Chapter 7

Electricity Bills and Income Taxes – Linear Functions

We use an electricity bill as a hook on which to hang an introduction to functions in general and linear functions in particular, in algebra and in Excel. Then we apply what we’ve learned to study taxes – sales, income and social security. You’ll also find here a general discussion of energy and power. Chapter goals:

Goal 7.1. Understand direct proportion as a linear equation with intercept 0.

Goal 7.2. Study situations governed by linear equations.

Goal 7.3. Stress the meaning and units of the slope and intercept.

Goal 7.4. View functions as tables, graphs and formulas.

Goal 7.5. Construct flexible spreadsheets to model linear equations.

Goal 7.6. Understand the piecewise linear income tax computations.

Goal 7.7. Sort out the confusing distinction between energy and power.

7.1 Rates

We began Chapter 2 with a discussion of the relationship

\[ \text{distance} = \text{rate} \times \text{time}. \]

or, with units

\[ \text{miles traveled} = (\text{rate, in miles/hour}) \times (\text{hours on the road}) \]  

\[ (7.1) \]

\(^1\text{See comment in instructor’s manual.}\)
There we worked with particular numbers; now we want to look at that relationship a little more generally. If we work with a fixed rate (say, 60 miles/hour), then we can find the distance traveled whenever we know the time. To say that with some algebra, write $D$ for the distance and $T$ for the time. Then

$$D = 60 \times T,$$

or, more generally,

$$D = r \times T,$$

where $r$ is the rate of travel, in miles/hour.

That formula says that distance traveled is proportional to travel time. The rate in miles per hour is the proportionality constant. If you drive for twice as long you go twice as far. If you drive ten times as long you go ten times as far. If you just sit in the driveway then $T = 0$ and you go nowhere: $D = 0$ too.

In Section 2.2 we introduced the units (gallons per hundred miles) as a useful way to measure automobile fuel economy: the amount of gas you use is proportional to the distance you drive:

$$F = r \times D$$

where $F$ is the amount of gasoline used, in gallons, $D$ is the number of hundreds of miles traveled, and the proportionality constant $r$ is the fuel use rate, with units gallons per hundred miles. If you drive 10 times as far you use 10 times as much gas. You don’t use any gas at all just sitting in the driveway.

The proportionality constant is always a rate: it appears with units. In these examples the units are miles per hour and gallons per 100 miles.

The unit pricing discussion in Section 2.4 provides more examples of proportionality.

Finally, sales tax is computed as a proportion. If you spend twice as much you pay twice as much tax. The tax rate is the proportionality constant. When it’s 5% then you pay five dollars of tax per hundred dollars of purchase.

## 7.2 Reading your electricity bill

The more electricity you use at home, the more you pay. But the relationship isn’t quite proportional. You don’t pay twice as much to use twice as much. Figure 7.1 shows a simple sample electricity bill.

This bill explains itself. We’ll study it before we look at a real one. It comes from England, so the costs are expressed in pounds and pence rather than dollars and cents, and it comes once a quarter (every three months) rather than once a month, but you can ignore that while you read it – from the bottom up.

The last line is the total bill, computed as

$$\text{Cost of electricity} + \text{fixed charge}$$

Checking the arithmetic:

$$\£34.62 + \£9.49 = \£44.11.$$
7.2. READING YOUR ELECTRICITY BILL

Electricity Bills

How to read your electricity bill

Tamworth Electricity

<table>
<thead>
<tr>
<th>Previous meter reading</th>
<th>5 6 4 7 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present meter reading</td>
<td>5 6 9 4 3</td>
</tr>
<tr>
<td>Number of units used</td>
<td>4 7 1</td>
</tr>
<tr>
<td>Cost per unit (pence)</td>
<td>7.35</td>
</tr>
<tr>
<td>Cost of electricity used</td>
<td>£ 34.62</td>
</tr>
<tr>
<td>Fixed quarterly charge</td>
<td>£ 9.49</td>
</tr>
<tr>
<td>Total Bill</td>
<td>£ 44.11</td>
</tr>
</tbody>
</table>

Cost of electricity + fixed charge
This shows the number of units used. This is the same as kWh.

