Chapter 9

Compound Interest – Exponential Growth

In this chapter we explore how investments and populations grow and radioactivity decays – exponentially.

Chapter goals:

Goal 9.1. Understand that exponential growth (or decay) is constant relative change.

Goal 9.2. Understand how compound interest is calculated.

Goal 9.3. Work with exponential decay.

Goal 9.4. Reason using doubling times, half lives, rule of 70.

Goal 9.5. Fit exponential models to data.

9.1 Money earns money

Imagine that you have $1,000 to invest. One of the points of investing money is to earn interest on your investment. Would you rather earn $100 per year in interest, or 8% per year in interest? In each case the interest is added into your principal (the balance in your account) each year, and you never make any withdrawals.

The first scenario is called simple interest. You find the new balance by adding $100 each year to the previous balance. After one year the balance would be $1,100, after two years $1,200, and so on.

The second scenario is a little more complicated. The interest each year is a fixed percentage of the balance. The one-plus trick does the job in one step: after one year, the new balance would be $1,080 = $1,000 \times 1.08$. After two, it would be $1.08 \times $1,080 = $1,166.40. This pattern is called compound interest.

Table 9.1 shows your balance in each case for the first four years.

So far simple interest offers a better return on your investment. What would the numbers be in 10 years? We could continue building the table a year at a time by hand (which is tedious), we could have Excel calculate
9.1. MONEY EARNS MONEY

Table 9.1: Simple and Compound Interest

<table>
<thead>
<tr>
<th>year</th>
<th>simple interest</th>
<th>compound interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>now</td>
<td>$1,000</td>
<td>$1,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$1,100</td>
<td>$1,080.00</td>
</tr>
<tr>
<td>2</td>
<td>$1,200</td>
<td>$1,166.40</td>
</tr>
<tr>
<td>3</td>
<td>$1,300</td>
<td>$1,259.71</td>
</tr>
<tr>
<td>4</td>
<td>$1,400</td>
<td>$1,360.49</td>
</tr>
</tbody>
</table>

for us (we’ll do that in a minute), or we could look for a pattern and find a formula for each scenario, so that we can compute for any year we like without having to do the work for all the years in between. We’ll do that first.

Simple interest leads to a linear equation. Each year the balance increases by a fixed amount, $100, so the slope is $100/year. The intercept is the starting value, $1,000. The linear function is

\[ B = 1000 + 100 \times T \]  \hspace{1cm} (9.1)

where \( B \) represents the balance, in dollars, and \( T \) the number of years. If you leave your money growing until \( T = 10 \) years, your balance will be \( 1000 + 100 \times 10 = 2000 \) dollars.

The function describing compound interest isn’t linear. The \textit{percentage} increase is constant but the \textit{amount} of interest changes from year to year. In the first year you earn $80, while in the second you earn $86.40. To see what kind of function to use, we unwind the arithmetic in the compound interest column of Table 9.1.

Year 1 1080.00 = 1000.00 \times 1.08
Year 2 1166.40 = 1080.00 \times 1.08 = (1000 \times 1.08) \times 1.08 = 1000 \times 1.08^2
Year 3 1259.71 = 1166.40 \times 1.08 = (1080 \times 1.08^2) \times 1.08 = 1000 \times 1.08^3

It’s clear that the function describing this growth is

\[ B = 1000 \times (1.08)^T \]  \hspace{1cm} (9.2)

where, as before, \( B \) represents the balance, in dollars, and \( T \) the time, in years. It’s an \textit{exponential} function, because the independent variable \( T \) is the \textit{exponent} of 1.08. The 0.08 in 1.08 = 1 + 0.08 is the constant relative change for each additional year. The 1000 is where we start: the value of \( B \) when \( T = 0 \). \footnote{You may remember but not have enjoyed the fact that 1.08^0 = 1. If so, perhaps it makes a little more sense in this context. After 0 years you’ve received no interest, so your balance should be multiplied just by 1.}

Suppose you want to compare the balances at simple and compound interest after year 10. With simple interest you will have \( 1000 + 10 \times 100 = 2000 \) dollars. But how can you compute \( 1000 \times 1.08^{10} \) without boringly multiplying by 1.08 ten times? The calculator in Figure 2.2 isn’t powerful enough. For that job you need a \textit{scientific calculator}, one with a key labeled \( y^x \) or \( x^y \).

There are many on line. Here are two: \texttt{http://www.math.com/students/calculators/source/scientific.html} \texttt{http://web2.0calc.com/}

Each will tell you that at the end of year 10 the balance will be about $2159, so the exponential growth has caught up with the linear.

You can do the computation with the Google calculator’s buttons (Figure 1.1 but to use the search bar or Excel you need to know how to enter the exponent from the keyboard without a \( y^x \) key. Both use the caret
character \( ^\wedge \) to raise a number to a power. That’s meant to suggest literally “raising” the exponent. You just type

\[
1000 \times 1.08 ^\wedge 10
\]

into the Google search bar, or as a formula (preceded by an equal sign) in a cell in Excel to check the arithmetic in the previous paragraphs.

In *Common Sense Mathematics* we rarely put things to remember in boxes, but the moral of this discussion deserves that treatment:

| In linear growth, the absolute change is constant. |
| In exponential growth, the relative change is constant. |

Interest isn’t the only place exponential growth happens. In Exercise 9.6.2 (page 220) we ask you to think about others.

