started. He got right to
work. After finishing the tetrahedron he proceeded to discover the
octahedron! And wanted its unfolding to be symmetrical, so he made the
butterfly.
```

Same exercise a year later with the next generation of third graders. The blue(weaker) club. And I forgot about the folding.

But was able to teach how to draw the tetrahedron by constructing K_4. Most kids get

quadrilateral with one diagonal

right away but can't see the back missing edge. Teaching them to discover it is hard. Sequence of suggestions: make the vertices really prominent. Label them. Keep looking back and forth from the model to the picture (presumably this is what artists do)

One piece of dialog:

"Which edge is missing?"

xxx gestures inconclusively at his math notebook.

"I don't know what you're trying to show me when you just wave your pen over your picture. You can't find the answer by looking at me. You need to look at the thing and at your picture and match up the pieces you've already drawn."

"This is hard."

"Yes, it is. If it were easy it wouldn't be as much fun, and we wouldn't need to do it in math club."

Shy smile.

The octahedron is next. I build it from the kit (It might be better to have sticks and balls, so the far edges were visible behind the near ones, but then it would be harder to enforce triangular faces.) triangular faces picture of the triangles, with octahedron built and unfolded

Teaching kids to draw this is hard (else why bother trying to do it ...)

How many corners does this figure have?

Next iteration - actually build the platonic polyhedra from their nets.

(Full size full color versions available on ItsElementary web site.)

Instructions for building the stellated dodecahedron are in the Answers chapter.

Moral of the story? I wonder what I will do this year. The most satisfying but in a way least mathematical version is the last one. Kids like cutting and pasting. But in a way they learn more by trying to draw the figures, and trying to count features. Perhaps this year I will try both.

Fall 2007. It's now this year and I know what I'm doing in at least one sense of the words. I can report on what I'm doing. But I'm still not sure I know what I'm doing. Each time I work with the kids on the platonic polyhedra I reinvent the lesson. I usually don't remember accurately what I did the last time, so reading about it here surprises me.

This time it's third grade math club with the same kids every week. Math club starts slowly. (Maybe I wrote this stuff earlier in this chapter since I did the counting words exercise on the spur of the moment the first day.)

Second week: four kids, and I was prepared with the yellow sticks and rubber connectors:

Photo of sticks and connectors,
polydron triangles,
assembled tetrahedra

With the stick and connector models it's easier for them to count edges (and vertices) since they can disassemble the model.

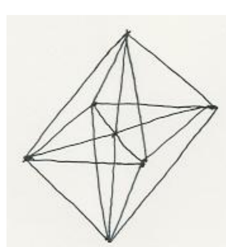
They built tetrahedra, and drew them. At the end of the hour we'd just started on octahedra. The kids wanted to take stuff home to work. I was reluctant (why?) but willing. Lorraine provided plastic baggies.

Homework has never worked in math club. But this time I found this message in my mailbox on Tuesday.

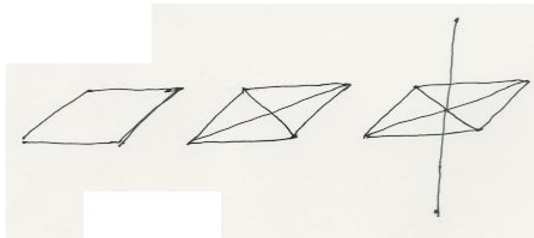
```
Hi Ethan
  We have one more addition to Math Club.  Harry Caffrey Maffei was not in
class when we discussed Math Club and really wants to do it!!  So if that is
ok he will join tomorrow.  He is loves MATH!  Btw I thought your last
homework assignment you gave them was GREAT !!

Maria
```

When I came in the next morning Rachel showed me her math notebook – with excellent attempts at pictures of the octahedron! So homework works this year.

Novak proudly showed me his perfect picture of an octahedron:



I pointed to one of the diagonals in his picture and asked which of the edges of the octahedron it represented. He didn't quite understand my question, so I asked him to show me how he'd drawn his picture. Here are the steps he followed:

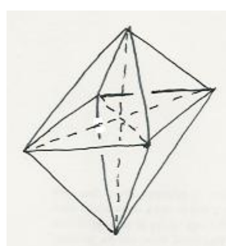


He drew the rhombus (a perspective view of a square), then its diagonals to find the center, then the vertical line through the center to locate the last two vertices of the octahedron. He finished the job by connecting those two new vertices to the four he already had.

“How did you figure that out?” I asked.

“My dad showed me.”

It’s clear to me that Novak and his dad had a good time. I don’t want to discourage that. But it’s also clear that he’s following his dad’s drawing algorithm, not seeing the figure as draws it. I try (almost in vain) to convince him to draw the auxiliary construction lines only lightly, and to break the edges that are in the back so that the end result will be



Note the difference between this dad’s intervention and the question from Christopher’s dad about how much to help with the problems. I don’t want to tell Novak’s dad not to do mathematics with his son. How can I engage him differently? I work with Novak on the other two ways to draw the octahedron, and suggest that he try to teach them to his dad. We’ll see what happens.

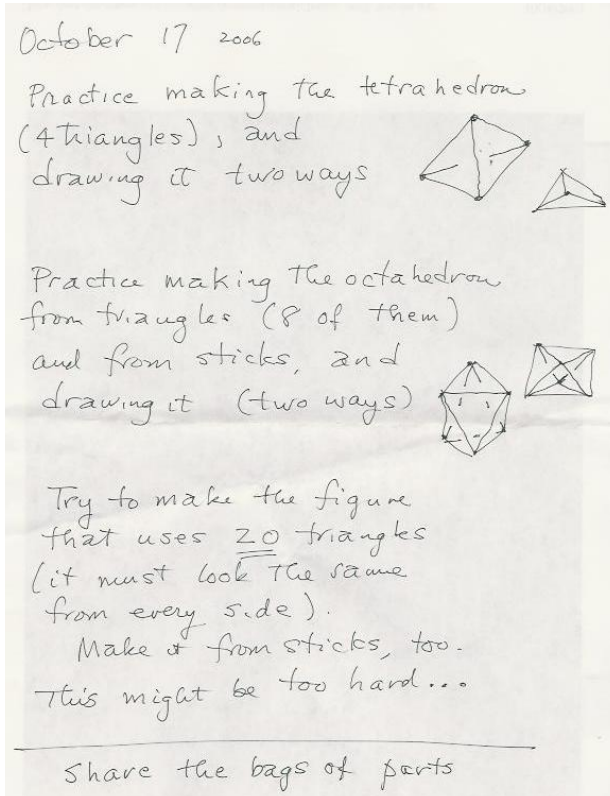
Catching up with Henry I asked Lorraine for a flashlight, to project the tetradron onto the page two ways.

pictures and description here

Next week’s homework. I rapidly assembled three baggies each of which contained 20 polydron triangles, at least thirty yellow sticks and at least a dozen connectors. I wrote out this assignment, made copies and asked Maria to distribute them to the math club kids, apologizing for adding work to her day.

“Anything for my kids,” she said.

We’ll see what happens next Wednesday.



October 17 2006

Practice making the tetrahedron (4 triangles), and drawing it two ways

Practice making the octahedron from triangles (8 of them) and from sticks, and drawing it (two ways)

Try to make the figure that uses 20 triangles (it must look the same from every side). Make it from sticks, too. This might be too hard...

Share the bags of parts

6.7 Base change redux

Excel spreadsheet, computers enter the picture.

Euler paths

Elaine, hint at solution to Persi's magic trick. Christopher likes these problems.

6.8 Geometric series

Now there are four math clubs. (Do they double every two years?) Fall is third grade Red and Blue. (Cardinals and Bluebirds?) Spring will combine grades four and five.

Here's the first set of exercises I proposed for the third grade Reds.

> *What comes next, and why?*
>
> Copy each problem into your math club notebook and work on it there. In each case try to find the pattern. Try to predict what the next ? will be before you work it out. Describe the pattern if you can.
> Work alone or together - whichever you prefer.
> Play with the ones that look the most interesting to you. You don't need to think about them in order.
> Some of these questions might be too hard – they might take us a few weeks to figure out.

1.

$$
\begin{aligned}
1 &= 1 \\
1+2 &= 3 \\
1+2+4 &= 7 \\
1+2+4+8 &= 15 \\
1+2+4+8+16 &= ? \\
1+2+4+8+16+? &= ?
\end{aligned}
$$

Hint. Can you see a connection between the last number you add and the sum on the *previous* line?

Could you use snap cubes like the ones you played with in first and second grade to explain *why* the answer comes out the way it does?

2.

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 = 2 \times 2 \\
1+3+5 &= 9 = 3 \times 3 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= ? \\
1+3+5+7+9+? &= ?
\end{aligned}
$$

3.

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+9 &= 13 \\
1+3+9+27 &= 40 \\
1+3+9+27+81 &= ? \\
1+3+9+27+81+? &= ?
\end{aligned}
$$

Hint: Look at the hint for #1.

4. I don't know how far you can go with fractions, but try this one:

$$
\begin{aligned}
\frac{1}{2} &= \frac{1}{2} \\
\frac{1}{2}+\frac{1}{4} &= \frac{3}{4} \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= \frac{7}{8} \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= \frac{?}{?} \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{?} &= \frac{?}{?}
\end{aligned}
$$

5. Here is a very different kind of problem. If there are two people in a room and they shake hands there's one handshake. If there are three people in a room and everyone shakes hands with everyone else there are three handshakes. If there are four people it takes six handshakes (check that please). How many

Notes: Problems 1 and then 2 didn't work well. Too hard. Great variation in the abilities. Billy is sharpest. Daniel is as impatient as his brother Nathan. Sister Lisa seems more composed.

Dialogue with Daniel about problem 1.

"I think it's 27."

"Why?"

"I just guessed."

How do I encourage guessing while discouraging wild guessing - the kinds that's trying to guess what's in the teachers mind rather than what the answer to the problem might be? I tried.

"How many numbers are there altogether?"

"A zillion gazillion."

"Even more than that. Infinitely many. So if you guess one at random what do you think the odds are of getting the right one?"

"Not very good ..."

From his tone of voice I think I made my point.

6.9 Handshakes

I suggested they move directly to number five. Billy drew

picture of K_4

and counted the six edges, but he couldn't get the count right for K_5.

Peter did that one by counting on his fingers. Thumb needs four handshakes, then index finger three, ... down to none for the pinky: $4 + 3 + 2 + 1 = 10$.

After a short while Billy saw the pattern.

But I don't see these kids for another two weeks.

And I can't do this with the Blue group. Maybe just the handshakes?

The next week I did handshakes with just the two Blue group kids who showed up.

Returned to them with the Red group a week later. Billy remembered where he was.

He wrote the table out to 10 kids. I asked him if he could figure out the answer for 20 without filling in the intermediate results. He (and Daniel) understood the problem, but couldn't solve it. I suggestd they look for the relationship between n and $2h$ first. Daniel found it: $n(n - 1)$.

I suggested they use a calculator to work out the actual arithmetic – distinguishing between the arithmetic and the mathematics.

6.10 Summing the odd numbers

This generalizes the handshake problem. Pattern can be guessed from numerical examples. Then can do the visual proof. Try this with math club in 2007-2008.

6.11 Peter's puzzle

Peter comes to math club with two pieces of origami paper and a pair of scissors. He draws eight regions on the orange sheet and cuts eight numbered squares from the yellow sheet:

The puzzle: arrange the eight numbers in the eight spaces so that each square space contains a number that's the sum of the numbers in the two adjacent round spaces.

The kids like to bring me challenges (Christopher did it often). This one wasn't too hard. The 1 and the 2 must be on corners. The 8 and 7 must be on sides. There seems to be just one way to finish.

As a mathematician, the next thoughts are about generalizations. It's clear how to state the puzzle for the numbers $1, \ldots, 2n$. There are no solutions for $n = 4$ and just one (up to symmetry) for $n = 6$. What happens next?

The closing words: Daniel says

"I think I know what you're doing. You're trying to mess with our brains."

6.12 You have to want to . . .

January 2008 – first day back from winter break.

Novak: "Sethi and Sarah don't want to come today.

"That's OK. Math club is only for people who want to do it. Have you got your math club notebook here?"

He opens the book to the next free page.

"Were you at the Manning for first grade?"

"Yes."

"Remember when you were learning to add small numbers by solving problems like 'You have three pieces of fruit. Some are apples and some are bananas. How many of each might you have?' I want to work on some problems like that now – to learn other things from them."

With some prompting, Novak writes in his book

	A	B	
	1	2	
3 peaces of fruit	2	1	four ways
	0	3	
	3	0	

"What if you had four pieces of fruit?"

Novak: "There's no one here to talk to."

I think "Don't I count?" but say "We're doing math. What do you need to talk about?"

Novak (pulling his iPod from his pocket): "There's just this to listen to. Can I go down to the schoolyard and tell Mrs. Brown that Sarah and Sethi need to come to the club?"

"The club is just for the people who want to be here. You can go downstairs and ask them to come up, but it wouldn't be right to ask Mrs. Brown to send them. If you would rather not be here that's OK – you can join them downstairs."

"I can't do that because my Mom wants me to come to math club."

"Well it looks as if the third grade math club is just about over for the year, since the club members seem to be thinning out. So we can make today the last meeting."

"But then you won't get paid, will you?"

"I'm not paid to run the math club. I do it just because I want to – the very same reason I hope kids have for coming.[8] In any case you needn't worry, because I was going to announce today anyway that this was the last of our meetings, because I want to start the math club for the fourth grade next week."

Novak (with obvious but unspoken relief that he need come no longer and need not tell his mother why) "Can I go get Sarah and Sethi now?"

6.13 Apples and Bananas

Sethi and Sarah come back with Novak. To my surprise, both are immediately engaged – Sethi, who's usually the quickest but didn't want to come today, and Sarah too, who's often easily distracted. Novak stops participating, but the two girls rapidly work out

number of pieces of fruit	number of ways
3	4
4	5
5	6

The girls know immediately that the answer will be 101 ways for 100 pieces of fruit.

"What if there were three kinds of fruit? We need a third kind to think about, starting the letter C." I am thinking about cherries.

Sethi: "Cantalopes." Sarah: "Clementines."

We settle on clementines.

We did enough examples to establish the recursion – i.e. guess the pattern in the table.

Then refer to final chapter for the proof, which the kids aren't interested in. The pattern is enough.

Remember to talk about number of pieces vs number of kinds, either here, or, better, where the first and second grade are working on attributes.

6.14 Girls' morning out

Five girls and no boys show up for the fourth grade math club in the spring of 2008. I announce

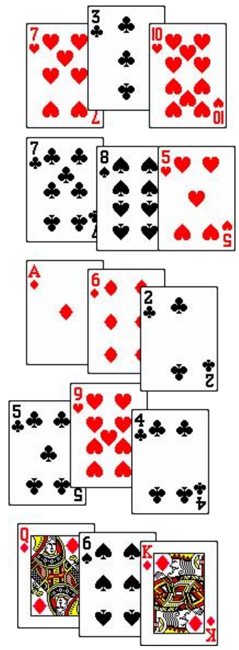
"If any boys want to come after this we'll tell them the club is closed."

First day: Persi Diaconis' card trick. All the girls learned it in an hour, for the eight card deck. I thought about the following week – tentatively promising to do the math for a 16 card deck, but not the trick.

The more I thought about that the less I thought it would work. So I looked around for other card tricks. There are lots on line, and several in Ball's Mathematical Recreations and Essays (a good excuse for me to look at that again). Finally, I invented my own (I'm sure it exists elsewhere). Here's how the next week went.

I took a deck of 15 cards, held them face down and dealt them into five rows of three, face up, like this, careful to overlap the cards as I dealt them in order from the first seven of hearts to the King of diamonds.

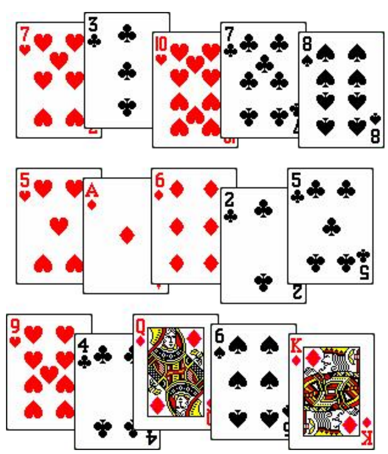
[8] Not strictly true, since I have convinced the University that the work I do at the Manning counts as a part of my teaching load. But true enough so that I can make the point about trying to spend time doing what one loves to do.



I asked xxx to choose one of the cards to remember and tell me which column it was in, calling the columns headed by the seven, the three and the ten "one," "two" and "three."

xxx "It's in column one."

Then I picked up the rows, starting with the last one, putting each set of three on top of the pile so far, so that when I turned the deck over the cards were in their original order. I dealt them again face up into three rows of five.



"Which column is your card in?"

xxx

"The second."

I pickep up the cards as before, so I was again holding the deck face down in its original order.

I thought about finishing the trick silently, but decided instead to engage them in the secret arithmetic right away.

"The magic numbers are 10 and 6. I'll tell you why later. Now what column was your card in the first time?"

"The first."

"The magic number for the three columns is the 10. 10 times 1 is 10. Remember that. Which column the second time?"

"Column 2."

"The magic number for the five columns is 6. Two times 6 is ..."

"Twelve."

"And 10 (you remember the 10) plus 12 is ..."

"22."

"Now I can tell you your card. Since there are just 15 cards, 22 is too big. What do you get if you take away 15?"

There's a pause while she does the arithmetic in her head.

"7."

"Good. Now watch."

I deal the cards face down from the top of the deck, counting as I go

"One, two, three, four, five, six."

When I reach the seventh card I turn it over - it's the Ace of diamonds.

"Was that your card?"

"Yes. How did you know."

We do the trick several more times. After you've done the mental arithmetic you take out as many 15's as you can. They understand that each 15 means dealing out the whole deck and getting ready to start again, so why bother. They have trouble doing the mental arithmetic, though.

It becomes much easier when I point out that the last column in each case will always mean a whole number of 15's, since 3×10 and 5×6 are both 30, which is 2×15.