Present reading – previous reading
Number of units \( \times \) cost per unit

That’s our old friend proportionality. The previous line gives the proportionality constant: the cost per unit as 7.35 pence per unit. Later on in the document we’re told that a unit is just a kilowatt-hour, abbreviated “kWh”\(^4\). There are 100 pence in a pound\(^5\) so we write the cost per unit as

\[
0.0735 \text{ £/kWh}
\]

In the current quarter this customer used 471 kWh of electricity – the difference between the meter reading before and after the quarter.

Here’s all the arithmetic, with units:

\[
£44.11 = 0.0735 \frac{£}{\text{kWh}} \times 471 \text{kWh} + £9.49.
\]

That English bill is easy to read. Figure 7.2 shows a real one that’s a little more complex, from NStar, in Boston.

We can identify the same two components. The fixed charge is the $6.43 labelled “Customer Charge.” It’s the part of the $145.26 total that does not depend on the amount of electricity used – in this case, 813 kwh. The six lines on the bill that do depend on that contribute

\[
(0.04432 + 0.01039 + 0.00468 + 0.00050 + 0.00250 + 0.10838) \times 813 = 0.17077 \times 813
\]

\[
= 138.83601
\]

to the total bill, which is

\[
$145.26 = 0.17077 \frac{\$}{\text{kWh}} \times 813 \text{kWh} + $6.43.
\]

(If you check the calculation you will discover that the electric company rounded $138.83601 down to $138.83 rather than up to the nearest penny. We should be grateful for small favors.)

\(^{4}\)It’s too bad the bill talks about “units” instead of just “kWh” since for us “units” has a more general meaning.

\(^{5}\)Like cents in a dollar. The computation would have been much more complicated before February 15, 1971 – the day England converted from pounds/shillings/pence to decimal currency. (See \url{http://news.bbc.co.uk/onthisday/hi/dates/stories/february/15/newsid_2543000/2543665.stm}).
7.3 Linear functions

So far Common Sense Mathematics has called for hardly any algebra. Now a little bit will come in handy.

Suppose you buy your electricity from NStar as in the example above and want to study how your bill changes when you use different amounts of electricity. The monthly $6.43 Customer Charge does not change. The rest of your bill is proportional to the amount of electricity. The proportionality constant is 0.17077 $/kWh in the sample bill above. We can assume that it does not change. If in a given month you use $E$ kWh of electricity your total bill $B$ can be computed with the formula

$$B = 0.17077 \times E \text{ kWh} + 6.43 \text{$.}$$

That formula captures how the dollar amount of your electricity bill depends on the amount of electricity you use, measured in kWh. The first term captures the part that’s proportional to the amount of electricity used. The second term (the amount $6.43$) is fixed. It represents the electric company’s fixed costs: things like generating the bill and mailing it to you and maintaining the power lines on the street in front of your house. Those are expenses they must cover even if you’re on vacation and have turned off all the appliances.

You probably encountered a similar formula once in an algebra class – it may look more familiar without the units

$$B = 0.17077 \times E + 6.43$$

It may look even more familiar if if we call the variables by the traditional names $x$ (for the independent variable) and $y$ (for the dependent) instead of $E$ and $B$:

$$y = 0.17077x + 6.43.$$

This is a linear function, which standard algebra texts write in slope-intercept form

$$y = mx + b.$$

In this example the slope $m$ is 0.17077 $$/kWh and the intercept $b$ is $6.43$. For the English bill the slope is 0.0735 £/kWh and the intercept is £9.49.

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6 In fact it may change slightly from month to month, but only when the electric company changes its rates.
7 See comment in instructor’s manual.
There are many everyday examples where a linear equation captures what’s happening: a total is computed by adding a varying part that’s a proportion to a constant. The proportionality constant is the slope.