### 9.2 Using Excel to explore exponential growth

The spreadsheet [http://www.cs.umb.edu/~eb/qrbook/Interest/exponentialGrowth.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/exponentialGrowth.xlsx) answers “what if” questions about exponential growth by allowing you to change the values 1,000 and 1.08 in Equation (9.2). Figure 9.2 shows two examples, for the equations

\[
B = 1000 \times (1.08)^T
\]

and

\[
B = 500 \times (1.16)^T.
\]

Each swoops upward at an increasing rate. That shape is the signature for exponential growth.

![Figure 9.2: Two exponential graphs](image)

Let’s take some time to see how Excel updates the calculations when we change the constants in the equation. Click on one of the cells in the exponential column, say B19. The formula bar reads

\[
= \text{START} \times \text{RELCHANGE} ^\wedge A19,
\]

(9.3)
which is Excel’s version of Equation (9.2). We labeled cells \textbf{A12} and \textbf{A13} as START and RELCHANGE so that we could use formula (9.3) instead of
\[ \text{A12}\times\text{A13}\textsuperscript{A19}. \quad (9.4) \]

The version using cell labels is much easier to understand than the one with cell references, and it doesn’t need the dollar signs to tell Excel not to change those references when we copy from one row to another. 

Figure 9.3 shows two screen shots of our spreadsheet, the first with cell values, the second with cell formulas.

The numbers in columns \textbf{B} and \textbf{C} are the same. Excel computes them in different ways. We’ve seen how \textbf{B19} uses the algebra in Formula (9.3). The value in cell \textbf{C19} comes from the previous value in \textbf{C18} instead:
\[ \text{C18}\times\text{RELCHANGE}. \]

The spreadsheet has a second tab (labeled “Compare two growth trajectories”) that shows two exponential curves on the same set of axes. Figure 9.4 provides an example. There it’s really clear how much faster growth is at 16\% than at 8\%.

In the hypothetical investment comparison at the start of this chapter linear growth starts out better but by year 10 exponential growth leads to a higher balance. To explore what happens in more detail, use the spreadsheet \url{http://www.cs.umb.edu/~eb/qbook/Interest/linearExponential.xlsx}. It extends Table 9.1 to cover 15 years. The graph in Figure 9.5 shows that starting at year 7, the value of the exponential function is larger than the linear.

Now we can answer “what-if” questions. Suppose, for example, our money earned 7\% interest instead of 8\% interest. To redo the calculations we need to change just one number: replace the 1.08 in cell \textbf{A9} with 1.07. Excel recomputes the values of the exponential function in column \textbf{C} and redraws the graph. Then you can see that with this lower interest rate, we have to wait 11 years before the exponential growth of compound interest gives us a better return.

\footnote{To label a cell, click on it. The Name Box at the left of the Formatting Toolbar will contain the address of the cell, so if you click on cell \textbf{B4} you will see \textbf{B4} there. You can highlight the contents of the box and type in your own name.}
9.3. DEPRECIATION

It’s always easier to think about increases (adding and multiplying) than decreases (subtracting and dividing) but sometimes things do decrease.

Suppose you buy a new car for $20,000. As soon as you drive it out of the dealer’s lot it’s worth less. In fact it’s worth less each year: it *depreciates*. Its value depends on its age.

If the car is a business expense you might choose linear depreciation for tax purposes – suppose the value decreases by $1,800 each year. The equation that determines the value $V$ as a function of the age $A$ is

$$V = 20,000 - 1,800A.$$ 

But a more realistic way to model the value of the car is to assume that the *percentage* decrease is the same each year. Suppose it’s 13%. Then each year its value is 87% of what it was the year before. The corresponding equation is

$$V = 20,000 \times 0.87^A.$$ 

We can use our old friend [http://www.cs.umb.edu/~eb/qrbook/Interest/linearExponential.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/linearExponential.xlsx) to draw Figure 9.6 showing what the car is worth over time in each case. Set START to 20,000, ABSCHANGE to

---

Figure 9.4: Two exponential graphs on the same set of axes

Figure 9.5: Linear vs Exponential Growth
9.4 Doubling times and half lives

How long will it take to double your money? (We'll assume you're clever enough to insist on compound interest.) The answer depends on the interest rate and the initial balance. The spreadsheet shows that at 8% interest with an initial investment of $1,000 the balance is $2,000 after 9 years (the table shows $1999.004627, which quite close enough to double).

If you change the initial investment to $100 then Excel shows a balance of $200 after the same 9 years. Experimenting with many different initial investments always shows the same doubling time. So the time it takes to double your money does not depend on the amount you start with.

What about a different interest rate? If you use 5% interest in the spreadsheet the doubling time seems to be between 14 and 15 years. We can do a calculation: $1.05^{14.5} = 2.028826162$, so 14.5 is a good guess.

At 2% interest it takes more than 30 years to double your money, so the spreadsheet doesn’t give us the answer. We could find it by adding some rows, but we’ll use another method instead. We’ll try to guess the value of $T$ in the equation $1.02^T = 2$ and adjust our guess until we’re close enough. Perhaps the answer is $T = 40$ years:

$$1.02^{40} = 2.208039664.$$ 

Too big, so we need less time. Try 35:

$$1.02^{35} = 1.999889553.$$ 

Bingo!