```
Math club will be all card tricks this year.

Set up the 3*15 problem.

Explain that it works because of "leftovers"

First leftovers mod 15, since you don't have to count around the deck
multiple times.


For 3*7, need two magic numbers.

For 3, want A == 1 mod 3 and 0 mod 7
For 7, want A == 0 mod 3 and 1 mod 7

The best way to find these (if you're a fourth grader) is to write the
table

n       1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21
mod 3   1  2  0  1  2  0  1  2  0  1  2  0  1  2  0  1  2  0  1  2  0
mod 7   1  2  3  4  5  6  0  1  2  3  4  5  6  0  1  2  3  4  5  6  0
                          x                       x

so the magic numbers are 7 and 15 and the unknown card is at

   (3 column)*7 + (7 column)*15 (mod 21)

The kids will be able to understand the formula, but not do the
arithmetic in their heads. Calculator? Maybe that would seem even more
```

magical.

What about a 35 card deck?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
mod 5	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1
mod 7	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0
															x						x

What happens with 4 and 5?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
mod 4	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0
mod 5	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
					x											x				

so the magic numbers are 5 for 4 and -4 for 5. The will work whenever you have an n by (n+1) deck.

6.15 Gergonne

6.16 Emily

On a cold April day I pull into the Manning parking lot just at 9. I see Emily get out of a car; her mother pulls up next to me on her way out.

"Thank you for having Emily in the math club. She so looks forward to each Wednesday morning."

Emily is a pudgy redhead who enjoys building houses with the plastic polygons I provide for constructing Platonic solids. She shows little interest in the mathematics. I just let her use the math club time as she wishes – nothing about the club is compulsory. But I do wonder what to say to her mother.

"We don't cover material in the math club that's at all connected to what she's learning in class. Have you noticed any differences there?"

"Oh yes. I don't need to spend nearly as much time with her this year on her math homework. She just seems to get it herself much more quickly. Thank you."

Of course I answer "You're welcome." And go home to ponder what has made the difference.

6.17 2008

November 20, 2008.

Second meeting. Second disaster. We meet after school rather than before (explain why). That's a bad idea in general -- kids are tired and edgy since they've been cooped up all day. And in this case it's a particularly bad idea because \Morris\ is hyperactive and can't sit still. His misbehavior is contagious. \Spencer\ and \Dashiell's mom came to pick them up and I talked about the difficulty. Then \Morris' mom came. She asked if he'd done his hundred jumping-jacks, which of course he hadn't -- he turned down my suggestion that he run around the yard twice between school and club.

I did manage to count handshakes -- only \Spencer\ was really paying attention, and he got the pattern. In fact \Morris\ saw the n*(n-1) right at the start, but couldn't hang on to it or remember it when I tried to introduce the double counting solution later in the hour.

Next time (after Thanksgiving) I will try platonic polyhedra. Maybe having the toys to play with will help.

Chapter 7

The Math Wars

In an early plan for this book this chapter on the Math Wars was just one page long. It began and ended with a single sentence I've seen in software manuals, centered and boxed:

This page intentionally left blank.

But my wife Joan convinced me that I had to write about the math wars, so I needed a real quote to start the chapter. I found three and chose not to choose among them. Here's the first:

> The Talmud tells about two parties to a dispute who seek the rabbi's help. He listens to one of them and says, "You're right." Then the other gives his side of the story, and the rabbi says, "You're right." The rabbi's wife, listening to all of this, says to her husband, "How can you say that he's right and he's right? How can they both be right?" The rabbi says "You're right, too."
>
> adapted from
> www.interfaithfamily.com/site/apps/nl/content2.asp?c=ekLSK5MLIrG&b=297399&ct=1283677

The second addresses the math wars dispute from the petitioners' points of view rather than the rabbi's or his wife's:

> ... their judgment was based more upon blind wishing than upon any sound prevision; for it is a habit of mankind to entrust to careless hope what they long for, and to use sovereign reason to thrust aside what they do not fancy.
>
> Thucydides, *History of the Peloponnesian War*, Chapter XIV,
> http://etext.library.adelaide.edu.au/t/thucydides/crawley/chapter14.html

Finally, one of my son Benjamin's favorites (he's a mathematical biologist):

> For every complex problem there is an answer that is clear, simple, and wrong.
>
> H. L. Mencken

@copyright Jim Haberman
(looking for permission to reproduce)

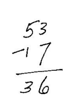
All three quotations (and the picture) are relevant. Teaching elementary mathematics is clearly a complex problem. In this chapter I'll collect and comment on the stories that seem to me particularly relevant to the math wars – they're mostly about "the right way to do arithmetic." I hope when you've read them you can claim me as an adherent of your faction (if you have one) only by quoting me selectively.

7.1 The right way to subtract (at parents' night)

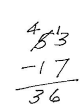
In my first spring at the ManningI went to a parents' night. One of the authors of the TERC materials worked with about half a dozen parents who wondered why their kids weren't being taught to calculate the "regular way." She handed out pencils and paper and asked each of us to find the difference between 17 and 53.

Here is what we offered.

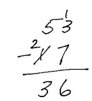
1. Just this statement of the problem and the solution, with no further explanation:



2.



3.



4. "From 17 to 20 is 3 Then it's 30 more to 50, and another 3 to 53. So the answer is $3 + 30 + 3 = 36$." Nothing written down on paper other than the answer.

5. "$17+40$ is 57, which is 4 more than what I want, so the answer is $40-4 = 36$." Again, nothing written down.

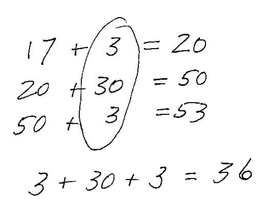
The variety stunned the group. People were surprised because they thought they knew *the* way to do the problem. All did agree that each method led to the correct answer.

Is any of these algorithms better than the others? I tell my software engineering students that all interesting questions have the same answer: "it depends." The boring questions are questions of fact, boring because you can look up the answer somewhere – nowadays easily on the internet. I think this is an interesting question.

The second and third methods are straightforward. Once you have practiced them they require very little thought (which is good). You probably do need a pencil and paper to work them out, even if you don't write down the borrowing. The third requires a little more pencil movement than the second.

The fourth or fifth you can do in your head after you've done it a few times on paper.

The fourth is the one the kids learn in the second grade at the Manning. When they start they're taught to write down the work this way:



circling the 3, 30 and 3 is how they remember that these were the things added to 17 on the way to 53.

Kids can understand why it works even before they are completely comfortable with the meaning of place value: why the 3 in 53 is not the same as the one in 36. It doesn't require as sophisticated an understanding of positional notation that the ones that call for borrowing do. In the second grade I think this is actually a pretty good way to start out subtracting. The kids think so too. My quarrel with it is my observation in the second grade classroom that what is at first presented as "a way to do the problem" soon becomes "the way to do the problem." In principle the TERC/investigations curriculum leaves the students with a choice of algorithms. But in fact there does not seem to be time in the curriculum to teach more than one.

This particular subtraction algorithm comes with often unacknowledged disadvantages too. Since positional notation is addressed only obliquely, it does not afford an opportunity to reinforce (or teach) place value. It's also longer than the others. You need to solve three small subtraction problems to find the 3, the 30 and the 3 and then an addition problem to add them up. And it grows proportionally longer still as the number of digits in the problem grows. With algorithms two and three it takes just twice as long to subtract four digit numbers and two digit numbers. But adding up as in algorithm four may take much more than twice as long.

I doubt that anyone who knows any of the "standard algorithms" would use this one regularly. I wouldn't. I doubt that the teachers do. I've seen some of the kids avoid it in favor of one of the standard ones they've been taught at home.

But I think that the Manning kids know more about positional notation in earlier grades than I was taught in the good old days. They've done a lot more arithmetic in much more varied contexts than I had done by a similar time.

In the early fifties the New York City elementary schools shifted from twice to once a year admission. So all the kids who'd already started grade 1A in the spring would have no B class to enroll in that fall. They would have to repeat the A class they'd just finished, or skip the B for that year. Since I was to skip 3B I stayed after school one afternoon to be taught "carrying" in addition. I remember thinking then "is this the *only* new thing I would have learned in 3B that I would actually need to know in 4A?"[1]

I hope that all sides in the math wars would agree that knowing any of these algorithms is better than needing a calculator to do the problem. I have never seen that happen at the Manning. [2]

7.2 The right way to subtract (second grade 2008)

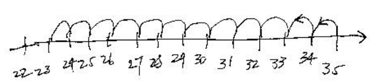
In January of 2008 (long after that parents' night) I was watching Ms. Windus [3] working with the second grade on a subtraction problem. This one was easy: no carrying:

$$35 - 12$$

As usual, she elicited several strategies from the class. The first is straightforward:

$$35 - 12 = 23$$
$$35 - 10 = 25$$
$$25 - 2 = 23$$

Then she tells them she wants them to do this problem by counting backwards on a number line they construct for the purpose:



This makes me unhappy – it looks to me like moving backwards, not just on the number line (no problem there) but from a more abstract to a less abstract algorithm. But I keep my mouth shut. And then she says she knows most of them will find this unnecessary, that some will find it useful, that all should do it now and that she'll be working soon on more sophisticated ways to do the problem.

Then she asks for other strategies. Declan proposes:

$$35 - 12$$
$$30 + 5 \qquad 10 + 2$$
$$30 - 10 = 20$$
$$5 - 2 = 3$$
$$20 + 3 = 23$$

[1] I can't remember whether the algorithm I was taught at Brooklyn's PS 139 was 2 or 3. Neither is what I do now when I need to subtract. If I need to do the problem in my head I use a variant on #?. If the problem is written down I think $7 + (6) = 13$ so the answer ends in a 6. There's [1] to remember to carry. $[1] + 1 + (3) = 5$ so the answer is 36. I don't really say those words to myself. I just know the parenthesized digits. I found reprogramming myself as an adult was surprisingly difficult. The only problem I face with my new algorithm is that I can't check my subtraction by "adding up" since that's just the way I do the problem in the first place.

[2] I have noticed a change over the years in the way the kids do use calculators when they use them. When I first came to the Manning they pushed the buttons with an index finger. Now it's with their thumbs. Game-boy practice, I think.

[3] She's the new second grade teacher – Susan hs retired)

Sharon asks "Why do you add the 20 and the 3?" It's clear to her that Declan knows what he's doing. She's probing for an explanation, so that she can help the rest of the class see that you need to understand the underlying counting reality to know when to add and when to subtract – just memorizing rules won't do.

Declan has trouble answering. That's not surprising. Sharon asks me for advice; we have a rare discussion of pedagogy, right in front of the class (usually we wait for a private moment to work out questions like this one). She tells the class I'm there to teach her.

I suggest that she leave the explanation out and move on – the kids who understand Declan's method need no further help, and the ones who don't yet will only become more confused. She likes that suggestion!

Afterwards I start to tell her that what makes Declan's argument work is positional notation – completely absent from the other two methods. But she's way ahead of me.

"I know that," she says. "Positional notation is on the agenda for the week after next. That's when the rest of the class will be able to understand Declan's work."

So I quickly shift gears from a careful attempt to introduce an important idea she'd not thought about to thoughts about how to help her when she gets to the week after next.

"When you do this problem that way then, how about using coins – dimes for the tens, pennies for the ones. Then it's clear from the physical evidence that you take a dime from the three dimes and two pennies from the five pennies, leaving two dimes and three pennies: 23 cents."

"What a good idea."

"Thanks. I think you should just explain using the coins as an idea. Don't give out the sample coins they worked with in the first grade – they'll spend more time sorting and counting and playing than they will on the mathematics. And don't even think about nickels!"

Reflection: why did this intervention work? (In fact, I'll try to see if it worked when the week after next comes 'round.) Sharon knows and trusts me. She's a better teacher – and surer of herself – than she was last year. I can adjust my comments on the fly to what I think she might find useful. I can connect parts of the curriculum that may not seem related to her, both because I know them, and because I'm a mathematician. In this case the connection was with the coin counting exercises I know are in the first grade curriculum.

7.3 Subtracting from 1000

In fourth grade in January 2009 Mr. Kearnan is working on positional notation. He's using the subtraction problem

