Overview

Regular LT properties

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  - ω-automata, i.e., acceptors for infinite words
  - ω-regular expressions

Regular expressions

remind: syntax and semantics of regular expressions over some alphabet $\Sigma = \{A, B, \ldots\}$

Regular expressions over Σ

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$$\mathcal{L}(\emptyset) = \varnothing \qquad \mathcal{L}(\epsilon) = \{\varepsilon\} \qquad \mathcal{L}(A) = \{A\}$$

$$\mathcal{L}(\alpha_1 + \alpha_2) = \mathcal{L}(\alpha_1) \cup \mathcal{L}(\alpha_2) \quad \text{union}$$

$$\mathcal{L}(\alpha_1 . \alpha_2) = \mathcal{L}(\alpha_1)\mathcal{L}(\alpha_2) \quad \text{concatenation}$$

$$\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^* \quad \text{Kleene closure}$$

ω-regular expressions

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regular expressions + ω-operator α^ω

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for $L \subseteq \Sigma^*$:

$$L^\omega \stackrel{\text{def}}{=} \{ w_1 w_2 w_3 \ldots : w_i \in L \text{ for all } i \geq 1 \}$$

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note: $L^\omega \subseteq \Sigma^\omega$ if $\varepsilon \notin L$

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ω-regular languages

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A language $L \subseteq \Sigma^\omega$ is called ω-regular iff there exists an ω-regular expression γ s.t.

$$L = \mathcal{L}_\omega(\gamma)$$

Provide an ω-regular expression for ...

alphabet $\Sigma = \{A, B\}$

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$$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha.\alpha^*$.

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$$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega \equiv (A^*.B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha.\alpha^*$.

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Let Φ be an invariant condition and let

$$\{A \subseteq AP : A \models \Phi\} = \{A_1, \ldots, A_k\}$$

Then: invariant "always Φ" $\widehat{=}$ $(A_1 + \ldots + A_k)^\omega$

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  Indeed: each invariant is ω-regular

- "infinitely often a"

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  $$\big((\emptyset + \{b\})^*.(\{a\} + \{a, b\})\big)^\omega$$

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$$(2^{AP})^*.(\{a\} + \{a, b\}).(2^{AP})^\omega$$

where $2^{AP} \mathrel{\widehat{=}} \emptyset + \{a\} + \{b\} + \{a, b\}$

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Symbolic notation

symbolic notation for ω-regular properties

... using formulas instead of sums ....

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Nondeterministic Büchi automata (NBA)

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syntax as for **NFA**

↑

nondeterministic finite automata

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semantics: language of infinite words

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run for a word $A_0 A_1 A_2 \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0 q_1 q_2 \ldots$ where $q_0 \in Q_0$ and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

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run π is accepting if $\overset{\infty}{\exists}\, i \in \mathbb{N}.\ q_i \in F$

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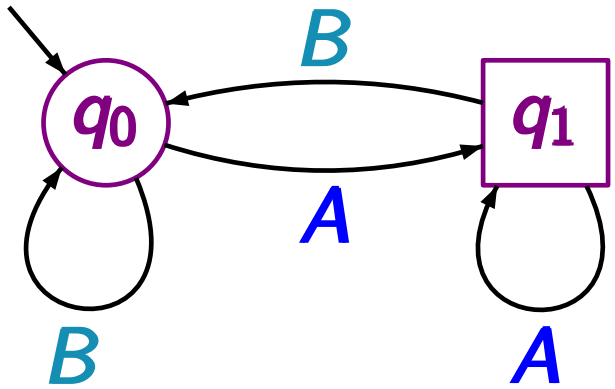
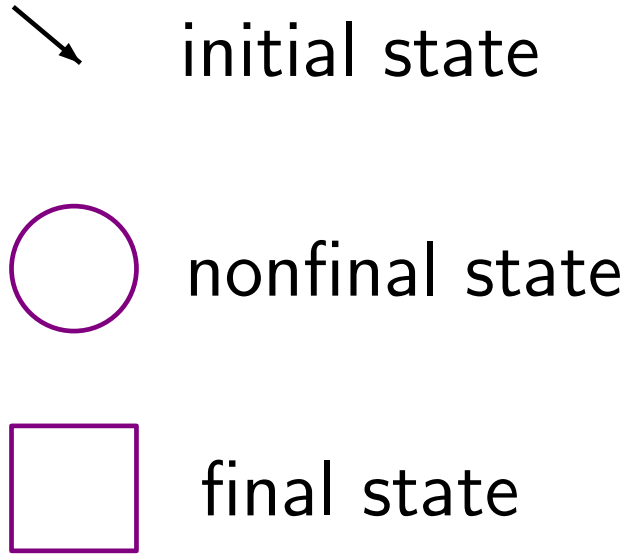
accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