We’ve collected up these results and a few more in the second column of Table 9.7.

---

3 The relative change is still positive. It’s a decrease rather than an increase because it’s less than 1.

4 See comment in instructor’s manual.
9.4. DOUBLING TIMES AND HALF LIVES

<table>
<thead>
<tr>
<th>interest rate (%)</th>
<th>approximate doubling time</th>
<th>70/rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>35</td>
<td>35.0</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>23.3</td>
</tr>
<tr>
<td>5</td>
<td>14.5</td>
<td>14.0</td>
</tr>
<tr>
<td>7</td>
<td>10.5</td>
<td>10.0</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>8.8</td>
</tr>
<tr>
<td>10</td>
<td>7.5</td>
<td>7.0</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>4.7</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>3.5</td>
</tr>
<tr>
<td>50</td>
<td>1.7</td>
<td>1.4</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 9.7: Double Your Money

The third column in that table shows the results from the “Rule of 70,” which says that you can estimate the compound interest doubling time by dividing the magic number 70 by the annual interest rate as a percent. The approximation is better when the interest rate isn’t too large; those are just the cases that matter most in everyday investing. The most commonly quoted consequence of the Rule of 70 is that money invested at 7% will double in 10 years.

Figure 9.8 shows how good the Rule of 70 is for interest rates up to 20%.

When a relative increase occurs repeatedly the doubling time is independent of the initial value. So if inflation is 5% per year, all prices will double in 14 years.

Knowing the doubling time helps you make quick calculations. Since 5% inflation doubles prices in 14 years it will quadruple them in 28 years. In 42 years they will be eight times as large. The Bureau of Labor Statistics inflation calculator says that inflation in the 42 years from 1968 to 2010 increased the cost of a $100 item to $626. That’s not quite eight times as much, so the average inflation rate for those years was not quite 5% per year.

The Rule of 70 applies to depreciation as well – it tells you the half life. That’s the equivalent for depreciation of the doubling time – the time until half the original value has disappeared. Like doubling time, the half life depends on the depreciation rate, but not on the original value. For 13% annual depreciation it’s approximately 70/13 = 5.38461538 ≈ 5.4 years.

The term half life comes from atomic physics, where it describes the way the quantity of a radioactive
9.5. **EXPONENTIAL MODELS**

An element decreases over time. The following quotation from [http://www.nirs.org/factsheets/hlwfcst.htm](http://www.nirs.org/factsheets/hlwfcst.htm) provides food for quantitative thought.

> After ten half-lives, one thousandth of the original concentration of a radioactive substance is left. After 20 half-lives, one millionth. Generally 10–20 half-lives is called the hazardous life of the waste. Example: Plutonium-239, which is in irradiated fuel from a nuclear power plant, has a half-life of 24,400 years. It is dangerous for a quarter million years, or 12,000 human generations.

This is the kind of quotation that begs to have its numbers checked.

First let’s look at “ten half-lives.” After one half-life the concentration is half what it was at the start. After two half-lives it’s half of a half, or 1/4. After three half-lives it’s 1/8. So after 10 half-lives it’s $\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{10}}$ of what it was. [5]

We saw when we studied the metric prefixes that $2^{10} = 1,024 \approx 1,000$. That’s why “kilo” means “1,000” most of the time but “1,024” when computers are involved. Now we use the same fact to see why $1/2^{10}$, which is exactly 1/1,024, is approximately 1/1,000 – one one thousandth.

What about twenty half-lives? In that time the original concentration will be reduced to 1/1,000 of 1/1,000 of what it was at the start. Since a thousand thousand is a million, that’s one one-millionth.

According the the quotation, Plutonium-239 will be dangerous for at least ten 24,000 year half-lives. That’s 240,000 years, which is indeed about a quarter of a million years. Is it 12,000 generations? Yes, if you calculate with $240,000/12,000 = 20$ years per generation. That’s perhaps a little low for the developed world, but good enough to highlight the danger of nuclear waste.

### 9.5 Exponential models

For compound interest and radioactive decay the equation for exponential change gives exact answers, just as the linear equation gives exact answers for simple interest and electricity bills.

When change is approximately linear a regression line may be useful. When it’s approximately exponential, we can construct an **exponential trendline**. Most elementary texts discuss the reproduction of bacteria as a system likely to be well modeled by an exponential equation. Bacteria reproduce by dividing, so each individual gives rise to 2, then 4, then 8 descendants, and so on. The number of bacteria grows exponentially – for a while. The constant relative change, in percent per hour, depends on the time between generations. Eventually, crowding or diminishing resources cause growth to slow, perhaps even to reverse as organisms die faster than new ones are born. [6]

Table 9.9 records the population of three different strains of the *E. coli* bacterium in a one day experiment conducted by Professor Vaughn Cooper at the University of New Hampshire.