```
1,000
- 357
        -----
```

to make his point. The students know that the 3 in 357 really means 3 hundred. There are no visible 100's in 1,000 from which to subtract those three. The teacher shows (using manipulatives) how to take the 1 in the thousands column and break it up into 10 hundreds, so he can rewrite the problem as

```
1x 10 0 0
-   3 5 7
---------
```

The "1x" is a crossed out 1. I will redo this in time with handwritten scanned text.

Then in order to have some tens from which to subtract the 5 that represents 50, he breaks one of the 10 hundreds into 10 tens:

```
    9
1x 10x 10 0
-   3   5 7
------------
```

and then one of the 10 tens into 10 units:

```
    9   9
1x 10x 10x 10
-   3   5   7
------------
```

Then the subtraction is easy, column by column:

```
    9   9
1x 10x 10x 10
-   3   5   7
------------
    6    4  3
```

I've watched the kids as he did his explaining – vividly, with karate chops to break a $10 \times 10 \times 10$ cube into ten 10×10 sheets, then one of the sheets into ten strips of 10, then one of the strips into ten small single cubes.

Along the way he says several times and in several versions:

"I know some of you learned this method from your parents or an older brother or sister. We're doing it now so those of you who know it can understand why it works, and those of you who don't can learn it. It's not the only way to do this problem – we'll see some others soon – but for now let's learn this one."

As is typical in these whole-class minilectures, some of the kids were paying attention, following the argument, learning mathematics. Some were not, despite his excellent best efforts to involve them all.

Then the kids went to their seats to work out some similar problems on their own. I wandered over and sat next to Elena, who said

"I can't understand this and I don't know why I have to. I will never need to do this kind of math, even in high school."

"You've just said several different important things I'd like to try to tease apart. Will you let me try?"

Hesitantly: "OK."

"First, you say you can't understand what Mr. Kearnan was just doing. I believe that you don't understand it, but I don't believe that you *can't* understand it. If you were willing to work at it I think I could probably explain it so that you do. But I do realize that you don't particularly want to understand it, because you don't know why it's important that you should. And you're right. It probably isn't important. If you ever have a real subtraction problem to work on, you could do it another way, or use a calculator. By the way, do you know another way to do this kind of problem?"

"Sure. It's easy." She writes:

```
1000 - 300 = 700.
700 - 50 = [counts on fingers] 750
crosses out the 750 and writes 650
50 - 7 = [counts on fingers] 43
```

"643."

"That took you a long time. I think the new way might be faster."

"It didn't take so long."

"Let's see how long it did take."

I put my pocketwatch on the table and pose a new problem. We wait until the second hand is at the 12; then she starts. She finishes two and a half minutes later. Is that a "long time?" The answer depends on how much the time matters, how often you need to do this kind of problem, and whether you know another faster way. But I say

"I think you could do it faster the new way."

So I try to teach her. She really thinks she can't understand it. I can't figure out on short notice either how to convince her that she can, and to explain it in some way that makes it possible. By this time, the class has completed the new sample problem and the teacher has put three solutions on the board: the one he's been teaching, this one, and a third: $357+3 = 360$, $360+40 = 400$, $400+600 = 1000$, $3+40+600 = 643$. I point that out to Elena

"Look. Your method is one of the ones on the board. Could you do it the third way too, adding up?"

"Sure." [She shows me that she can.]

"So you do in fact know two ways to do this problem – three, if you count using a calculator. I still think you could learn the new way too, but maybe there's no need. Two out of three is pretty good."

Should Elena learn the new way? If she actually understood it (rather than learned it as an algorithm) it would help her understand positional notation. If she ever needed to do similar computations with much longer numbers it would be faster. The two methods she knows work well enough for subtracting from 1000, but are a little clumsier for subtracting from, say 1352. They have more steps, so more chance for careless error. But they can be done in her head; the new algorithm really needs pencil and paper.

Moreover, the second method is in fact quite common.. It's the best way to make change when someone pays with a $10 bill for an item that costs $3.57. The change comes out right and there's no need ever to know the actual amount of change delivered.

My conclusion: I think the "standard algorithm" should be taught, but not required.

Finally: near the end of the class I told the teacher that one other way to do the problem was to pretend that you were subtracting from 999 rather than from 1000. Then there'd be no borrowing ("breaking") required. When you were done you'd just add one to the answer. He thought that was cool, and put it on the board for the class as yet another possibility.

7.4 Counting Pockets

We've just spent a fair amount of time thinking about ways to do one rather uninteresting problem in subtraction. How about addition? Here's one example from the second grade.

Susan asks each kid in turn how many pockets in what he or she is wearing that day. She records the numbers on the easel as a list. [4]

7 + 3 + 4 + 4 + 8 + 5 + 3 + 10 + 4 + 11 + 6 + 3 + 14 + 5 + 7 + 4

Then the class uses the data two ways: to construct a histogram and to practice addition.

We'll take a look at the histogram in a minute. First lets look at the addition. How many pockets in all? Susan asks the class to start the computation, and kids volunteer. They've been taught to look for combinations of 10 and for doubles, and there are a few right at the beginning of the list. The computation starts this way:

~~**7**~~ + ~~**3**~~ + ~~**4**~~ + ~~**4**~~ + 8 + 5 + 3 + 10 + 4 + 11 + 6 + ~~**3**~~ + 14 + 5 + ~~**7**~~ + 4
7 + 3 = 10
4 + 4 = 8
3 + 7 = 10

The next reductions took a little longer to find.

~~7~~+ ~~3~~+ ~~4~~+ ~~4~~ + ~~**8**~~ + ~~**5**~~ + 3 + 10 + 4 + 11 + 6 + ~~**3**~~ + 14 + ~~**5**~~ + ~~**7**~~ + 4
7 + 3 = 10
4 + 4 = ~~**8**~~
3 + 7 = 10
8 + 8 = 16
5 + 5 = 10

[4] I've invented this list of 16 numbers. There are more than 16 kids in the class!

This is painful to write about, because it was painful to watch. In class it didn't go this smoothly – it never does. Susan was confused at one point about what had been done so far and needed to go back and correct an earlier error. I can't bear to finish the calculation now – it gets harder when you've run out of easy combinations of 10.

In the real world there are several ways that are much better – faster and less error prone.

- Write the numbers in a column rather than a row and add the old fashioned way, working sequentially rather than cleverly.
- Use a calculator!

When I balance my checkbook (do you still do yours?) I use the first method when the columns are short, the third when they're long. Math wars conservatives would have us teach just the first. I think it's wrong/foolish not to teach it. But it's more foolish to teach only that. The new curriculum at least offers the opportunity to suggest several strategies for a problem.

My journal tells me this confusing lesson happened in 2003. In the fall of 2007 I watched Andrea and Sharon do the pocket counting lesson again. This time it succeeds much better. Each kid gets a postit with his or her pocket count, and sticks it on a copy of the class list next to her name. She also takes the same number of snap cubes to contribute later to a total count. Then Andreaasks the kids to find combinations of 10. As they do, she removes the corresponding postits and writes the appropriate total on the whiteboard.

Toward the end of the computation when combinations of 10 have been exhausted kids suggest clever solutions like

"There's an eight and a three. Cross off the three on the postit and make it a one. Use the other two together with the eight to make another 10."

Since finding doubles aren't part of the strategy this time it's easy to find the total at the end just by counting by 10 and adding on the last few.

I suggest quietly that Andrea bag the cube count. She agrees, saying that Susan might have balked at my suggestion since she (Susan) always thought she should follow the rules.

I ask Andrea why the lesson went better this time. She says

"I'm learning, the lesson plans are better, Susan was rigid, the kids know more coming out of first grade, luck."

My last reflection? Yes the lesson went much better. As practice with groups of 10 it worked. Is counting pockets a good use of class time? I'm still skeptical.

7.5 Use a calculator?

Here's a calculator story that surprised me. In the fourth grade in January 2009.

Mr. Kearnan asked the class to sum six numbers, one in the thousands, the other five in the hundreds. I walked around the room looking over shoulders. The kids who did it the "up and down way" (i.e. the standard algorithm) mostly got the right answer. When it was wrong, it was easy to help them find their mistake. The ones who used another strategy fared a lot less well.

I talked briefly with Mr. Kearnan, pointing out that although the "break up the numbers" algorithm was conceptually cleaner than the standard one, it involved so many more steps and so much more writing that the probability of an error increased a lot. And that, moreover, if such a problem came up in the real world you'd probably use a calculator to solve it.

He was sufficiently interested in my thoughts to change the direction of the rest of the lesson. He did the problem on the whiteboard the up and down way, then gave out calculators and asked the students to check. *Very few got the right answer!* The shortcomings of the calculator, which should have been predictable, were quite clear. It provides no record whatsoever of the process. If you key in a number incorrectly, or skip one, or enter one twice, or subtract instead of adding, you get the wrong answer and have no way to know or even suspect. Even if you've followed advice and know the order of magnitude of the expected answer you can't detect errors in the less significant digits.

What's the lesson to be learned?

PS Next time I think that those kids who carry cell phones should be encouraged to find and use their cell phone calculators rather than the ones available in the classroom.

7.6 Applied mathematics

One of the best parts of the new new math (as opposed to the old new math Tom Lehrer sang about) is its focus on mathematics as part of the world rather than as a separate set of abstract procedures. (The bad news that accompanies that good news is that the idea the mathematics can be studied because it's beautiful/cool/fun seems not to appear anywhere. Of course it didn't in the good old days either. But that's what really drew me to the profession in the first place. Note that the same trend is there in college too, where applied mathematics has replaced liberal arts mathematics. Hey, I even wrote an applied mathematics book. Perhaps this book is my return to my roots. (But my elementary applied math book has digressions into fun stuff too - the quotes that start the chapters and the problems they generate.)

In the old old days school mathematics was applied - it was for prospective shopkeepers and carpenters and housewives who really did need to compute and measure. [5]

sample problems from one of my old high school texts

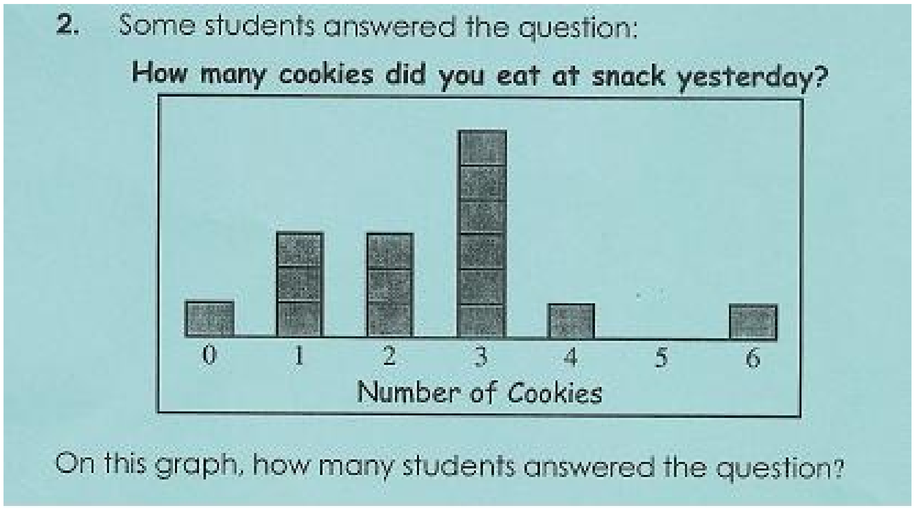
The "how many pockets" survey prompted a nice discussion of data collection and interpretation ("do the pockets in my jacket hanging in my cubby count? What about pockets that are buttoned shut and just for show?")

From my journal when the second grade was learning about the kind of survey discussed above.