$$\mathcal{L}_\omega(\mathcal{A}) \stackrel{\text{def}}{=} \text{set of infinite words over } \Sigma \text{ that have an accepting run in } \mathcal{A}$$

Notations in pictures for NBA





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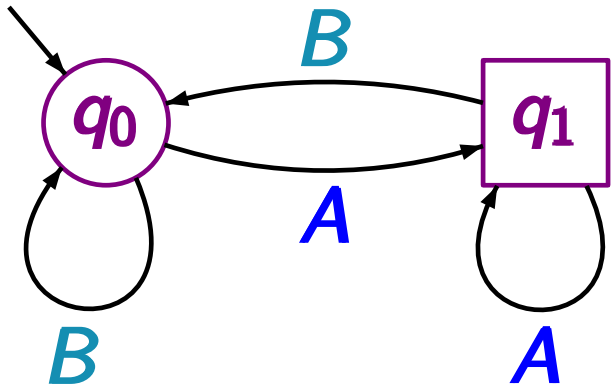
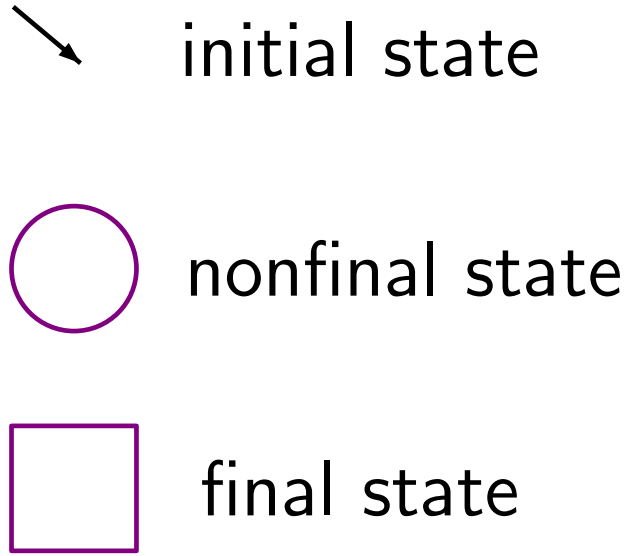