- The most familiar examples are the ones where the intercept is 0: all the ones in Section 7.1.
- When renting a truck, the amount you pay is
  \[(\text{rate in dollars/mile}) \times (\text{miles driven}) + (\text{fixed charge})\]
- Your monthly cell phone bill might be
  \[(\text{rate in dollars per minute}) \times (\text{number of minutes}) + (\text{fixed fee})\]
- If you work as a salesperson and your commission is 15% of total sales your total wages are
  \[0.15 \times (\text{total sales}) + (\text{your base salary})\]

The pattern is

\[\text{total} = (\text{rate}) \times (\text{amount of some quantity}) + (\text{fixed constant})\]

In each case the slope is the rate and the intercept is the fixed constant. The units of a slope are always those of a rate. In the truck example, the slope is the rate in dollars per mile; in the cell phone example, the units of the slope are dollars per minute; in the salesperson example the slope is 0.15 dollars of commission per dollar of total sales.

Think of the intercept as an initial or starting value; it’s what happens when the input is zero. It has proper units too – in each of these examples that unit is dollars. If you rent a truck but don’t drive it anywhere, you still pay the fixed charge. If you make no cell phone calls you still pay the fixed fee. If you don’t sell anything that month, your commission is $0 but you still earn your base salary.

### 7.4 Linear functions in Excel

In Section 7.2 we saw that the amount you pay for electricity in a month is a linear function of the amount you use. In this section we’ll use Excel to calculate electricity bills and to draw a picture of the results.

Figure 7.3 is a screen shot of the Excel spreadsheet [http://www.cs.umb.edu/~eb/qrbook/ElectricityBill/TamworthElectric.xlsx](http://www.cs.umb.edu/~eb/qrbook/ElectricityBill/TamworthElectric.xlsx). We put the slope 0.735 in cell C4, with its units £/kWh in cell D4. We put the intercept 9.49 in cell C5 and the units (£) in cell D5.

Then we entered column labels in cells B7:C8 and a few values in rows 9 through 13 in column B. Finally, we asked Excel to calculate the electricity bills in column C. To do that, we started with the formula

\[\text{=C4*B9+C5}\]

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8A real cell phone bill will probably be much more complicated, perhaps with separate charges for phone minutes, text messages and data transfer, perhaps with some of each kind of use built in to the fixed fee.
7.4. LINEAR FUNCTIONS IN EXCEL

in cell C9. The = sign tells Excel to multiply the numbers in cells C4 and B9 and add the number in C5. The result is 9.49, as expected.

The next step is to copy that formula from cell C9 to cell C10. With any luck Excel will guess what we want to do, change B9 to B10, compute

\[ =C4*B10+C5 \]

and display 44.11. *But that’s not what happens!* Excel shows 4469.79 instead! If you look at the contents of cell C10 you will find

\[ =C5*B10+C6 \]

so Excel added 1 to the row numbers for cells C4 and C5 as well as to B9. There’s nothing in cell C6. Excel treats that as a zero and adds it to 471 × 9.49 to get 4469.79.

That’s *not* what we want. Changing B9 to B10 is right, but we want Excel to *leave the references to cells C5 and C6 alone*. The trick that makes that happen is to put a $ in front of the 5 and the 6. This is not something you could have figured out. There’s no particular reason why this trick should work. Just remember it. The right formula to use in cell C9 is

\[ =C$4*B9+C$5 \]

When we copy that formula from row 9 to rows 10:13 we get Figure 7.3.

Figure 7.4 is a screen shot of the same spreadsheet – after we asked Excel to show the *formulas* for each cell instead of the values.

In Chapter 6 we learned how to use Excel to draw bar charts and histograms so that we could visualize data organized into categories. The x-axis displayed category names, with corresponding values on the y-axis. That won’t work for the the data in Figure 7.3, since there both the x- and y-axes have numerical values.
Instead, after selecting cells B9:C13 we must ask Excel for a chart of type \textit{XY(Scatter)}. Figure 7.5 shows the result.

The graph is a straight line – that’s why the function is called “linear”. The slope tells us how steep the line is, the intercept tells us where it crosses the vertical axis – in this case at the value £9.49, the total bill when you use no electricity at all.