```
Second grade. Data collection. swimming/painting/running/sleeping
survey. When it was my turn I asked for two check marks since I
_really_ liked swimming. A few years ago that would have thrown Susan
- this time she took it and ran with it. The kids all agreed that they
could not honor my request, since it would confuse the final count of
respondents. There were several suggestions designed to make me feel
better - a large check mark, or two check marks but in a circle so
they would count as just one. I felt a little like Nathan Dickler
working at being a troublemaker.

When the result came out 21/5/3/3 and they had finished the check that
21+5+3+3 was the same as the count of respondents, I raised my hand
and said that I thought that more people preferred swimming than all
other choices combined. Susan worked with that for a while, then said
she had another idea for a question. So did I. She asked me for mine,
which was "do twice as many people prefer swimming to anything else?".

That was her question too - I told her afterwards that she's beginning
to think like a mathematician.

xxx (who carefully pronounces the "ty" in "twenty" and doesn't say
"twenny" said "almost" - thinking that 2*11 = 22 which is just a
little larger than 21.
```

Here's a question from the second grade assessment examination (not the MCAS!) that tests whether the kids have understood the pocket-counting exercise. (The numbers are a lot smaller than the ones they learned with, since the point here is not addition practice.)

[5] The old old days aren't the good old days that people now seem to remember, when students could all read, write and calculate.



The histogram is properly done – in particular, it includes the empty columns.

Daniel, who is a bit of a wise guy sometimes, said

"I ate six cookies. 15 students answered the question."

Susan was surprised, and a little bit annoyed. She asked him why he thought that was the answer. So he reminded her that she was always reminding the class to read every test question carefully and be sure to answer completely. In this case there were *two* sentences that ended with question marks, so he should really answer both for full credit.

In fact that answer is hard to argue with. It's only wrong because "everyone knows that the first question is a description of the survey, not part of the test." But there's no way to know what everyone knows. The author of the test has just made an assumption. There are always assumptions.

consider moving this to the everyone knows section of the explanations chapter

The histogram suggests another way to solve the "how many took the survey" question: compute the sum as

$$x \times 0 + x \times 1 + x \times 2 + \ldots$$

This isn't really available in second grade since the kids don't know about multiplication, but I suggested it in class anyway ...

Interesting note: the question asked is *not* the one they practiced when counting pockets. The analogous question here would be "how many cookies were eaten?" But no one remarked on that at the time. (I just noticed it while writing this part of the book.)

Why doesn't the lesson continue with data analysis. The histogram makes it easy to compute the mode and the median. The mean is really hard, since the kids don't know multiplication, let alone division. But I think an approximation is in sight once they have the total. They can try to guess...

The data themselves are useless and boring. I've started trying to work with the teachers and the science teacher to replace pocket counting with a similar exercise using meaningful numbers.

Over the years we've come up with several possibilities.

- There is a science experiment going on in the second grade classroom at this very moment ...
- The first grade measured the circumferences of the pumpkins. Why not use those data? Or kids' heights?
- In the first grade you might combine beginning literacy skills with data collection: how many of the letters of the alphabet have round elements? Corners? The need for good ground rules (definitions) would become apparent: upper or lower case? Which fonts ?

```
From: Ethan Bolker <eb@cs.umb.edu>
To: ugima81060@aol.com
CC: ltheroux@gmail.com, eb@cs.umb.edu
Subject: data collection in the first grade
Date: Thu, 9 Aug 2007 14:59:54 -0400 (EDT)

Cynthia

Maybe there's time this year to plan a useful modification of the data
collection part of the grade 1 terc stuff. Something like replacing
the survey of how many kids like chocolate vs vanilla ice cream with a
question like how many letters have a round part to them, how many
not?

yes: BCDGJOPQRSU
no:  AEFHIKLMNTVWXYZ

(there might be a discussion about upper and lower case here ...)

Trickier: which of the letters can you write without removing the
pencil from the paper and starting somewhere else?

yes: BCDIJLMNOPRSUVWZ
no:  AEFGHKQTXY

(Of course the fact that you can do it doesn't mean it's the best way
- consider B, D, P, R, which would require some part of the path to go
unnaturally from bottom to top.)

And (or) consider doing something with the numbers you get when you
measure the pumpkins. That's a little tricker since the data are
numbers, not just a division into two categories. I wonder if there's
something in the science curriculum that does make two categories.

Looking forward to the year.

Ethan
```

But these are improvements that can be made only over time, either one classroom at a time when the teachers are ready, or by getting back to the TERC people and incorporating them in the printed materials. That's a big political task. I'd have to go to work for TERC rather than consulting at the Manning to make that happen. I'm not planning to do that. Maybe the TERC people will read this book.

7.7 One child at a time

Counting requires algorithms too.

On a warm November day I'm playing "towers of ten" with first grader Vanessa. We take turns rolling dice, collecting the indicated number of snap cubes and adding them to our joint production. [6]

[6] I notice for the first time after several years at the school that there are *two* kinds of cubes. One kind has one male and

Vanessa counts 34 cubes correctly, one at a time. (I need to help her a bit keeping track of the eye hand voice coordination - she doesn't move really carefully from cube to cube - more on that later)

xxxxxxxxxx xxxxxxxxxx xxxxxxxxxx xxxx

I flinch at the inefficient and error prone strategy. They have been taught about counting on, but it hasn't registered for her yet.

I ask if she can count by tens. She says yes. Then she counts

"ten, twenty, thirty" for the three towers of 10, then "forty, fifty, sixty, seventy" for the four single cubes.

I try gently to suggest that there can't be both 34 and 70 cubes - but she has no idea what I am talking about. And I can't explain it in terms she can understand. I don't try very hard.

She has not grasped the fact that the number she gets should reflect some fact about the world - the number of cubes.

When I ask Cynthia about this afterwards she thanks me for the observation. Knowing what to do next is *not* a job for a mathematician – it's a job for a first grade teacher.

The next week when I return I note that three other kids make very similar mistakes. Katie seems a little more collected than Vanessa – less shy. First she counts the three piles of ten and announces the answer as thirty. When I ask her about the other four cubes she says there aren't enough there. Since she didn't know what to do with them she ignored them.

When I ask her to include them she makes a tower of four, places it next to the three towers of 10 she already has, and counts

"ten, twenty, thirty, forty."

At a loss for what to do next, I asked her to count all the cubes one at a time. She did it very nicely, using two fingers to walk through the towers so she'd know she wasn't missing any. She ended at 34.

"Are 40 and 34 the same number?"

"No."

Last week I gave up on Vanessa, but this time I kept at it with Katie. I put down a single tower of tens.

"How many cubes here?"

"Ten."

Then I put down one more cube and ask "How many?"

"Eleven."

I add three more loose cubes one at at time.

"Twelve, thirteen, fourteen."

Then I put down two towers of ten. Katie says "20." When I add four cubes one at a time she counts

"21, 22, 23, 24."

One more iteration produces the correct answer correctly:

"ten, twenty, thirty, 31, 32, 33, 34."

Some kids know this right away. Clearly some struggle. I speak with Cynthia, suggesting that perhaps counting by tens has started too soon. (Note how many days in school felt board.)

But the incident does suggest some philosophical thoughts. I hope they're not too abstract for "It's Elementary".

At the foundation of arithmetic mathematicians distinguish between *cardinal* numbers, which capture the notion of "how many", and *ordinal* numbers, which, as their name suggests, capture the notion of counting in order.

The traditional way we "count" is to name the ordinal numbers in order, ticking off the objects we are counting. The number we stop at we then declare to be the cardinal number that tells us how many things we counted. That's Vanessa's 34. We know, but rarely say, that if we counted the same objects in a different order we'd still end at 34.

Philosophically speaking, cardinals are simpler than ordinals. With cardinals you don't have the extra order information to throw away. In the pseudohistory of mathematics the shepherd story shows that the

five female faces, so that you can build all kinds of structures. The other has one male face and an opposite female face, so you can build only towers.

idea of a one to one correspondence makes it possible to compare cardinals without being able to "count". For a while a number of years ago ("the new math") mathematicians even tried to persuade teachers that therefore cardinality should be taught separately.

Well, it could be. But in evolutionary/developmental terms that's not how we actually count. Kids memorize/learn the counting words "one, two, three, ..." in sequence and use them to count – of course. What Vanessa shows is that you can have memorized the sequence – a useful tool – without having internalized what makes that tool useful.

7.8 Vanessa one year later

In January of 2008 I'm back in the second grade classroom. (Susan has retired; Sharon is the new teacher.) I watch Vanessa work on

$$5 + 3 + 6 + 7$$

She counts correctly on her fingers to 21. Then, in order to "show her work" she writes