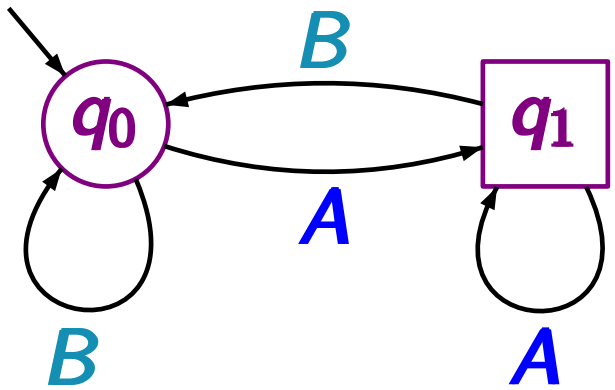
NBA with state space $\{q_0, q_1\}$

q_0 initial state

q_1 accept state

alphabet $\Sigma = \{A, B\}$

Examples for NBA over $\Sigma = \{A, B\}$



accepted language: **?**

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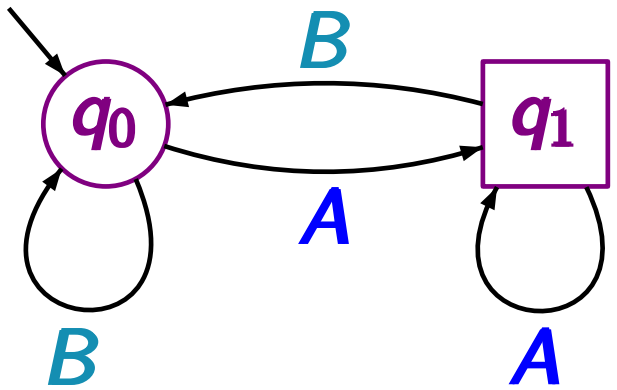
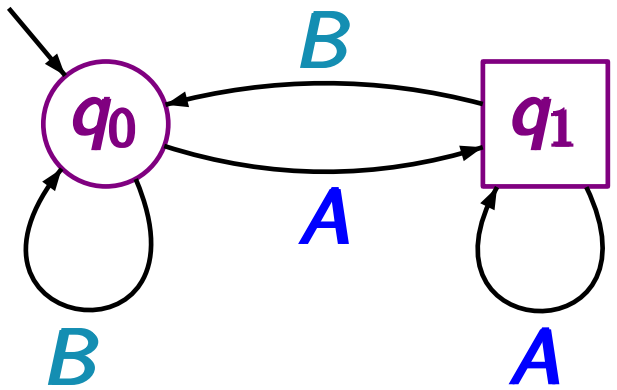


accepted language:

set of all infinite words that contain infinitely many A's

Examples for NBA over $\Sigma = \{A, B\}$



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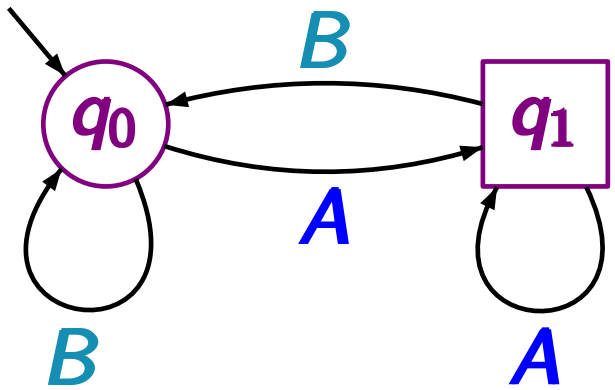

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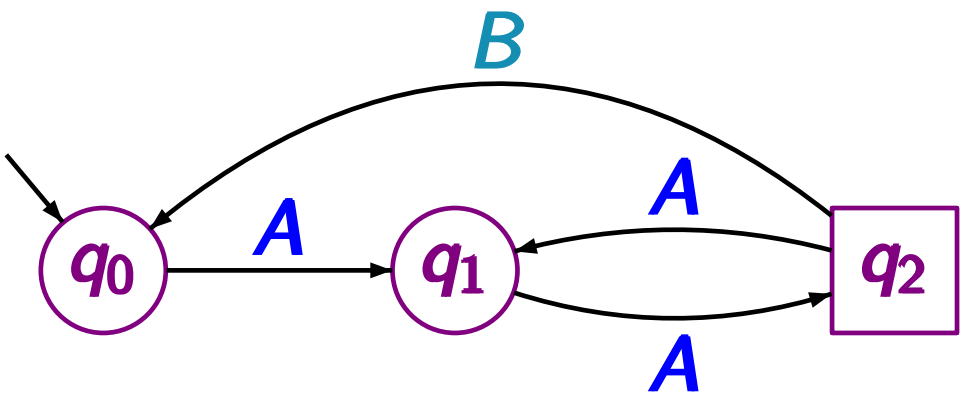