Excel will let you change the type of a chart once it’s built. If you change the chart in Figure 7.5 to a \textbf{Line Chart} Excel will use the data in the column B as \textit{category labels} rather than as the numbers of kilowatt-hours. It will space them evenly along the x-axis, whatever their values, and draw the nonsense you see in Figure 7.6.

\footnote{See comment in instructor’s manual.}
If you select the two columns of data and build a line chart first, things are even worse. Excel thinks each row is a category for which you have two pieces of data. It labels the categories 1, 2, ... and shows a line for each. If you change the chart type to scatter you get two scatters, one for each column. You can get Figure 7.5 only if you start with a scatter plot.

### 7.5 Comparing electricity bills

Which company charged more for electricity – Tamworth or NStar?

Excel and the slopes and intercepts will help us answer this question.

To compare costs, we’ll convert the British currency to dollars. Both of the bills are from 2007; the conversion rate then was about 2.20 $/£. \(^{11}\)

Tamworth charges 0.0735 £/kwh. In dollars that becomes 0.1617 $/kwh. That’s less than the 0.17077 $/kwh at NStar.

To find the intercept, we start with the conversion: £9.49 is $20.88. Since the Tamworth bill is quarterly and the NStar bill monthly, we divide by three to find the Tamworth fixed monthly cost: $6.96 – a little more than NStar’s.

We did the work in spreadsheet [http://www.cs.umb.edu/~eb/qrbook/ElectricityBill/TamworthNStar.xlsx](http://www.cs.umb.edu/~eb/qrbook/ElectricityBill/TamworthNStar.xlsx) – the screenshot in Figure 7.7. We’ve labelled the slopes and intercepts of both graphs because they provide important information. The intercept is the value of the dependent variable \(y\) when the value of the
independent variable $x$ is 0. It’s the height at which the graph crosses (“intercepts”) the $y$-axis. In these examples it’s the amount you pay when you use no electricity at all.

As its name suggests, the slope measures how steep the graph is. The larger the slope, the more you pay per kWh, the faster the line rises as you move to the right.

In each case the units of the intercept are the same as the units of $y$. The units of the slope are

\[
\text{units of } y \quad \text{units of } x
\]

In this example the two graphs are very close, which we find interesting and surprising. The NStar graph starts out lower (the intercept is smaller) but rises faster (the slope is larger). The lines cross at about 60 kwh. If you use fewer than about 60 kWh in a month you’ll pay less to NStar than to Tamworth. For more than 60 kWh per month you’ll pay more to NStar. The data table confirms that observation.

Figure 7.8 shows how we arranged the formulas in the spreadsheet to compute the electricity bills for both companies. Column B lists the values for the different amounts of electricity. It starts in row 10 at 0, to show the intercept. The formula $=B10+20$ in cell B11 fills column B when we copy it to cells B12:B15. The formula in cell C10 is

\[
=C5*B10+C6
\]

It uses three $\$\$ signs to keep Excel from adjusting references for rows 5 and 6 and for column B. That allowed us to copy it to all of the range C10:D15.

The problem we’ve just solved is typical of situations where you have to decide between two options, one with a small startup cost but a high ongoing rate, the other the reverse. Here are some examples:
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- Insulate your house (high initial investment compared to doing nothing) in order to pay less for heat in the winter (lower rate for use),
- Buy a hybrid instead of a conventional car (higher initial cost, lower rate of fuel consumption),
- Buy energy efficient light bulbs (more expensive to start with, but they use less electricity to run),
- Select a phone plan with unlimited text messaging (more expensive than pay-as-you-text, but the slope ($ per text) is zero).

Each of these can be thought of as looking at two linear equations to see where their graphs cross. You can do that by building a table of values or by drawing the graphs (in Excel, or with pencil and paper). You can also do that by writing down the equations and solving them with remembered algebra, or by guess-and-check.

But you can’t rely on just this mathematics to make a decision. There are always other important things to think about. How long does the more expensive purchase last? Can you afford the initial high payment? If so, what else might you rather do with that money? Do you need to take depreciation or inflation into account?

7.6 Energy and power

How much electrical energy does a 100 watt light bulb use? That depends on how long it’s on for. When it’s switched off, it doesn’t use any at all. If it’s on for two hours it must use twice as much as it does in one hour. Figure 7.9 (from the second page of the Tamworth bill) shows the proportion lurking there.