---


9.5. EXPONENTIAL MODELS

### Table 9.9: Bacteria Growth

<table>
<thead>
<tr>
<th>time</th>
<th>W</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7.9</td>
<td>8.5</td>
<td>17.5</td>
</tr>
<tr>
<td>8</td>
<td>17.3</td>
<td>13.5</td>
<td>42.1</td>
</tr>
<tr>
<td>12</td>
<td>44.7</td>
<td>48.9</td>
<td>225.2</td>
</tr>
<tr>
<td>17</td>
<td>119.3</td>
<td>268.7</td>
<td>407.3</td>
</tr>
<tr>
<td>20</td>
<td>41.3</td>
<td>98.0</td>
<td>149.3</td>
</tr>
<tr>
<td>24</td>
<td>41.3</td>
<td>64.0</td>
<td>160.0</td>
</tr>
</tbody>
</table>

The raw data points in the graph on the left in Figure 9.10 (built in spreadsheet [http://www.cs.umb.edu/~eb/qrbook/Interest/bacteriaGrowth.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/bacteriaGrowth.xlsx)) suggest that the growth of each strain was exponential until about hour 17. To construct the graph on the right we plotted the data for each strain for that period. In the resulting chart we right-clicked on a data point for each strain, selected Add Trendline ..., and chose Exponential on the Type tab. As usual, we asked for the equations and the R-squared values. Those are all pretty close to 1; best for strain W, worst for strain S.

Let’s look at the equation for the exponential trendline for strain W, which you can see in the figure:

\[ y = 3.3663 e^{0.2109x}. \quad (9.5) \]

What is the “e” in this equation? The complete answer to that question calls for much more mathematics than you need to know to apply common sense to quantitative arguments. But since Excel uses it you may encounter it somewhere so we’ll discuss it briefly.

The simplest explanation is that e is just a particular number – approximately 2.7183. Like \( \pi \approx 3.1416 \) it’s one of those numbers whose decimal expansion “goes on forever,” so showing just a few decimal places is only

---

7 In Chapter 8 we discovered that the linear trendline equation reported on the graph might not provide enough significant digits. The same is sometimes true for the equations for exponential trendlines, but recovering the more accurate values is trickier, and not worth trying to master. In this example the displayed numbers are good enough.
9.6 Exercises

Exercise 9.6.1. [S][R][Section 9.1] [Goal 9.1] [Goal 9.2] Compound interest computations

If you invest $1500 at 7% interest compounded every year, how much will you have at the end of 10 years? 15 years? 20 years? Use the formula for exponential growth; then check your answers with the http://www.cs.umb.edu/~eb/qrbook/Interest/exponentialGrowth.xlsx spreadsheet.

Exercise 9.6.2. [S][Goal 3.1] [Section 9.1] [Goal 9.1] When do you expect exponential growth?

In each of the following situations, explain why you would expect linear or exponential growth.

Think about whether the change is best described as an absolute rate (like dollars per hour or gallons per mile) or a percentage (like percent per year or percent per washing).

Write the units for the kind of rate you decide on.

(a) Price increases from year to year due to inflation.

(b) How the amount of gas you use depends on how far you drive.

(c) The amount of money left on your public transportation debit card as the days go by and you commute to school or work.

---

8 If you experiment with EXP you will find that \( e^{0.7} = 2.0137527 \ldots \approx 2 \). It’s the 0.7 in the exponent that leads to the Rule of 70. To learn just how, go on to take a course in calculus.

9 A newspaper reporter wanting to emphasize the drama might say, incorrectly, that the populations dropped “exponentially.”
(d) The amount of sales tax you pay, depending on how much you buy.
(e) The amount of dirt left in your kid’s filthy jeans when you wash them over and over again.
(f) The population of the world as the years go by.
(g) Your credit card balance if you stop making payments\textsuperscript{10}.
(h) The height of the snow as it accumulates in a big storm.
(i) The number of people sick in the first weeks of the 'flu season.
(j) The number of subscribers to a hot new social network in its first days.

Think about your answers before you look at the hints.

[See the back of the book for a hint.]

\textbf{Exercise 9.6.3.} [S][Section 9.1] [Goal 9.1] Is it really exponential?

In everyday usage the phrase “growing exponentially” is just a vibrant synonym for “growing rapidly.” It’s rare that it really means a constant relative change.

Find instances of “exponential” growth in the media where what’s meant is just very rapid growth.

\textbf{Exercise 9.6.4.} [U][Section 9.1] [Goal 9.1] [Goal 9.2] Double your money slowly

Show that the time it takes to double your money when you’re collecting simple interest depends on the annual interest \textit{and} on the initial investment.

\textbf{Exercise 9.6.5.} [S][Section 9.1] [Goal 9.1] Health care spending.

In Chapter 3, Exercise 3.10.17 (page 76), we used data from the 2010 National Health Expenditures report to compute the absolute and relative changes in health care spending per person from 2007 to 2008.

(a) Use the results of those calculations to build linear and exponential models for the growth of health care spending per person.

(b) Use each model to predict when health care spending will reach $10,000 per person per year.