```
  5  +  3  + 6  +  7
  |     \    |      /
  |      \   |    /
   \      \ /    /
    \      /\   /
     \    /   10
almost doubles
```

and then her attention wanes.

replace this with scanned handwriting.

I talk to Sharon afterwards about this weird inversion. I note that it's hard to figure out how to move Vanessa along to some more efficient algorithm without taking away a tool she knows how to use.

I like Vanessa, and she thinks in interesting ways even when she's behind the class, so I tend to look over her shoulder when I'm visiting the class. In the January subtraction lesson discussed above she's working on the assigned subtraction problem: $45 - 13$, with instructions to solve it by counting back on a number line. She started her line from zero and got as far as 20 at the end of the paper when I show up. I suggest she start over again, working back from 45. So she turns the paper over and does that. I'm delighted to note that she can count backwards to label the ticks on her line, even when going from 40 to 39. But when she starts to count backward steps to do the problem, she starts with the tick mark at 45 rather than with the step from 45 to 44. When I ask her why, she tells me the rule: "I count one for each mark."

I explained that she should be counting steps, not tick marks. So she did. I'm not sure she understood that my method was correct and hers incorrect. I suspect she did it my way just because I was a teacher telling her a different rule. But she did it correctly, and was pleased to get the right answer.

When she was done, she started to do the work over again on the proper side of the page. I asked her instead to write "I did the work on the other side" instead. I think she liked that idea.

When I talked with Sharon afterward about Vanessa's error and identified it as a fence post problem she knew just what I meant.

Grab stuff from the qr book about new algorithms to insert here.

Maybe asking the kids to sit on their hands would make a game of it rather than a deprivation. Or asking her to count on her fingers as a check, only after doing it a new and different way.

Before this the kids were playing "close to 20." I suggested to Sharon that they might learn blackjack instead – same skills, more fun. Why not real playing cards instead of the number cards that come in the classroom? She was amused but probably won't/can't do it.

7.9 What about those flash cards?

I can still recall that in elementary school I always struggled with 6 and 7. I had the hardest time memorizing both $6 + 7 = 13$ and $6 \times 7 = 42$.

Not long after I started in at the Manning I was talking with Cynthia, who was frustrated with the TERC materials. She felt the kids weren't learning facts fast enough, and wanted to introduce some flash card work - "just a little." I rather agreed with her, but felt I should clear this with Casel first.

"Clearing with Casel" turned into a more interesting exercise than I suspected. When I raised the question with her, she agreed with me – then continued

"But I don't want Cynthia to do that, and I don't want you to tell her it's OK."

This was my first encounter with Casel as the boss. (The discussion in the chapter on the math club happened much later.) I was taken aback. I felt compelled to honor her wish, but I wanted to know why, if she agreed with me, we shouldn't act. Here's what she said (my reconstruction of her words, perhaps with some of what was between the lines added as part of the quote).

"When I started as principal here I knew that both the math and the literacy curricula needed to change, and that I needed to bring the teachers along with me in making the necessary changes. That was xx years ago. I was new to the job, and new to the teachers. I knew they were more comfortable teaching reading and writing than teaching mathematics, so I decided to start with them reworking the literacy curriculum. We did that together.

"Now they trust me more, so I can ask them to do with math what they've just successfully done with literacy – rethink how and what they teach. But they are reluctant still, and doubtful, and doing this at least in part just because I say they must. It would be easy to decide early – that is, now – that the new ideas aren't working. I want to insist that they do at least one full year of TERC 'by the book' before letting them make changes on the fly."

Five years later, the flash cards are indeed back in Cynthia's first grade classroom. She tells me

"This year's group of kids is much more passive/receptive/normal/quiet than last year's bunch. So I need to ground them in the TERC materials first, before I introduce the flash cards. But they are coming."

So both teaching styles have a place in the classroom, even if the TERC people never mention flashcards and conservatives abhor TERC style teaching. There's no reason why both sides can't be right. I think the same kind of dichotomy has come up in the teaching of reading. Sounding words out is important – but it's also important to just recognize some words without having to work at them. Phonics and look-see aren't quite as much at war.

The only problem is that there's never enough time for math – and for everything else too.

This is probably the section in which to note that flashcards might be a good thing for weak teachers. The new kind of curriculum is not teacher-proof. Note that by "weak" I mean weak as teachers, not weak in mathematics. I think my Manning experience proves the latter can be addressed. The former is much more of a problem.

7.10 Dinner table conversation

Soon after I started this chapter I found myself at a noisy restaurant dinner next to a woman who works full time as a math tutor in California. She asked me what I did for a living; when I said I was a math professor who spent some time in a local elementary school we naturally fell into a conversation about our mutual enthusiasm. She described her job as helping kids learn the mathematics they are not learning in school that they need to do well on standardized tests. She's distressed that fifth and sixth graders can't do long division.

A lot of the math wars conservatives talk about long division. How often do you need to do that? But have you ever thought about why whatever your "standard algorithm" for long division works? Probably not. I hadn't, until I asked my class of prospective high school teachers to prove it. It isn't easy. I remember learning (with difficulty) an algorithm for square roots. I've no idea why it worked and can't remember it.

How about the algorithm for square roots? If you're as old as I am you may have learned one in high school. All I remember is that you set it up on the page so that it looked like long division, and then you did something with two digits at a time instead of one.

I suggested to my dinner companion that perhaps it was at least as important to be able to do easy everyday problems in ones head than to know how to write out one of the old fashioned straightforward algorithms with crossing out and borrowing, or even to do long division.

I wondered how many people ever had to do long division at all. She said she did, occasionally - to keep in practice. I do to, from time to time, for much the same reason. A calculator would be faster but I like the feel of pencil on paper.

She agreed that it's pretty clear why either of the two algorithms above produces the correct answer. But it's not at all clear why the standard long division algorithm works. I asked her if she knew, or if she'd ever thought about it. She said not. I told her I hadn't either, until just two months ago, when I asked my class of prospective high school teachers to prove it. Turns out that it's very hard to prove - so we all "just do it" because we were taught it. And it does work. I can no longer remember the algorithm I was taught in high school for extracting square roots - let alone why it gives the correct answer.

Her final words were that I'd made her think.

My final words are those of my mentor Andy Gleason, in a talk he gave in 1982 (I think) to the National Acadamy of Sciences.

> Right now there is debate apparently existing as to how mathematics should react to the existence of calculators and computers in the public schools. What should be the effect on the curriculum?..and so on. Now the unfortunate point of that is that there is even a very serious debate as to whether there should be an impact on the curriculum. That is what I regard as absolutely ridiculous. Let me just point out that right now in this country there are probably 100,000 fifth grade children right now learning to do long division problems. In that 100,000 you will find very few who are not thoroughly aware that for a very small sum of money (like $10) they can buy a calculator which can do the problems better than they can ever hope to do them. It's not just a question of doing them just a little better. They do them faster, better, more accurately than any human being can ever expect to do them and this is not lost on those fifth graders. It is an inpult to their intellence to tell them that they should be spending their time doing this. We are demonstrating that we do not respect them when we ask them to do this. We can only expect that they will not respect us when we do that.

7.11 There's never enough time

Kids and teachers don't really choose one algorithm after discussing possibilities. After a certain amount of discussion, Susan "teaches" the one TERC seems to recommend (number XX above).

There's no time in the curriculum for the luxury of choice, of really learning more than one way to do each problem.

But there's lots more controversy - it's not just about algorithms. And time is important - what you have/create time for.

I have to take a nuanced stand here. The conservatives want to teach the rules. The liberals want to teach the concepts. But it's not that easy.

Algorithm choice is good. We all routinely select different routes depending on the circumstances (driving analogy. What if you don't have a sense of direction? Observe traffic, vary travel time.)

> There is nothing quite so violent as a war based on differences in faith ... Good teaching ... is a day by day experiment in which the teacher tries to find a combination of new and old that works with some student.

Ron Ferguson
quoted by Judith Roitman in
Beyond the Math Wars
Contemporary Issues in Mathematics Education

MSRI Publications
Volume 16, 1999

Someplace I want to discuss the 40? year old analogue of the math wars – "The New Math". (Refer back to Tom Lehrer in math club chapter.) In each case the right wing conservatives say that the good old days were the best. The innovators want to innovate. Then it was rigor (of a sort). Now it's applied mathematics and an algorithmic free for all.

Chapter 8

What is the moral of this story?

Turns out, *I'm* a mathematician!

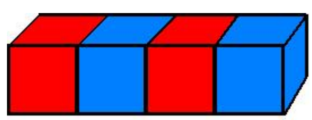
Meg Thomas
Algebra student and high school math teacher
UMass Boston, Spring 2006

8.1 Guess what comes next?

Cynthia is teaching patterns for the first time from the TERC materials. She sits with the book open in her lap, looking down at it from time to time to see what she's to do next. The book says

```
Find right passage from Investigations for here.
Need permission to quote? Probably not.
```

She builds a short stick of four snap cubes,



hides the end in a cardboard tube,

and asks

"What color comes next?"

I wince, silently, and scribble notes to myself. Why must she read from the book? Hasn't she prepared the lesson? Didn't she see the kids see her full stick of cubes before she hid the last two? Doesn't she realize that after a blue and a red *any color* could come next? I try just to watch and listen.

Kids eagerly press to answer - most of them seem to know it's "blue."

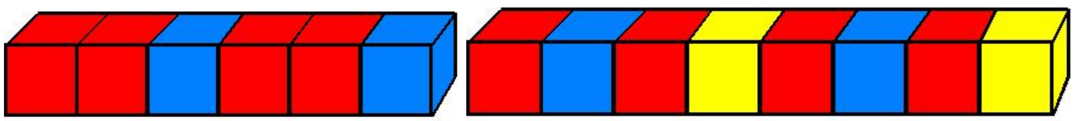
dialog here about how we behave in first grade. Raising hands, sitting quietly. Get these quotes next year when I observe Cynthia again - if I can, since she may be on leave taking care of her adopted kid.

I could see confusion on some faces, But no restlessness. That was my first glimpse of a first grade teacher's real job - teaching the social skilol that create a a place and a way for kids to learn. I've often complimented her on how well she does that job. She's shyly but proudly agreed with me.

Back to the mathematics. She did a few more examples, with different pairs of colors, then set the kids to work in pairs. Each was to create a pattern and ask her companion to guess what came next. But she had neglected to provide tubes. I knew enough of my way around her classroom to find the tape and blank paper and roll some. When she saw what I was doing she joined me and we soon had enough.

The kids started on their teamwork practice. Most followed Cynthia's lead and built sticks with alternating colors. No one had much trouble guessing what came next.

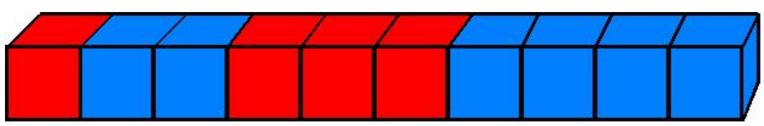
While they were practicing I walked around the room, looking over shoulders, listening, joining the game from time to time. I tried out more complex patterns when it seemed right:



The kids caught on quickly to the fact that they could guess what came next only after they'd seen two full repeats of the pattern, however long that took. After class I showed Cynthia what I'd been doing, hoping she'd catch on. I didn't say much because I really didn't know what I could say that would not sound as critical and discouraged as I felt.

Fast forward a year. By this time Cynthia and I had established a friendship/collaboration we were both pleased with. She told me one day that she had just done that pattern lesson again, and how she remembered everything I'd said the year before and could do it much better. She was prepared with enough cardboard tubes to supply all the students and didn't need to look at the book during the lesson. Now she knew, she said, that patterns were about repetitions.

We were well enough acquainted by then for me to push her. I built



and asked

"What comes next?"

"Five reds."

"So 'pattern' can mean 'predictable,' not just 'repetitive.' "

"Cool. I can't wait to try it with the kids tomorrow."

One of the common complaints (if that's the right word) I hear often from my mathematician colleagues is that elementary school teachers don't know enough "real" (if that's the right word) mathematics. A cure they propose it that they be taught more, either when they're preparing for their profession or while they are teaching. If we interpret "taught" loosely enough, that's just what happens frequently at the Manning. But I think it takes time, patience, trust, waiting for the right moments - not simply better courses and curricula.

In this chapter I'll talk about teachers learning. I'm particularly fond of the (mathematical) relationship Cynthia and I have developed over the years, so many of the examples will be from her classroom. But other teachers learned too. In particular, I learned a lot - not just about teaching. I learned some of what I hope my mathematician colleagues would classify as real mathematics.

8.2 Learning takes time (years)

Here's another story that stretches over years. We've already encountered "finding combinations" when we met the distributive law in the Vocabulary chapter.

When Cynthiawas literally "teaching from the book" she dutifully did lots of exercises asking the kids to find combinations that summed to some particular number. From day to day the number varied, as did the problem in which the arithmetic was embedded (sometimes it was vegetables, sometimes fruits). Both my instincts and my current observations in the second grade suggested to me that she should be asking for combinations that summed to 10 more often than those for other random small totals. So I said so, tentatively, after class one day. She said "of course, you're right. Why didn't I think of that" and immediately incorporated it in her curriculum.

By year five she'd reached new levels of sophistication. One day when working at the board (markers on an easel with paper, not a blackboard or whiteboard) on combinations that sum to 6 [1] she asked

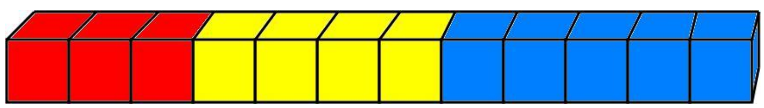
"Do you think we have found them all? There are five."

She reminded them that there *were* five:

$$1 + 5$$
$$2 + 4$$
$$3 + 3$$
$$4 + 2$$
$$5 + 1$$

because "you can use each number up to six as the first number."

Then for fun she asked them to find combinations of three numbers that added to twelve. They played with that problem for a while, using snap cubes and drawing pictures. One of the combinations the kids suggested led to this stick:



which she recorded on the easel as

colored scan here with equation and barely visible yellow

I thought the kids were having trouble with the pale yellow marker, so I raised my hand (that's what you do in first grade). Cynthia noticed and called on me:

"Dr. Bolker, do you have something to say?"

"I think the yellow marker is hard to see on the paper. Do you think you could use a different color - maybe orange instead since it's brighter?"

"OK. Anything else?"

"Do you think it matters that the marker color doesn't match the cube color?"

"Interesting ... class, what do you think?"

She calls on Declan, who says

"No, because the math is the same."

8.3 Cynthia is a mathematician

Afterward in our usual debriefing I told her how delighted I was at her response to my question (asking the class), at Declan's answer, and at hearing her ask about finding all the combinations, and then showing how you could find them all in that systematic way. That's real mathematics, elementary but quite as serious as what professionals do in journals. She told me that the idea was Rachel's yesterday - and that I would have been proud. I agreed.

Then I pushed much harder:

"Do you know how many combinations of three numbers are there that make 12, so that you could know that the class had found them all?"

[1] She doesn't use 10 every day, just often.

"No."

"Finding out is a little tricky. Last week I taught just that to my junior math and computer science majors at UMass. I think I'll put the question to them on their exam. But I think you could do it. Are you interested?"

"Yes."

"I think the place to start is with a small example - perhaps finding all the ways to make 5 with three numbers, and then three numbers that make 4 or 6."

"Yes, that's how I thought I would begin."

"Good. Let me know next week what you find out. I'm looking forward to it."

Well next week came and went and that busy first grade teacher hadn't had time to spend on mathematical research. But she and I both knew that she could do it if she wanted to. Nothing in the dialog or the thought of tackling the problem was frightening. And I carried through on my promise to ask my students to solve it:

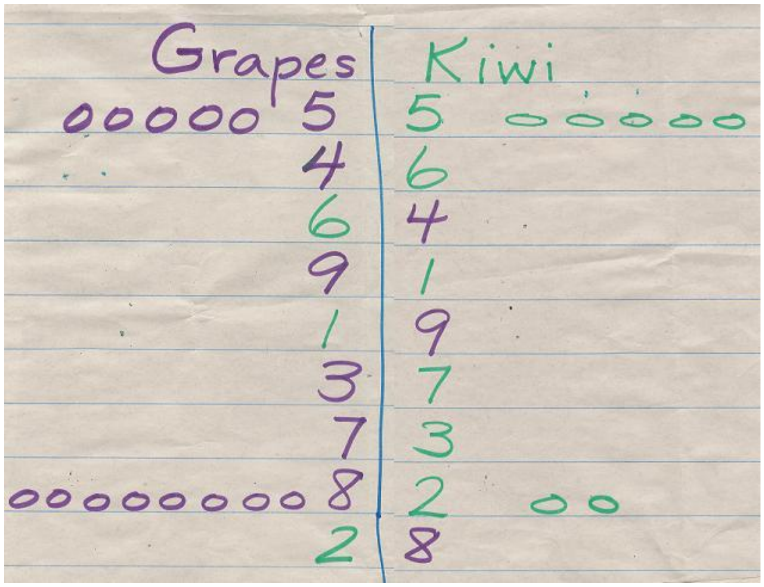
> 5. *A true story.* The first grade class at the J. P. Manning Elementary School is learning the beginnings of arithmetic. On October 19 the teacher, CynthiaTerry-Edd, asked her students to look for combinations of numbers of dogs, cats and rabbits that total 12 pets. The kids rapidly found $7 + 1 + 4$, $3 + 6 + 3$, and others. After class I asked her if she thought she could figure out how many combinations were possible, so she'd know when the kids had found them all. She's not a mathematician, but was excited by the idea of trying. I told her I'd ask you, on this exam. So do it. (You don't have to write an argument that she could follow. But you do need to write one for me. I echo Ms. Terry-Edd, who regularly tells her students "I need more words!". Just the answer isn't sufficient.)

Cynthia was delighted when later her husband found this reference when googling for her on the internet.

Since Cynthia didn't get around to figuring out how many combinations there are that sum to 12 I won't do it here for you. But you can do it yourself, just as she could. If you must know tha answer and haven't managed to solve the problem you can find the answer in the back of the book.

8.4 Comfort in the classroom

The next year Cynthia integrated finding all the combinations (just for two summands) smoothly and easily into the curriculum very early. In the first week of October she asks after a while if this list of combinations that sum to 10 the class has constructed is complete (after carefully and explicity excluding zero for one of the summands):



Amelia gets up from the rug and comes to the easel. Searching the column on the left she carefully counts from 1 to 9. It's a wonderful exercise in recognizing the digits when they're out of order. The class follows along.

I raise my hand.

Cynthia: "Yes, Doctor Bolker?"

"I see the green six in the Grapes column, which is OK since there are green grapes as well as purple ones. But I've never seen a purple kiwi."

Cynthia (smirking, because she'd anticipated my question and asked it the day before) "Class, what do you think? Does the color matter?"

Caitlin: "No. The math is the same."

Next paragraph may be out of place.

When $5+5$ came up the class realizes it's the only one for which "switcheroo" makes no difference.[2] The conversation turns to doubles. $6+6=12$. Then Morris says "Eight and eighty is a hundred and sixty."

[3]

Where does the cs320 exam question go? Here or grade school grad school chapter - probably there.

The variations on the combinations theme continue, with Cynthia a receptive listener and participant. Just this year (while writing this book) I watched the class play double compare with the numbered cards. (Cynthia was distressed because the decks had been mixed when put away. She beamed when – after raising my hand – I pointed out that she didn't need complete decks to play the game, just decks of approximately the right size. This incident belongs in the learning chapter since it's the "beaming" that counts.

What I connected this time was her reminding the kids that they knew about the cards because they used them for go fish. After class:

"I think I just thought of a new way to play go fish. When you have a seven then instead of asking your partner for sevens, ask for threes. "

"Cool! Go fish for tens. Or twelves, or anthing else. I can't wait to try it out."

I asked Cynthia whether my perception that the kids were doing larger numbers this October than in previous years. She agreed. I see two reasons. One is the demographics of the class - a much larger fraction of better prepared kids. But another is Cynthia's maturity. She's no

[2]Of course that's a better way to talk about it than to invoke the commutative law of addition.

[3] A small pedantic point I learned in graduate school from Andy Gleason: the proper way to read "160" aloud is as "one hundred sixty", without the "and".