The second line in that figure displays the units for the quantities in the first line. Time is measured in hours, of course. Power is measured in kilowatts. Energy is measured in kilowatt-hours: the product of the units for power and for time.
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Energy transferred = power \times time

\begin{align*}
\text{kilowatt-hour, kWh} & : \text{kilowatt, kW} : \text{hour, h} \\
\end{align*}

Figure 7.9: How much electrical energy?

That tells us right away that \textit{energy} and \textit{power} are not the same thing. Comparing the figure to Equation 7.1, you can see that \textit{energy} is like \textit{distance} – a thing that’s consumed or traveled, while \textit{power} is like \textit{speed} – the \textit{rate} at which energy is used or distance covered. [12]

If you turn a 100 watt light bulb on for 2 hours the formula in Figure 7.9 tells you how much electrical energy you use:

\begin{align*}
100 \text{ watts} \times 2 \text{ hours} & = 200 \text{ watt-hours} \quad (7.2) \\
& = 0.2 \text{ kilowatt-hours} \quad (7.3)
\end{align*}

Line 7.2 is just multiplication. Line 7.3 changes watt-hours to kilowatt-hours.

How much does it cost to leave a 40 watt bulb on all the time in your basement for a year? There are about 9,000 hours in a year. That is 9 kilo-hours, so you’ll use about

\begin{align*}
40 \text{ watts} \times 9 \text{ kilo-hours} & = 360 \text{ kilowatt-hours.}
\end{align*}

If you pay $0.10 per kWh for electricity you will pay about $36 per year to guarantee that you don’t fall down the basement steps in the dark.

The electrical energy that flows through the wires in your house to your appliances probably comes to you from a power plant, which might be burning coal or natural gas or extracting the energy from nuclear fuel. [13] So power plants produce energy, not power. The power of a power plant is the \textit{rate} at which it can produce energy. [14] The web site for Chicago’s Cook Nuclear Plant says that

The 1,048 net megawatt (MW) Unit 1 and 1,107 net MW Unit 2 combined produce enough electricity for more than one and one half million average homes. [15]

Let’s check this. The combined total power is 2,155 megawatts. This is the \textit{rate} at which that plant produces electrical energy when it is running at full power. (When it’s not running it’s still just as powerful, but not producing any energy.) When it’s running, how many average homes could it produce electricity for?

If the Cook plant ran all year (about 9,000 hours) it would produce

\begin{align*}
2,155 \text{ megawatts} \times 9,000 \text{ hours} & \approx 18,000,000 \text{ megawatt-hours} \\
& = 18,000,000,000 \text{ kilowatt-hours}
\end{align*}

of electrical energy. Googling “average household electricity usage” finds

6,000 kWh per household per year for 3 residents average per household.

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[13]You might have a wind turbine in your neighborhood, or a hot spring, or solar panels on your roof, but these are unlikely power sources for most people.
[14]So “energy plant” would be a better name than “power plant”.
from [http://www.physics.uci.edu/~silverma/actions/HouseholdEnergy.html](http://www.physics.uci.edu/~silverma/actions/HouseholdEnergy.html). The source is a physics professor’s web site, so it’s probably reliable. At 6,000 kWh per household per year Cook could power 3 million homes. The quotation claims half that, so it’s clearly in the right ballpark. The 6,000 kWh per household per year is a southern California average – households in northern Illinois might well use more electricity.

Energy comes in many forms besides electric. The Cook plant *converts* the energy in its nuclear fuel to electricity. Driving a car uses the energy stored in the gasoline. Running a marathon uses the energy in the food you eat. Each form of energy has its own units. We’ve seen that electrical energy is measured in kilowatt-hours. If you cook on a gas stove, the energy in the gas is measured in *therms*. The energy in the oil that heats your house is measured in *British Thermal Units* or BTUs. The energy in the food you eat is measured in calories. Physicists measure energy in *ergs* or *joules*; you rarely see those units in everyday life. You can look up conversion factors for these units – for example, the energy in a barrel (42 gallons) of oil is about 5.8 million BTU, which is equivalent to 1700 kilowatt-hours. So it would take about a fifth of a barrel to keep that 40 watt light bulb burning for a year.