\textbf{Exercise 9.6.6.} [S][Section 9.1] [Goal 9.1] Malthus

Thomas Malthus, an English economist and clergyman in “An Essay on the Principle of Population” in 1798:

\begin{quote}
I think I may fairly make two postulata.
First, That food is necessary to the existence of man.
Secondly, That the passion between the sexes is necessary and will remain nearly in its present
state.
These two laws, ever since we have had any knowledge of mankind, appear to have been fixed
laws of our nature, and, as we have not hitherto seen any alteration in them, we have no right to
conclude that they will ever cease to be what they now are, without an immediate act of power in
that Being who first arranged the system of the universe, and for the advantage of his creatures,
still executes, according to fixed laws, all its various operations.
\end{quote}

\textsuperscript{10}We will study credit cards in the next chapter.
Assuming then my postulata as granted, I say, that the power of population is indefinitely greater than the power in the earth to produce subsistence for man.

Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison of the second.

The power of population is so superior to the power in the earth to produce subsistence for man, that premature death must in some shape or other visit the human race. The vices of mankind are active and able ministers of depopulation. They are the precursors in the great army of destruction; and often finish the dreadful work themselves. But should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague, advance in terrific array, and sweep off their thousands and ten thousands. Should success be still incomplete, gigantic inevitable famine stalks in the rear, and with one mighty blow levels the population with the food of the world.

Malthus claimed that the food supply grows in a linear fashion. As a unit of food supply he used the amount of food needed for one person for one year. He estimated food production in Britain in 1798 as 7,000,000 food units and that food production might increase by a constant 280,000 units each year.

Malthus also believed that the population of Britain was growing at a rate of 2.8% each year. In 1798, the population was about 7,000,000.

(a) Write a linear function that models food production.

(b) Write an exponential function that models population growth.

(c) Was there enough food for each individual in Britain in 1798?

(d) Using Malthus’ models, determine whether there would be enough food for each individual in Britain in 1800.

(e) Malthus claimed that the population in Britain would eventually outstrip the food supply – a prediction we now call “the Malthusian dilemma.” He didn’t have Excel to do the arithmetic for him, but we do. Use it to estimate when Malthus’ predicted disaster would occur. Was Malthus right?

Exercise 9.6.7. [S][Section 9.1] [Goal 9.1] [Goal 9.2] The pawn shop business model.

On April 9, 2011 The New York Times reported on a pawn shop that opened in an ex-Blockbuster store:

La Familia’s business model is to make loans, almost entirely to people without bank accounts, using a customer’s ring or television or iPod as collateral. The borrowers are given 60 days to pay back the loan, and La Familia charges a 20 percent interest rate per month. (So for a $100 loan, the borrower would need to pay back $140 after 60 days.)

(a) Explain why 20% interest per month on a $100 loan for two months would actually require repayment of a little more than $140.

(b) What is the annual interest rate when this business lends money?

Open the spreadsheet [http://www.cs.umb.edu/~eb/qrbook/Interest/exponentialGrowth.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/exponentialGrowth.xlsx) and describe what happens to the graph when you make each of the following experiments. If you can see easily what happens to the numbers, describe that too.

(a) Change the value of \( \text{START} \) from 1,000 to 10, then 100, then 10,000. Change it to some other random positive values that aren’t as nice.

(b) Change the value of \( \text{START} \) from 1,000 to -1,000.

(c) Change the value of \( \text{RELCHANGE} \) to 1.

(d) Change the value of \( \text{RELCHANGE} \) to 1.01 (1% growth). Fit a linear trendline to the data. What is the R-squared value? What does it tell you?

(e) Change the value of \( \text{RELCHANGE} \) to 2. Why does the graph look flat at 0 as far as \( T = 20 \)? Is it really flat?

(f) Change the value of \( \text{RELCHANGE} \) to 10.

(g) Change the value of \( \text{RELCHANGE} \) to 0.9 (a 10% decrease).

(h) (Optional) Can you figure out how we got the label on the graph to incorporate the values of \( \text{START} \) and \( \text{RELCHANGE} \)?


If you try to use [http://www.cs.umb.edu/~eb/qrbook/Interest/linearExponential.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/linearExponential.xlsx) to see when exponential growth at 5% catches linear you see that it’s still behind at 20 years, which is as far as the table goes.

Modify the spreadsheet to determine when it catches up.


(a) How does changing the initial investment change the time it takes for the exponential function to catch up with the linear function?

(b) Show that if you double or triple both the initial investment and the absolute change the time it takes for the exponential function to catch up stays the same.


The data in this problem aren’t real. But the problem is interesting and instructive, so it’s worth spending time on.

Suppose you are shopping for a car and find three deals advertised:

- Make a $10,000 down payment and pay only $100 per month for two years.
- Just $5000 down, monthly payments start at a low $50 and increase by $50 each month for two years.
- Give me $1.00 today and take the car home! Pay 1 penny for the first month. Then double your payment each month. After two years, the car is yours.
9.6. EXERCISES

(a) Before you do any calculating, which deal do you think is best? Why?

(b) What would your monthly payments be in the second and tenth months if you take the second dealer’s offer?

(c) What would your monthly payments be in the second and tenth months if you take the third dealer’s offer?

(d) For each deal, write an algebraic expression that gives the monthly payment.

(e) Use Excel to calculate your total payments for the 24 months. Set up four columns as in Table 9.11. Then tell Excel how to fill in the columns to 24 months. Finally, use the SUM function to add up the payments.

<table>
<thead>
<tr>
<th>Month</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(down) 0</td>
<td>10,000</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>:</td>
</tr>
<tr>
<td>24</td>
<td>:</td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

Table 9.11: Three Car Deals

(f) Now use what your calculations tell you to compare the three deals. Which is best? Which worst?