```
longer TERC-tentative, and is willing to stretch the kids beyond the
material in the text, while still following TERC syllabus and style.
```

8.5 What I learned ...

8.6 The 2006-2007 school year

2006 did not go well (for me) at the Manning.

Susan was distressed that I hadn't told her about this book, and asked me not to come to her class. Casel was sorely missed by all. The new principal couldn't fill her shoes.[4] The Math club was fragmented. I tried to accomodate all comers by meeting every other week with two groups of third graders in the fall and then alternating fourth and fifth graders in the spring. But no one ever knew when it was their turn, and everyone (including me) forgot what we had done last time. When I started asking about the fall of 2007 I found out that Cynthia's long hoped for adopted daughter arrived, and she planned to be in Washington with her husband and child, so our long nurtured relationship seemed about to end. I thought about trying the third grade but Maria said she wanted her classroom to herself in the fall – if she was at the Manning at all. I think I was not the only one who felt the school disintegrating in Casel's absence. Even Casel seemed to think so, when I finally arranged to meet her at her downtown Boston Public School office in the late spring.

In June I found out that Susan was retiring. I felt I needed to mend fences there even if they weren't broken. (Good fences make good neighbors?) So with some trepidation I arranged to meet with her and Andrea.

First dealing with their (particularly Susan's) discomfort with my having written at all. I needed to reassure them that I hadn't planned to at the start. Casel said the other day that the staff couldn't believe I would spend all that time at the Manning without an agenda.

Susan repeated her first question to me: "What can I do for you?"

In spite of all the grumblings about the TERC curriculum, they wouldn't go back to the old one. They say the kids have a much better number sense, and can attack word problems much better.

I think we have to trust the teachers. (That's one of the morals.)

And I found out that Genteen won't be back either. So the fall is up in the air. Maybe this book won't have to be elegiac, all in the past tense, as I feared all year.

August 26 (2006) thoughts, in Berkeley:

8.7 2007

Now it's September 2007. There's a new principal who seems promising – I met her over the summer. Maria returned and is helping me find third graders for the math club – I hope just one group.

Cynthia didn't move to Washington. When I met her briefly before school started she showed me in the new edition of the TERC materials. The first grade curriculum goes *much* faster. They're doing combinations to twenty in the second or third week.

"I'm glad the TERC people are speeding things up. The kids will be more interested" she said.

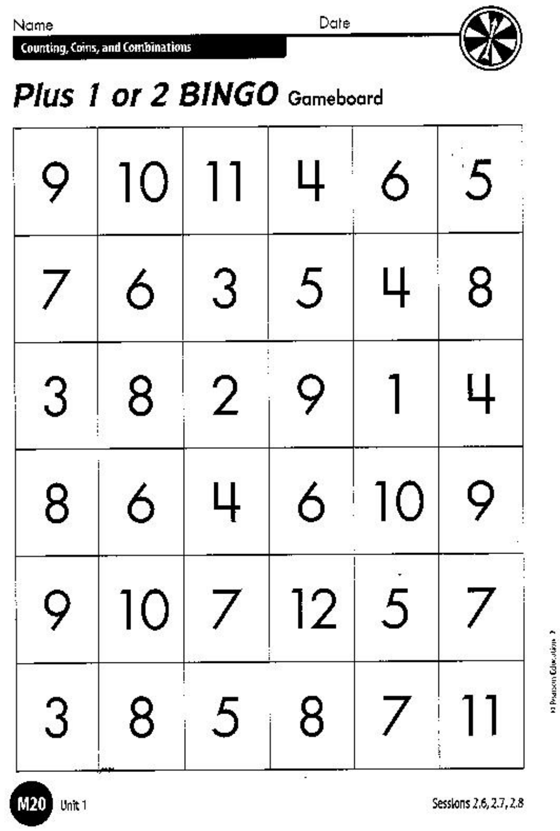
"Yes, but you were moving in this direction even before the new edition."

expand this dialog to show how she's taken charge.

Then I spoke briefly with Andrea on my way out the door. She likes the new TERC edition less - says they sold out to the publisher and made the book all fancy (colored printing, sidebars). It used to be easy to understand and use. But we did more than talk about the new TERC. I asked tentatively about returning to the second grade – since Susan had retired. She said she'd check with Sharon Windus.

And I am back in the second grade. Here's a typical day, from the second day of school.

[4] The first one after a star is always a problem - c.f. Jay Featherstone's difficulty succeeding Walter Merrill at Commonwealth.


Name Date

Counting, Coins, and Combinations

Plus 1 or 2 BINGO Gameboard

9	10	11	4	6	5
7	6	3	5	4	8
3	8	2	9	1	4
8	6	4	6	10	9
9	10	7	12	5	7
3	8	5	8	7	11

M20 Unit 1 Sessions 2.6, 2.7, 2.8

Rules of the game
My skepticism
Should there be a zero in the deck?
xxx silence
extensions: what card would you like to get?

With Sharon: double the card. Subtract. Any other rules you like. Her openness to the ideas is what's important.

8.8 2+3+5, 20+30+50, 200+300+500

January 2009. I walk into the first grade at 1:00 and Ms. Conti says "I'm going to put you to work today." She needs to administer individual reading assessments

The class is starting on adding three numbers; the problem on the whiteboard is

```
Tim has 2 marbles.
John has 3 marbles.
Alicia has 5 marbles.

How many balls all together?

Show your THINKING.
```

"I'm good at mathematics and arithmetic but not nearly as good as Ms. Conti at managing a room full of first graders. So please help me out – try to pay attention quietly."

I should try to write the rest of this in dialog, but I don't have the time right now.

I ask one kid (whose hand is up) to read the problem. He does.

Most seem to know the answer. I call on someone who says "10."

"How did you do it?"

She counts on her fingers. (Not counting on, counting all of them.)

I mentally prepare to ask later how the finger-counting will work when the sum is greater than 10.

We spend a little more time on this. I ask if marbles and balls are the same thing – could the answer really be zero balls, although there are 10 marbles? We agree that a marble is a kind of ball. But I have a plan for later.

Then I pick up a marker and squeeze three zeroes into the problem:

```
Tim has 20 marbles.
John has 30 marbles.
Alicia has 50 marbles.

How many balls all together?

Show your THINKING.
```

Hands go up. About half the class knows the answer is 100; they can count by tens.

I squeeze in three more zeroes:

```
Tim has 200 marbles.
John has 300 marbles.
Alicia has 500 marbles.

How many balls all together?

Show your THINKING.
```

The class oohs and ahs. They are clearly excited by looking at such big numbers.

I ask to have the problem read aloud. Then for an answer. I get two:

“Ten hundred”

“One thousand”

This is wonderful.

“Both answers are right! Ten hundred *is* one thousand. But look – you can get the answer even if you don’t know that.”

This is the distributive law once again.

$$2x + 3x + 5x = (2 + 3 + 5)x$$

where x can be marbles or hundreds!

I’m really into this class now, and about half the class seems to be with me, so I try more variations on the theme.

```
Tim has 7 marbles.
John has 6 marbles.
Alicia has 9 marbles.

How many balls all together?

Show your THINKING.
```

“What’s the answer?”

“22?”

“How did you do it?”

“I counted on my fingers.”

“What did you do when you ran out of fingers?”

"I started over again and kept track."

And, finally:

