1. Let $S = \{ f : \mathbb{N} \to \mathbb{N} | f(n) \text{ is divisible by } n, \text{ for all } n \}$. Use diagonalization to show that $S$ is uncountable.

**Solution:** Suppose that $S$ is countable. Since $S$ is clearly infinite, this means that $S$ has the same size as $\mathbb{N}$, so all the elements of $S$ can be listed as $f_1, f_2, f_3, \ldots$. Define a function $d : \mathbb{N} \to \mathbb{N}$ by $d(n) = f_n(n) + n$ for all $n \in \mathbb{N}$. Then $d$ is in $S$ because for all $n$, $f_n(n)$ is divisible by $n$, so $f_n(n) + n$ is divisible by $n$, and $d$ is different from all the $f_n$’s because $d(n)$ and $f_n(n)$ are different numbers. Thus, the list $f_1, f_2, \ldots$ does not list all elements of $S$. This contradiction shows that $S$ is uncountable.

2. $HALT_{TM}$ is defined as $\{ \langle M, w \rangle | M \text{ is a Turing machine, and } M \text{ halts on } w \}$. Prove that $HALT_{TM}$ is Turing recognizable.

**Solution:** $HALT_{TM}$ is recognized by the following Turing machine $V$.

$V =$ “On input $\langle M, w \rangle$ where $M$ is a Turing machine and $w$ is an input string

1. Simulate $M$ on $w$.
2. If $M$ halts, accept.”

3. In class we showed that $HALT_{TM}$ is not decidable by reducing $A_{TM}$ to $HALT_{TM}$. For this problem, you are asked to show that $HALT_{TM}$ is undecidable by using diagonalization instead of using a reduction. Your proof should be similar to but not the same as the proof that $A_{TM}$ is not decidable.

**Solution:** We assume that $HALT_{TM}$ is decidable and obtain a contradiction. Suppose that the Turing machine $H$ decides $HALT_{TM}$. This means that

$$ H(\langle M, w \rangle) = \begin{cases} 
\text{accept} & \text{if } M \text{ accepts } w \\
\text{accept} & \text{if } M \text{ rejects } w \\
\text{reject} & \text{if } M \text{ loops on } w 
\end{cases} $$

Using $H$, we define another Turing machine $D$

$D =$ “On input $\langle M \rangle$ where $M$ is a Turing machine

1. Run $H$ on $\langle M, \langle M \rangle \rangle$.
2. If $H$ accepts, go into an infinite loop. If $H$ rejects, accept.”

We have

$$ D(\langle M \rangle) = \begin{cases} 
\text{loop} & \text{if } M \text{ accepts } \langle M \rangle \\
\text{loop} & \text{if } M \text{ rejects } \langle M \rangle \\
\text{accept} & \text{if } M \text{ loops on } \langle M \rangle 
\end{cases} $$
Applying this to $M = D$, we get

$$D(\langle D \rangle) = \begin{cases} 
\text{loop} & \text{if } D \text{ accepts } \langle D \rangle \\
\text{loop} & \text{if } D \text{ rejects } \langle D \rangle \\
\text{accept} & \text{if } D \text{ loops on } \langle D \rangle 
\end{cases}$$

No matter what $D$ does on $\langle D \rangle$, we get a contradiction, so $D$ can’t exist, which means that $H$ can’t exist and $HALT_{TM}$ is undecidable.