Exercise 9.6.12. [S][C][Section 9.2][Goal 9.1] Green Giant: Beijing’s crash program for clean energy

In the December 21 & 28 2009 issue of The New Yorker Evan Osnos wrote in his essay with that headline that

R & D expenditures have grown faster in China than in any other big country – climbing about twenty per cent each year for two decades, to seventy billion dollars last year.

(a) Use Excel to build a chart of annual Chinese R&D expenditures for the years 1989 – 2008.

(b) Add a data column showing the annual expenditures adjusted for inflation (use the United States cost of living index) and display that data on your chart.

Exercise 9.6.13. [S][Section 9.3][Goal 9.1] Car excise tax

In Massachusetts you pay excise tax each year on the current value of your automobile. Assume for the sake of this problem that the rate is 3%, so you would pay $600 in excise tax in the first year you owned a new $20,000 car.

Use Excel to answer the following questions.

(a) Suppose the car depreciates linearly at a rate of $1,800 per year. Write a formula for the amount of excise tax you pay as a function of the age of the car.

(b) If you own the car for ten years, what will the car be worth then and how much total excise tax will you have paid?
(c) Answer the same questions if it depreciates at the rate of 13% per year.

(d) Find real data on the way a new car depreciates in value. Is an exponential model a good approximation?


The web site [http://www.nirs.org/factsheets/hlwfcst.htm](http://www.nirs.org/factsheets/hlwfcst.htm) we quoted earlier offers much more information about radioactive waste.

The majority of high-level radioactive waste is the fuel from the hot core of commercial nuclear power plants. This irradiated fuel is the most intensely radioactive material on the planet, and unshielded exposure gives lethal radiation doses. It accounts for 95% of the radioactivity generated in the last 50 years from all sources, including nuclear weapons production. Uranium is processed into fuel rods and loaded into nuclear power reactors where it undergoes the nuclear fission reaction. This increases the radioactivity due to the formation of intensely radioactive elements known as fission products, such as cesium and strontium, resulting from the physical splitting of uranium-235 atoms. Heavier elements, known as transuranics, are also formed – including plutonium. Each 1000 megawatt nuclear power reactor annually produces about 500 pounds of plutonium, and about 30 metric tons of high-level waste in the form of irradiated fuel. After several years, when removed from the reactor core, the fuel is about one million times more radioactive than when it was loaded. This irradiated fuel is currently stored at the reactor sites.

Ask and answer some interesting Fermi problems suggested by the data in this paragraph. You could consider what it says about a nuclear power plant near you.\(^{\text{13}}\)


An article in *The New York Times* on April 6, 2011 discussed levels of radioactive Iodine (Iodine 131) in fish caught near Japan. The article noted that Japan recently revised the safety limit for Iodine 131 in fish to 2,000 becquerels per kilogram. (A becquerel is a measure of radiation.)

Radioactive iodine has a half life of about 8 days.

If a fish contained 10,000 becquerels of Iodine 131 per kilogram, how long would it take for the Iodine to decay to a “safe” level?

**Exercise 9.6.16.** [U][Section 9.4] [Goal 9.1] [Goal 9.4] Quadrupling time

(a) Explain why the quadrupling time in exponential growth is just twice the doubling time.

(b) Show that quadrupling time is given by a “Rule of 140” analogous to the rule of 70.

**Exercise 9.6.17.** [S][Section 9.4] [Goal 9.1] [Goal 9.4] [Goal 9.2] Tripling times

Suppose you invest $1000 at 10% interest compounded every year. (That’s a pretty good rate of return if you can get it – don’t trust a Madoff promise!)

(a) How long will it be until your balance is $3000?

\(^{\text{13}}\)See comment in instructor’s manual.
9.6. EXERCISES

[See the back of the book for a hint.]

(b) Do the same for some other interest rates.

(c) Check that the tripling time in exponential growth is given (approximately) by a “Rule of 110.”

(d) Check that $e^{1.1} \approx 3.$


(a) Calculate the effective rate for 8% annual interest when it’s compounded weekly, daily, hourly, and once every second.

(b) Estimate the effective rate if the interest is compounded every instant.

[See the back of the book for a hint.]

(c) Redo the calculations starting with a 25% annual increase. (Not realistic for interest on a bank account!) Show that the Rule of 70 for doubling times is more accurate the more frequently you compound the interest.


In the Preface to the Carnegie Corporation report Writing to Read Vartan Gregorian wrote

In an age overwhelmed by information (we are told, for example, that all available information doubles every two to three years), . . .

(a) What growth rate in percent per year would lead to a doubling time of two to three years?

(b) Who is Vartan Gregorian?

(c) Can you verify his assertion?

[See the back of the book for a hint.]

Exercise 9.6.20. [S][Section 9.5] [Goal 9.5] [Goal 9.4] Bacteria doubling time

Find the approximate doubling times for strains R and S in the bacteria growth example in Section 9.5.

Exercise 9.6.21. [S][Section 9.5] [Goal 9.1] [Goal 9.5] When will R catch S?