```
2 apples
3 apples
5 bananas

How many apples?
How many pieces of fruit?
```

Hands go up. Everyone wants to say "10." I ask them to read the question. Clearly they hadn't. It's near the end of the 45 minute session and attention is waning. I think they could learn to read the question *before* answering it, but that would have to be the point of the lesson. It'd be worth doing, though, since what they are doing instead is trying to apply a pattern: when you see three numbers, add them.

I wish I could discuss this with Andy Gleason.

When I tell Ms. Conti what I've done and what fun I've had she's very receptive. She'll absorb some of the ideas. But she can't do this every day – nor should she! I've gone so far so fast that the kids haven't had time to *absorb* much of what I said. Maybe it remains as a glimmer in the subconscious of some of them.

8.9 Does it scale?

The question comes from my friend Tom Sallee, a mathematician at the University of California at Davis who has spent a good part of his career actively and productively engaged in working with K-12 teachers and mathematics curricula. It was one of the first things he said in response to my enthusiastic stories.

If what I am doing at the Manning school is good for kids and good for mathematics, what might be done to spread that good around?

At one of our recent lunches together my thesis advisor (many years ago) Andy Gleason and I thought about that question. We started out thinking about how many consultants it might take to do for each school what I'd been doing at the Manning. We wanted to know the number of elementary school classrooms. Since we were at a restaurant we had no access to data. The napkins were cloth, so we couldn't do what mathematicians often do and scribble on them. We had a Fermi problem[5] . Andy's quick estimate went something like this: 300 million people, perhaps 1030 million school kids. At 25 kids per class that's 1.2 million classrooms.

8.10 How many mentors?

In Chapter (explanations) we saw how I was available to bless common sense, my credibility guaranteed by my credentials. Should we have a PhD mathematician in every school? Would there be enough of them who wanted the job and could do it well - i.e. humbly?

Where do I put stuff about the atmosphere Casel creates in the school. Probably here, since it's what makes the adventure possible - a necessary if not sufficient condition.

8.11 Acting locally

I can't answer Tom's question, or Andy's. But maybe I don't have to. A cousin of mine who's worked extensively in second and third world development projects told me she is frequently frustrated by governent

[5] "... the estimation of rough but quantitative answers to unexpected questions about many aspects of the natural world. The method was the common and frequently amusing practice of Enrico Fermi, perhaps the most widely creative physicist of our times. Fermi delighted to think up and at once to discuss and to answer questions which drew upon deep understanding of the world, upon everyday experience, and upon the ability to make rough approximations, inspired guesses, and statistical estimates from very little data." Philip Morrison, Letters to the Editor, Am. J. Phys., August 1963, v31n8 p626-627

agencies who look at a successful project, decide that it can't be replicated on a nationwide scale (whatever the nation) and therefore dismiss it. "Acting locally" is worthwhile for its own sake. I haven't written this book to try to change the way mathematics is taught everywhere. My goal has been to offer what I've seen and learned in hopes that you have enjoyed my adventures as much as I have, that you have learned some mathematics. I hope too that some of what I've said comes to mind when you read about the math wars or your own kid comes home from school excited (or unhappy) about mathematics (if you're a parent) or in your own classroom (if you're a teacher).

Now describe the QR class approach, using the web ...?

When *It's Elementary* was between half and two thirds done I started searching for a literary agent. (I'd decided that I wasn't writing a textbook, so didn't go to those publishers, or a book just for my university colleages, so didn't approach the Mathematical Association of America.) I was fortunate to find Dystel and Goderich right away. They wanted me to rewrite my proposal (a version of which is the preface now). One of the things the agent wanted was a marketing analysis. I remembered the lunchtime conversation with Andy Gleason, so thought to say in the proposal that the 1.2 million elementary school teachers represented a large potential market.

To check that computation I asked my the students in my UMass Quantitative Reasoning class (everyday useful mathematics for ordinary people - not science or math majors) to redo the Fermi problem. Just a few did it as one. Most went right to the internet (the classroom was equipped with networked computers) and found that there are 1.5 million elementary school teachers in the United States. (Several places, one of which is the Bureau of Labor Statistics: http://stats.bls.gov/oco/ocos069.htm#emply.)

8.12 What is mathematics?

One of the themes I've hoped to develop is the "we are all mathematicians," in parallel with "we are all authors" (of our own speech). I think the (a) way to make what I have found/done scale is to encourage that idea. In particular, the teachers need to discover it, or be led to discover it. I don't know how to do that, but I do know some ways not to do it. Mathematics can't be viewed as (merely) a school subject. After all, reading isn't, nor is speaking. There's an obvious connection between the reading and writing kids do in school and what they and adults do "in real life." One would hardly expect a teacher to leave literacy behind on the way out the schoolhouse door at the end of the day. Reading, writing, *word play*, all our part of our everyday life. I'd like the same to be true of mathematics.

This is a very different proposition from the common current cliche that mathematics must be relevant, or useful. We've seen from lots/some of the TERC materials at the Manning that some of what's claimed to be "real" or "useful" is in fact phony. Yes, pages of drill have been replaced by – by what? The data collection problems, the fraction problems – they are neither particularly interesting nor accurate models of daily occurrences in the real world.

Discussion here, or somewhere, about what applied mathematics really is – Mackey's anecdote?

For sure what the TERC book has isn't real applied mathematics. It's fake applied mathematics with the same goal as the drill problems it replaces. It's thought of as a way to make mathematics palatable or interesting – but it doesn't seem to occur to anyone that the mathematics might be interesting "for its own sake"! At UMass Boston there used to be a course called "Liberal Arts Mathematics," analogous to the "Physics for Poets" courses common in those days too. It's disappeared. What has replaced it is a course that pretends to be practical.

How sad it would be if the only reason to teach reading and writing to kids was so that they could do practical things with language – read and write the directions for using their cell phones (?), understand street signs, carry on conversations about the weather, ... but not teach them (or encourage them, or show them) poetry, novels, puns and anagrams, crossword puzzles, ... that language is fun. It's worldly in that it connects us to each other and to a world that in some sense we create with our words – beyond the senses.

For me, mathematics does the same thing. I see the world mathematically (as well as with words, or with direct sense experience). We can all do that if we're encouraged. Some of us are luckily born with that natural ability/hunger. Just as some of us are born potential poets, or athletes, or writers. But for those of us not so blessed at birth, the experiences are still open to us. I can thrill to poetry, even pretend to write some. I read novels even though I can't write them. I move my body – am addicted to exercise. We can all become math addicts too.

Chapter 9

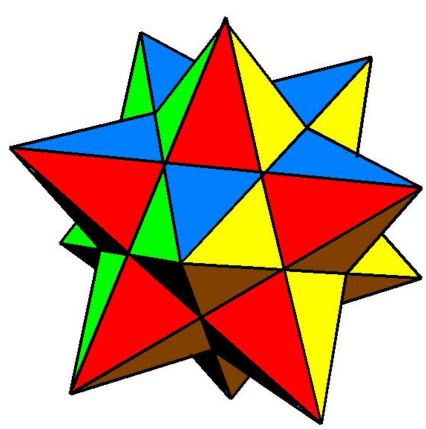
Answers to some problems

> In guessing a conundrum, or in catching a flea, we do not expect the breathless victor to give us afterwards, in cold blood, a history of the mental or muscular methods by which he achieved success; but a mathematical calculation is another thing.
>
> Lewis Carroll, *A Tangled Tale*

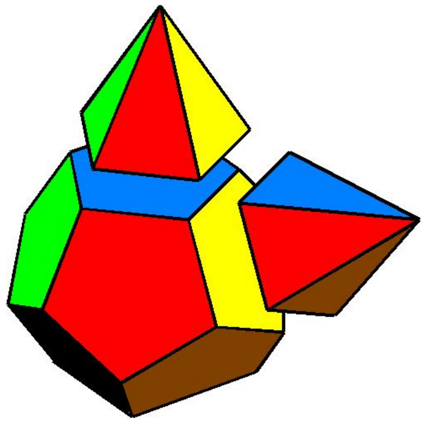
> Three minutes thought would suffice to find this out; but thought is irksome and three minutes is a long time.
>
> A. E. Houseman

9.1 How to build a stellated dodecahedron

Here's the picture of the stellated dodecahedron we saw in Chapter 1.

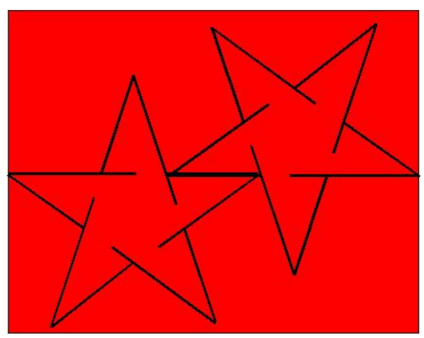


You can think of it as an ordinary dodecahedron with a pentagonal prism of jut the right height built on each of the twelve faces. This picture shows two of the prisms:

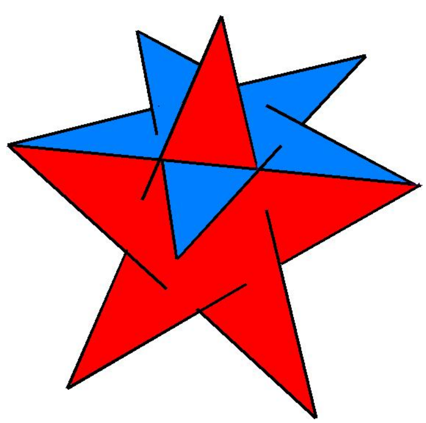


In principle that works but in practice it's not a good idea. The engineering is impossible - particularly for kids. I rather suspected that before I began the project, and our experience with the ordinary dodecahedra confirmed my suspicions:

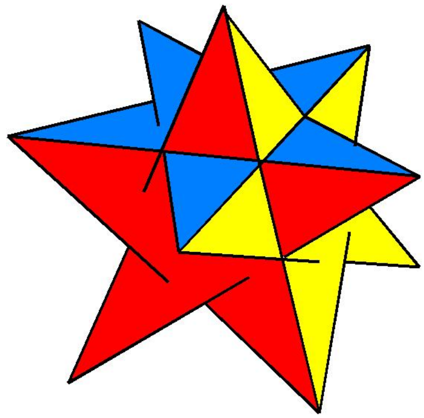
The colors in the picture of the stellated dodecahdron suggest the strategy I actually used. Those colors show that there are twelve partially hidden five pointed stars. Six show in the picture; the other six are in the back. You can actually build the stellated dodecahedron from a dozen pentagrams. Start with six sheets of cardboard in your six favorite colors, and cut two pentagrams from each using the pattern below. Note that the points of the stars are notched just a little more than halfway each.



Now take two of the pentagrams - say, a red one and a blue one, and slide the notch on one red point into the notch on one blue point. You'll get a figure like this one.



Now take a yellow pentagram and slide two of its notches into red and blue notches. This may require a third hand or a little bending. You'll get a figure like this one.



Now I hope you get the idea. Add pentagrams one at a time. When you get to number six, which might be the second blue one, make sure it's opposite the blue one you've already used. That way each point will be surrounded by five colors none of which matches the color of the pentagram the point seems to rise from.

This is harder than it sounds. The night before my second week at with the third grade I'd cut out all the pentagrams and practiced putting them together and despaired of doing it in the class. But was committed.

Refer back to math club, platonic polyhedra.

9.2 Factors of zero

Every number (every integer) is a factor of 0. To see why, you need to think clearly about the definition of "factor:" *The integer a is a factor of the integer b when there is some integer c such that $b = c \times a$.* If you start with the particular value $b = 0$ then by taking $c = 0$ the equation $b = c \times a$ becomes $0 = 0 \times a$, which is true whatever value a happens to have. So every integer a is a factor of 0.

x/y is the solution to ?*y = x, which is why you can't divide by zero.

9.3 Who has three factors?

The numbers with exactly three factors are the squares of the primes. (cite reference in vocabulary chapter)

9.4 10^{100}

Googol

Kasner, Edward, and Newman, James Roy Mathematics and the Imagination (London: Penguin, 1940; New York: Simon and Schuster, 1967; Dover Pubns, April 2001, ISBN 0-486-41703-4).

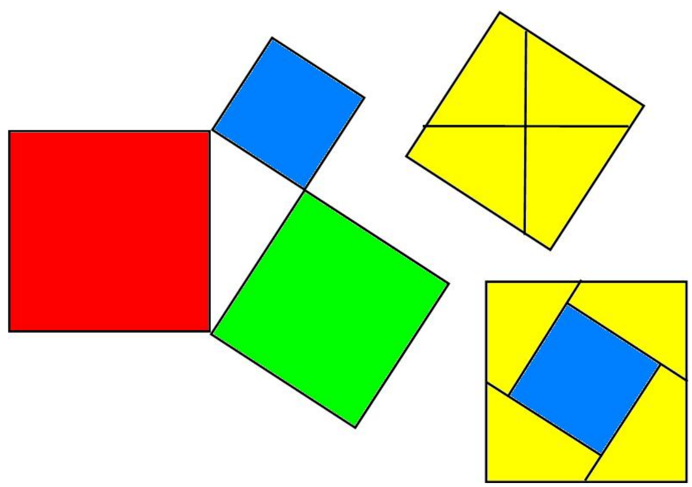
The search engine was named after this number.

9.5 gcd and lcm

9.6 Six pencils

I couldn't figure out this puzzle either when I first heard it at a bar in Havana on a trip there as a teenager with my parents. There it was asked as a question about six straws. Here's the solution with pencils. The trick is to think in three dimensions. I'd hoped that would occur to Christopher since he'd just been practicing drawing three dimensional figures.

9.7 The Pythagorean Theorem



9.8 Peter's puzzle

Here's a solution:

picture here

Although there are lots of ways to put the numbers into the spaces (about 7!, in fact) it's not too hard to find this more or less unique solution if you think a little bit first The 1 can't go on an edge, so put it in a corner. The 8 can't go on a corner, and it can't go on an edge next to the 1, so put it on one of the other two edges – it doesn't matter which since they are symmetrical with respect to the corner with the 1. Now the 7 can't go in any of the corners, so it must be on an edge. ...

The puzzle has natural generalizations. There's no solution for four numbers. It's easy to find the unique solution for six numbers. There are none for ten numbers (my proof is an ugly case by case analysis) and, I suspect, none for twelve or more.

Chapter 10

Not as Elementary

P2C2E
Process Too Complicated Too Explain

10.1 Equals