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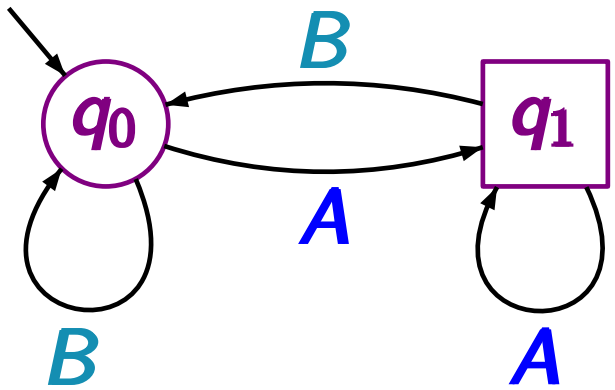

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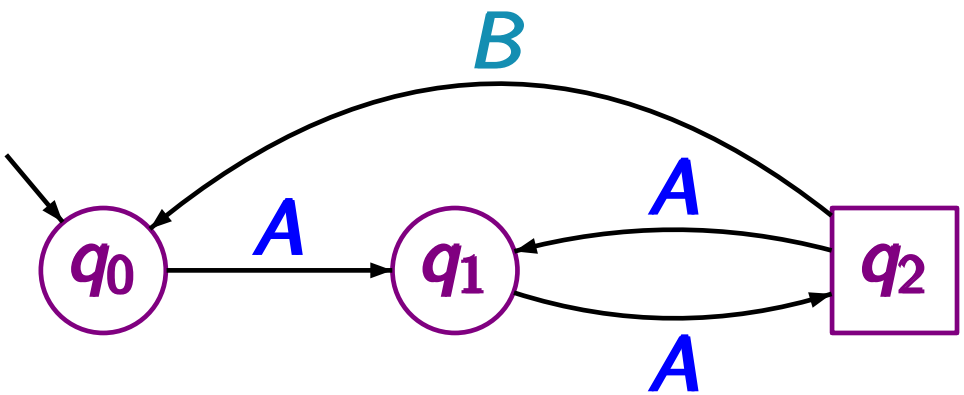

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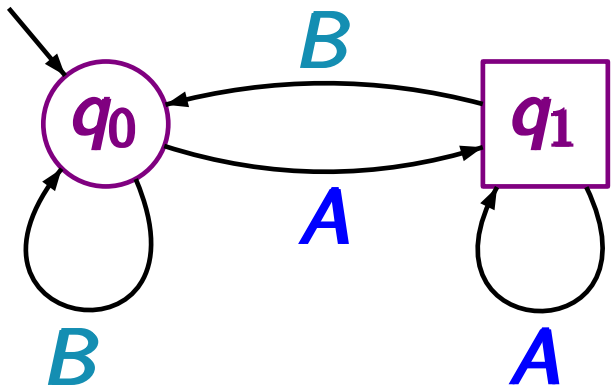

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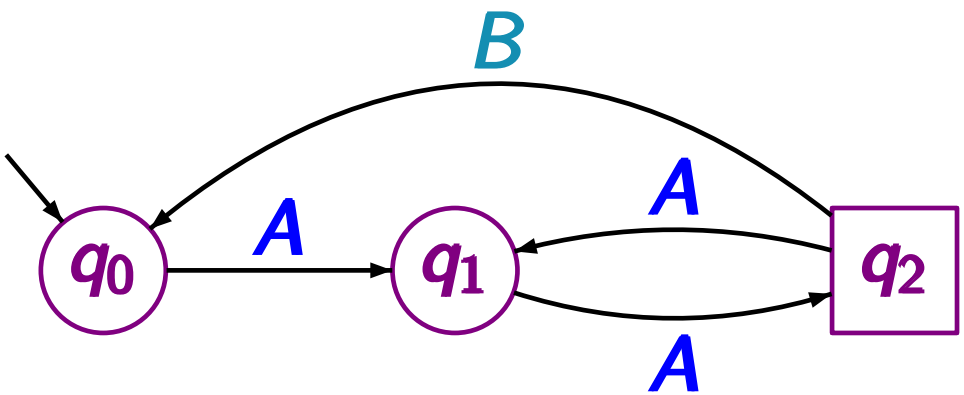