The population of strain S outnumbers that of strain R for the entire first 17 hours of the experiment discussed in Section 9.5. But the exponential trendline equation shows that strain R is growing faster. If the exponential growth were to continue (which it didn’t) when would strain W catch up?

[See the back of the book for a hint.]

Exercise 9.6.22. [U][Section 9.5] [Goal 9.5] The magic number $e.$

\[ \text{http://www.all4ed.org/files/WritingToRead.pdf} \]

© 2014 Ethan Bolker, Maura Mast 226
(a) Find the value of $e$ in Excel using the formula $\text{EXP}(1)$

(b) Find the value from Google with the same formula (without the equal sign). Check that the answers agree as far as they go together.

(c) Which provides more digits?

(d) Can you get more precision from Excel by formatting the cell in which the number appears?

(e) Find even more digits with an internet search.

Exercise 9.6.23. [S][Section 9.5] [Goal 9.1] *Educating mothers saves lives, study says*

From *The Boston Globe*, September 17, 2010

By using statistical models, the researchers found that for every extra year of education women had, the death rate for children under 5 dropped by almost 10 percent. In 2009, they estimated that 4.2 million fewer children died because women of childbearing age in developing countries were more educated.

In 1970, women aged 18 to 44 in developing countries went to school for about two years. That rose to seven years in 2009.

(a) How much did the death rate for children under 5 decline from 1970 to 2009?

(b) Build as much as you can of the exponential model implicit in this quotation. What are the independent and dependent variables? What is the annual relative change?

Exercise 9.6.24. [S][Section 9.5] [Goal 9.1] [Goal 9.5] *email*


In the early 1990s there were some 15 million e-mail accounts worldwide. By the end of 1999 there were 569 million. Today there are more than 3 billion.

(a) Is this exponential growth?

(b) Can you use these numbers to make predictions?

[See the back of the book for a hint.]

Exercise 9.6.25. [U][N][Section 9.5] [Goal 9.5] *Cash, credit card, and . . . smartphones accepted*

Figure 9.12 appeared in *The Boston Globe* on November 2, 2011, along with the quote

Mobile payments are just a sliver of retail sales. They accounted for $3 billion in sales in 2010, or 1 percent of e-commerce transactions, but they are expected to double this year and reach $31 billion by 2016, according to Forrester Research of Cambridge.

We’ve entered the data in a spreadsheet at [http://www.cs.umb.edu/~eb/qrbook/Interest/MobileCommerce.xlsx](http://www.cs.umb.edu/~eb/qrbook/Interest/MobileCommerce.xlsx).

---


9.6. EXERCISES

The percentage on mobile devices is projected to grow linearly at one percent per year. The amount is growing faster – because the total is growing faster.

Compute total e-commerce, fit an exponential trendline.

Exercise 9.6.26. [S][Section 9.5] [Goal 9.5] When will India pass China?

In an article dated April 1, 2011 on the website About.com you could read that

> With 1,210,000,000 (1.21 billion) people, India is currently the world’s second largest country.
>
> Demographers expect India’s population to surpass the population of China, currently the most populous country in the world, by 2030. At that time, India is expected to have a population of more than 1.53 billion while China’s population is forecast to be at its peak of 1.46 billion.
>
> Although India has created several impressive goals to reduce its population growth rates, the India and the rest of the world has a long way to go to achieve meaningful population controls in this country with a growth rate of 1.6%, representing a doubling time of under 44 years.  

(a) Is the article correct in stating that an annual growth rate of 1.6% means India’s population will double in 44 years? Back up your answer with appropriate calculations.

(b) Assuming that India’s growth rate remains 1.6% annually, what will its population be in 2030 when it surpasses China’s?

(c) Assuming that India’s growth rate remains 1.6% annually from 2011 on, what will its population be in the year 2100? Compare that figure to the current population of the world. Do you think India’s growth rate can in fact continue at 1.6% for the 89 years from 2011 to 2100?

Exercise 9.6.27. [S][Section 9.5] [Goal 9.5] Health care expenditures grow


\[ \text{http://geography.about.com/od/obtainpopulationdata/a/indiapopulation.htm} \]

\[ \text{http://www.geography.about.com/od/obtainpopulationdata/a/indiapopulation.htm} \]

We apologize for the grammatical peculiarities of the last paragraph in the quote.

© 2014 Ethan Bolker, Maura Mast 228
(a) Calculate both the absolute change and percentage change in health care spending per person from 2006 to 2007.

(b) Using 2006 as your starting year (2006 = year 0), determine an exponential equation that calculates the amount of health care spending over time assuming the annual percentage change stays the same. Clearly identify the variable names and symbols in your equation.

(c) Using 2006 as your starting year (2006 = year 0), determine a linear equation that calculates the amount of health care spending over time assuming the annual absolute change stays the same. Clearly identify the variable names and symbols in your equation.

(d) Create an Excel spreadsheet to compare the two growth models’ predictions for health care spending through the year 2021. Include a chart showing both models.

(e) Which model first predicts that U.S. health care spending will reach a level of $10,000 per person? In what year will that occur?


Joe Seeley died at age 50 in the fall of 2012.

He was a brave and witty blogger at http://joes-blasts.blogspot.com/ throughout his hospitalization, creating virtual lemonade from the sourdest of lemons. We think his words helped him; I know they helped those who cared for him to cheer him on. They will help the hospital staff care better for patients who come after him. And they will help you learn a little mathematics.