There are other meanings for "equals" that I haven't taken up at the Manning.

In what I sometimes refer to there as my day job I teach computer science as well as mathematics. In some computer programming languages "=" used to assign a value to a variable. In that case *none* of the equations at the start of the vocabulary chapter makes sense, the first one because 10 isn't a variable. I've never encountered this meaning in the elementary school curriculum. Some kids come across it on their own, perhaps playing with an Excel spreadsheet. The new meaning is implicit in the context; they don't need to have it explained – although programming students at the University sometimes do.

I can't think of an everyday usage for "equals" that is correctly interpreted as an assignment statement. If you can, please let me know.

Once you start to think about the word you find other everyday uses. My wife remarked on this sign on a parking meter:

$$25 \text{ cents} = 30 \text{ minutes}$$

Exploring that one would lead us into a discussion of direct proportionality and unit conversions.

Second graders solve problems by breaking them into pieces – an excellent strategy. One small example might look like this:

$$\begin{aligned} 4+5+6 &= ? \\ 4+6 &= 10 \\ 10+5 &= 15 \end{aligned}$$

If they are comfortable using "equals" to mean "is the same as" then these computations can be written as a neat chain of equations:

$$4+5+6 = 10+5 = 15$$

It's sadly common to see the chaining misused unknowingly by beginning (and sometimes not so beginning) algebra students. In the equation

$$6x+4 = 10$$

the equals sign means "is the same number as" and the problem is to discover the value of the unknown number x. I often see this solution:

$$6x + 4 = 10 = 6x = 6 = x = 1$$

The answer is correct: x does equal 1. But the chain of equations also says that 10 equals 1. The problem is that the first, third and fifth equals signs mean "is the same *number* as" while the second and fourth mean "is the same *equation* as". I try to convince my students that to write mathematics they need words as well as equations. The solution to this problem should be

> The given equation
> $$6x + 4 = 10$$
> implies
> $$6x = 6$$
> so
> $$x = 1.$$

10.2 Quotations

The punctuation convention in standard English requires that the period in the following sentence be inside the quotation marks.

Nowadays equations are often called "number sentences."

But logically, the quoted text should be ' "number sentences".', since the period isn't part of the new name for an equation. As a mathematician, I prefer the logically correct version. As a computer scientist I know that the distinction is sometimes quite significant. But in this book I've usually followed the common convention and put the punctuation inside the quotation. Otherwise sharp readers would often be distracted by what they thought were frequent proofreading errors. This discussion could go on for a long time. If you are interested, you can Google "punctuate quotations", or start at
www.oreillynet.com/mac/blog/2004/09/editorial_ps_and_qs_punctuatio.html.

Discuss punctuating quoted sentences inside sentences.

10.3 Counting from zero

In everyday affairs counting starts at 1 and continues on $2, 3, 4, \ldots$. Computer programmers have learned that it's almost always better to count this way:

$$0, 1, 2, 3, \ldots,$$

starting with zero. That makes three the fourth number, which computer science students find disturbing when they first encounter it.

The issue about where to start counting comes up (for me) in elementary school in connection with the hundred's chart.

The standard version starts with 1. If you start counting columns with column 1 then the units digit for all the numbers in a column is the column number. But the row number doesn't tell you the digit in the tens place – its off by one. That's a good reason for starting to count rows with zero.

In the hundred's chart that starts with 0 if you count both columns and rows starting with 0 the the row number matches the tens digit and the column number matches the units digit: the number in the third row, fifth column is 35.

That makes the row and column indices into the start of a good Cartesian coordinate system for the plance.

10.4 Rubik's dodecahedron

10.5 The fundamental theorem of arithmetic

Counting handshakes

This isn't too technical at all. Move it from here to math club chapter, where in fact kids solve it regularly.

Suppose there are 14 kids in the first grade class, and each kid shakes hands with every other one. How many handshakes are there altogether?

Ask the kids to stand in a line in alphabetical order by first name (they practice this on the way to the lunchroom). Then ask Aaron to walk down the line, shaking hands and counting, then sit down. He will count to 13. Next Brittany walks down the line, shaking 12 hands. Then it's Charlize's turn. Finally only Maddy is left, standing alone. There isn't any line. She initiates no handshakes – but it's clear that she's shaken each kid's hand as he or she passed by. Adding up the counts, we see that the number of handshakes is just

$$13 + 12 + \cdots + 1$$

How does that help solve the problem. (Remember the problem - it's to find that sum, without (as the math club kids would say, doing all the work).

Well, think about some kid in the middle of the list, George, say. He has shaken hands with every one of his 13 classmates, some while he was standing in line and they came to him, some when it was his turn at the head of the line to shake the hands of the kids after him. So each of the 14 kids shakes 13 hands, for a total of $14 \times 13 = 182$ handshakes. But wait. That total of 182 counts George's handshake with Charlize twice – once for him and once for her. Every handshake is counted twice in the total of 182, so the number of handshakes, counted correctly, is just $182/2 = 91$.

[1]

To summarize:

$$13 + 12 + \cdots + 1 = 14 \times 13/2$$

or, more generally (because there's nothing special in our argument about the 14)

$$n + (n-1) + \cdots + 1 = (n+1) \times n/2$$

Give the visual proof too – perhaps first.

Gauss' proof too?

10.6 Counting combinations

Solution to the apples, bananas, clementines problem that came up both in first grade and in math club.

You will need to establish your ground rules. Can you use 0 as one or two of the pieces? If so then

$$0 + 0 + 12 = 12$$

counts. If you can't then it doesn't.

Does the order matter? If it does then

$$\begin{aligned} 1 + 1 + 10 &= 12 \\ 1 + 10 + 1 &= 12 \\ 10 + 1 + 1 &= 12 \end{aligned}$$

[1] I like the number 91 because it's the only number less than 100 that looks like a prime but isn't.

count as three different solutions. If not, then they count as 1.

Among the various interpretations of this problem the easiest to solve is the one in which 0 is allowed and order matters. That's also the most reasonable model when we're following the first grade applied mathematics problem counting fruit. Five apples, four bananas and three cherries is not the same as four apples, five bananas and three cherries. And one might have no cherries at all. (Of course I've chosen apples, bananas and cherries since as a mathematician I want variables whose names start with 'a', 'b' and 'c'.)

The answer (by observation) satisfies the recursion

$$c(n+1) = c(n) + (n+2)$$

– professionals know this is $\binom{n+2}{2}$. At the Manning Cynthia and the kids in the math club recognize it as the recursion for the handshake numbers.

Until I encountered this problem repeatedly at the Manning the only proof I knew for this counting problem was the standard one, so clever that I never could imagine how anyone thought of it. To count the number of partitions of n into three parts when 0 is allowed you imagine a string of $n+2$ possible locations. You choose two places to put a blank character, then put 'a' in each space to the left of the first blank, 'b' in each space between the blanks and 'c' in the spaces to the left of the second blank. That clearly counts the partitions, and there are $\binom{n+2}{2}$ ways to place the blanks.

But I wanted a proof that didn't depend on this *deus ex machina*.

I discovered this one. I'm sure it's not new. I would probably find it if I looked through the discrete mathematics books in my library. There are several reasons to put it down here. First, it answers a question left unanswered earlier in *Its Elementary*. Second, it illustrates the fact that teaching at the Manning led me to learn things as well as to teach things.

Suppose there are N combinations for n pieces of fruit. There are two kinds of combinations for $N+1$ pieces – those that use at least one apple and those that use none. If you add one apple to each of the combinations of N pieces of fruit you get all of the first kind of combinations of $N+1$. But we know there are $n+2$ ways to use a total of $n+1$ bananas and clementines So the total number of combinations of $n+1$ pieces satisfies the recursion.

10.7 The irrationality of $\sqrt{2}$

Several traditional proofs - using the fundamental theorem of arithmetics, one the standard parity argument leading to a contradiction, a geometric descent.

Generalize so we know $\sqrt{3}$ is irrational – useful in the next section.

10.8 No equilateral triangles

From the fifth graders' perspective this was an interesting research project. From my mathematician's perspective it turned into a research project of a different kind. Here I describe where that has led me. Most (but not all) of what's in this section isn't really elementary.

The first thing I realized was that there is *no* equilateral triangle with lattice points for vertices. Even stating that realization I've fallen into my natural language. A lattice point is a point with integer coordinates - just the kind the kids were working with. In their experiments they looked for triangles with one edge parallel to the x axis. In that case the other edges of a real equilateral triangle would make an angle of 60 degrees with that axis. Now a little trigonometry shows that there is no such triangle; the tangent of the actual angle is rational but $\tan(60) = \sqrt{3}$ is irrational. [2]

What about arbitrary triangles, which don't have an edge parallel to the x axis? None of those can be equilateral either. The proof that occurred to me uses *Pick's Theorem*, which conveys interesting information in its own right. It says:

[2] I won't try to prove that here. I've always enjoyed the fact that $\sqrt{3} \approx 1.732$, which is easy to remember because George Washington was born in 1732. Of course for a mathematician this coincidence is how you remember Washington's birth year.

> Suppose P is a polygon whose vertices are lattice points. Suppose there are I lattice points inside P and B lattice points on its boundary. Then the area of P is
>
> $$I + B/2 - 1.$$

The next picture illustrates Pick's theorem. [3]

Pick's Theorem picture here.

I think Pick's Theorem would be a cool thing to introduce in a classroom when kids had begun to learn about the areas of rectangles and triangles. You can even use it to check some cases of the Pythagorean theorem:

Pick's Theorem checking Pythagoras'.

Back to the problem of the equilateral triangles on the lattice. The argument goes this way. Suppose there is such a triangle. Then the length of its side is $s = \sqrt{someinteger}$. Then the area is $s^2\sqrt{3}/2$, which is half an integer times $\sqrt{3}$ and thus irrational. But Pick's theorem implies that the area is rational. No triangle can live with that contradiction,

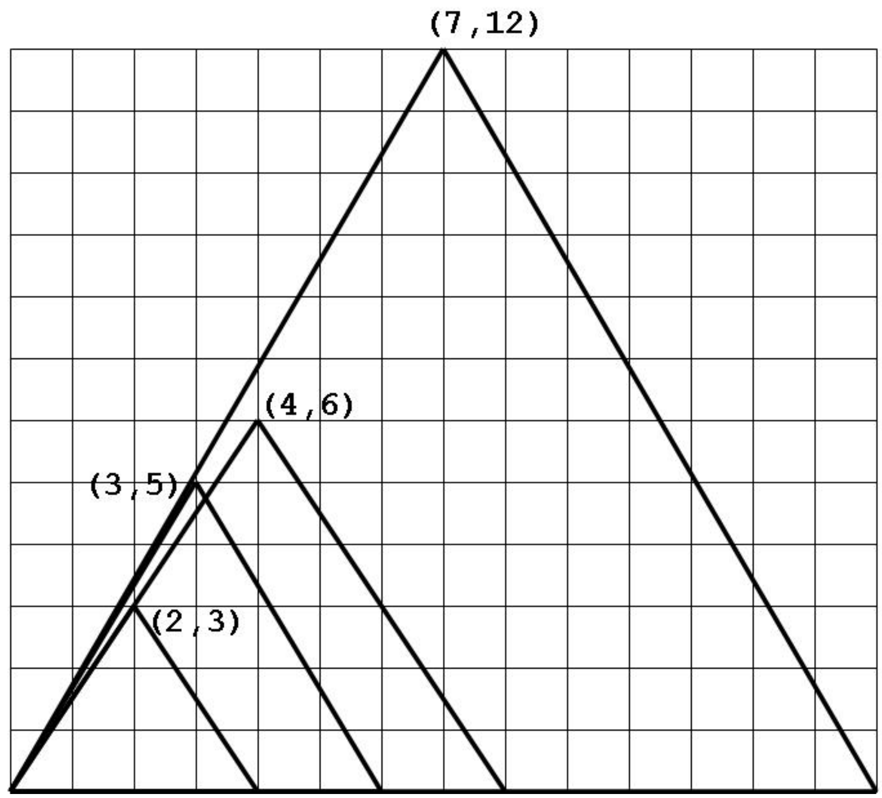
I have challenged my mathematical colleagues to find a proof that is simpler than this one. All the ones I've found rely essentially on the irrationality of $\sqrt{3}$ although none of them uses Pick's Theorem.

Having established the nonexistence of equilateral triangles on the lattice, I wondered about how close it was possible to come. In particular, I wanted to know how good were the examples the kids found. Were they accidents? Turns out not.

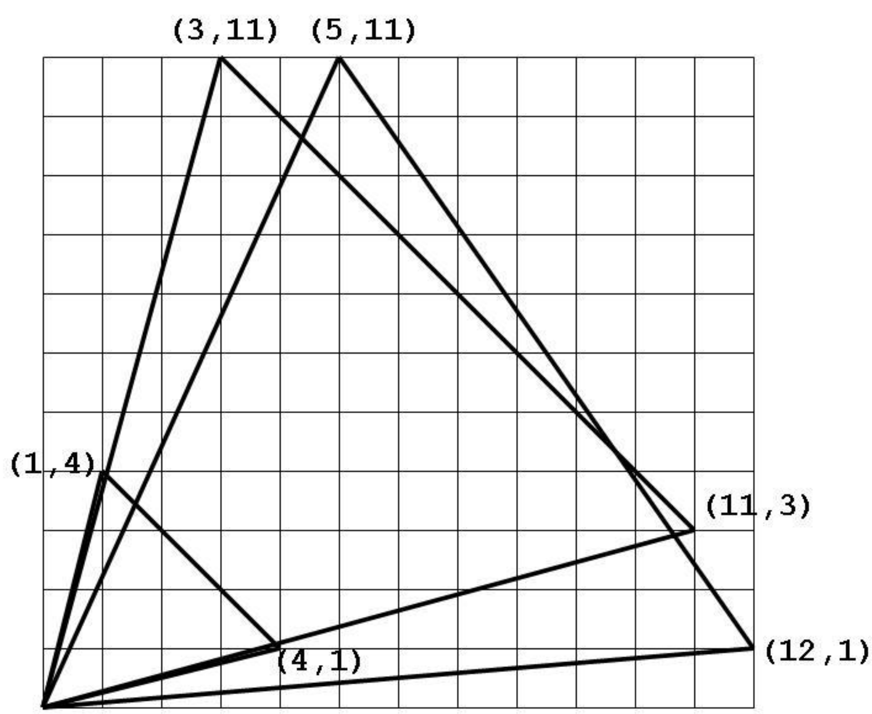
An isosceles triangle with one side on the x axis with lattice point vertices may as well have one of those vertices at the origin $(0, 0)$. Then the other two vertices will be at (a, b) and $(2a, 0)$. It will be approximately equilateral when the angle between a side and the x axis is approximately 60 degrees. That will happen when the tangent of that angle, which is b/a is approximately $\sqrt{3}$. Now we've reduced the problem to one we know how to solve. Mathematicians have known for years how to find good rational approximations to $\sqrt{3}$.

recursion here

[3] I won't try to prove it here. You can find lots of proofs on the web.



What about other almost equilateral triangles? I wrote a computer program to find a few. Here's some of what turned up.



The triangles symmetric about the line $y = x$ will work when we have good rational approximations to tan(15).

Is there a general pattern here? Left to the reader.

Index