Converting among the units for energy is just like converting among the units for length (meters, feet, yard, miles, . . . ). You can use a table, an online calculator like the one at the National Institute of Standards and Technology ([http://physics.nist.gov/cuu/Constants/energy.html](http://physics.nist.gov/cuu/Constants/energy.html)) or the Google calculator.

Possibly the most interesting energy conversion is the one that Einstein discovered in 1905: mass and energy are the same thing, measured in different units. The conversion factor is the square of the speed of light – hence the famous equation

\[ e = mc^2. \]

To see that at work, look again at the yearly energy output of the Cook plant. The Google calculator tells us that

\[ 18 000 000 000 \text{ kilowatt hours} = 6.48 \times 10^{16} \text{ joules} \]

The National Institute of Standards and Technology website says that corresponds to a mass of about 0.72 kg, which is 720 grams. That means just about 1.6 pounds of matter must be converted to energy to power millions of Chicago homes for a year. [*16*

### 7.7 Taxes: sales, social security, income

Taxes are a part of life. So it’s only common sense to learn how they work. In Section 3.6 and Section 7.1 we studied sales taxes. They are collected by cities and states and are computed as a percentage of the purchase price.

In this section we’ll explain two important federal taxes which depend on your income, not on how you spend it.

The first of these is the *FICA* tax. “FICA” is the acronym for the “Federal Insurance Contributions Act”. Those taxes pay for social security and medicare. In 2014 the tax rate was 6.2% for social security and 1.45%
7.7. TAXES: SALES, SOCIAL SECURITY, INCOME

for medicare. The social security tax is collected only on the first $117,000 of your earnings. Up to that income level the combined rate is 7.65%.

Here are some sample computations.

- If you earn $1,000 your FICA tax is $0.0765 \times 1,000 = $76.50.
- If you earn $50,000 your FICA tax is $0.0765 \times 50,000 = $3,825.
- If you earn $117,000 your FICA tax is $0.0745 \times 117,000 = $8,950.50.
- If you earn $500,000 your FICA tax is $0.062 \times 117,000 + 0.0145 \times 500,000 = $14,504.

Once you make more than $117,000 the amount of social security tax remains constant; the medicare tax continues at the rate of 1.45%. That means the percentage of your earnings collected for FICA taxes decreases as your earnings increase. That percentage is called the effective tax rate. Up to $117,000 the effective rate is 7.45%. For $500,000 the effective rate is just $14,504/$500,000 = 2.9%. For higher incomes, the effective rate is even smaller. Because they decreases as earnings increase, social security taxes are regressive. Once you reach the social security maximum your effective tax rate decreases. So the more you earn, the smaller a percentage you pay.

Figure 7.10 is a screenshot of the spreadsheet http://www.cs.umb.edu/~eb/qrbook/ElectricityBill/SocialSecurityTax.xlsx showing a data table and a chart illustrating social security tax computations. The formula bar shows how Excel computes the tax on $500,000.

Income tax is a little more complicated. It’s a progressive graduated tax. When you make more money you not only pay more tax, some of your income may be taxed at a higher rate. Table 7.11 shows the 2014 tax brackets for single taxpayers.

That table tells you that the first $9,075 of your income is taxed at 10%. If you make exactly that much, you pay $907.50 in tax. If you make more, the extra income is taxed at a higher rate – you have moved to a higher tax bracket. For example, if you make between $9,075 and $36,900 you will pay 15% of the amount you earn over $9,075. If you earn more than $36,900 you start paying at a 25% rate.

\[ \text{The actual rules are a little more complicated. First, the tax applies only to wages. Other income (like stock dividends or interest) are not subject to this tax. Second, the real rates are twice the quoted amounts, but your employer is required to pay half. If you’re self employed you pay it all.} \]

\[ \text{Improved exposition (Shirley Elliot, Spring 2014)} \]