Figure 9.13 appeared in the blog at a hopeful moment in his odyssey. It shows Joe’s white blood cell counts on days following a stem cell transplant. He chose white for the bars, to symbolize white blood cells, and red for the background, for blood in general. I wrote him about it:

> Figure 9.13: Proliferating White Blood Cells

March 18, 2011 6:58 AM
Ethan Bolker said . . .
Exponential growth is good! . . . Will you still have a daily double after the predicted short dip? May I use your data for my quantitative reasoning class at UMass Boston?

March 18, 2011 9:48 AM
Joseph Seeley said . . .
I will not see doubling again, unless something is wrong. Over the next few months, the counts will rise and fall, sometimes for no reason that the doctors can determine.

I hereby authorize the use of my blood count data for any and all educational purposes.

(a) Enter the data in Excel. Reproduce Joe’s chart. Match the formatting (labels, colors, sizes, fonts) as well as you can.

(b) Create an exponential trendline for the data.

(c) Use common sense or your trendline to predict when Joe’s white blood cell count will be in normal range.

(d) His white blood count on March 17 was 5100. Does that match your prediction?

(e) Modify your chart to include this new information. Mark it with a suitable exclamation!

Exercise 9.6.29. [U][Goal 9.3] MIT grad led team that built faster YouTube player

The Boston Globe reported on September 24, 2012 that

[Andy Berkheimer] found that viewers start closing out if there’s even a two-second delay. Every one-second delay after that results in a 5.8 percent increase in the number of people who give up. A 40-second wait costs a video nearly a third of its audience. 19

Show that at this rate more than ninety percent of the viewers would give up after 40 seconds – not the “nearly a third” in the quote.

Exercise 9.6.30. [U][Goal 9.1] Even Nate Silver makes mistakes

On page 32 of his otherwise excellent and highly recommended The Signal and the Noise Nate Silver writes

[Over the] 100-year-period from 1896 through 1996 . . . sale prices of houses had increased by just 6 percent total after inflation, or about 0.06 percent annually.

Explain Silver’s mistake.

[See the back of the book for a hint.]

Exercise 9.6.31. [U] Cuba, you owe us $7 billion

On April 18, 2014 Leon Neyfakh wrote in The Boston Globe that property confiscated by the Cuban government in the 1959 revolution was

... originally valued at $1.8 billion, which at 6 percent simple interest translates to nearly $7 billion today. 20

(a) Is the simple interest calculation in the quotation correct?

(b) What would the value be today at 6 percent compound interest?
(c) What would the value be today simply taking inflation into account?
(d) Discuss which of the three valuations makes the most sense?

Exercise 9.6.32. [U] As Time Goes By

On December 13, 2012 you could read in *The New York Times* that the piano from Rick’s place in the 1942 movie Casablanca is up for auction.

Sotheby’s expects it to sell from $800,000 to $1.2 million in the auction on Friday. That is between 34 to 48 times what [Ingrid]Bergman was paid for sharing top billing with Humphrey Bogart.

(a) How much was Ingrid Bergman paid for her role in the film?
(b) Would adjusting her pay to take inflation into account allow her to bid on the piano in 2012?
(c) What compound interest rate would she have to have earned on her pay to bid on the piano in 2012?

Review exercises

Sample routine review questions. When *Common Sense Mathematics* is published these and others may be available in an online homework system/

Exercise 9.6.33. [A] You invest $500 in an account that earns $10 in interest each year.

(a) At the end of 24 months, what is the balance?
(b) At the end of 30 months, what is the balance?
(c) At the end of 5 years, what is the balance?
(d) Find the linear equation that gives the balance after $t$ years.

Exercise 9.6.34. [A] You buy a car for $15,000 and for tax purposes you depreciate the car at a rate of 11% per year.

(a) At the end of 24 months, what is the value of the car?
(b) At the end of 5 years, what is the value of the car?
(c) Find the exponential equation that gives the value of the car after $t$ years.
(d) Does the value of the car ever reach $0$?

Exercise 9.6.35. [A] For each of the following, calculate the percentage.

(a) What is 8% of $2000?

(b) What is 108% of $2000?
(c) What is 3.25% of $800?
(d) What is 103.25% of $800?

Exercise 9.6.36. [A] Use a calculator to evaluate these expressions using exponents. (You may find typing into the Google or Bing calculator much faster than using one that requires you to press keys, either with your fingers or with a mouse.

(a) 1.03^4
(b) 0.89^5
(c) 140 \times 1.03^4
(d) 80 \times 0.89^5
(e) \frac{1}{3^\pi}
(f) (\frac{1}{3})^8
(g) 1.25^0
(h) 1.25^1
(i) e^2
(j) e^{15}

Exercise 9.6.37. [A] In the exponential functions below, identify the relative change and the initial amount.

(a) \ P = 100 \times (1.05)^T
(b) \ y = 400 \times (0.88)^x
(c) \ S = 550 \times (1.22)^Q
(d) \ P = 96 \times (0.50)^T

Exercise 9.6.38. [A] Excel gives the following best-fit exponential function for a set of data: \ y = 2.099 \times e^{1.344x}. Find the constant growth rate and re-write the function without using e.