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"every B is preceded by a positive even number of A's"

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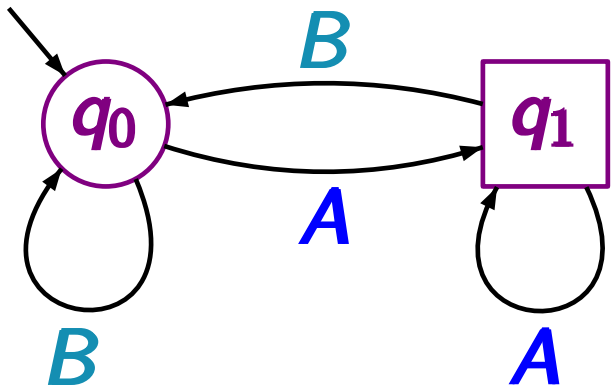

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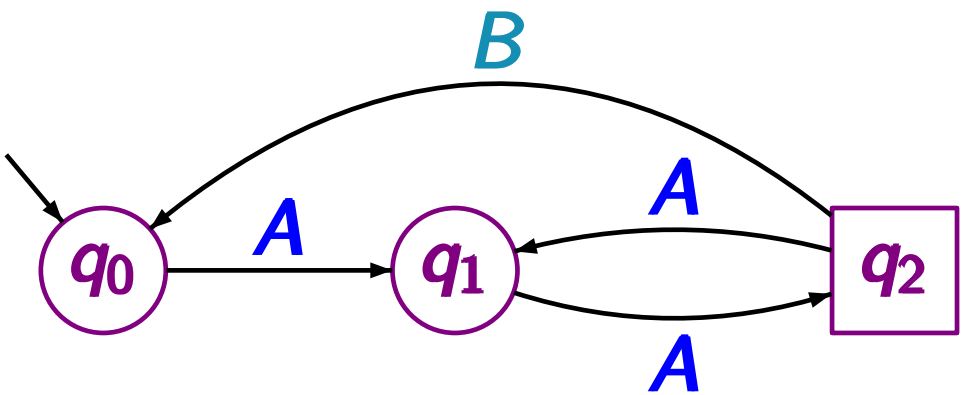

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$$((A.A)^+.B)^\omega + ((A.A)^+.B)^*.A^\omega$$

NBA for LT properties

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NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

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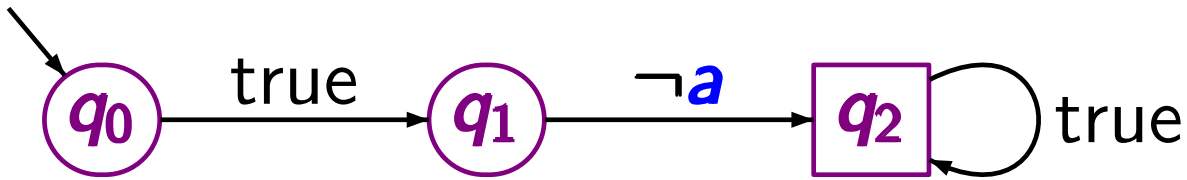
accepted language $\mathcal{L}_\omega(\mathcal{A})$ is an LT-property:

$\mathcal{L}_\omega(\mathcal{A})$ = set of infinite words over 2^{AP} that have an accepting run in $\mathcal{A}$

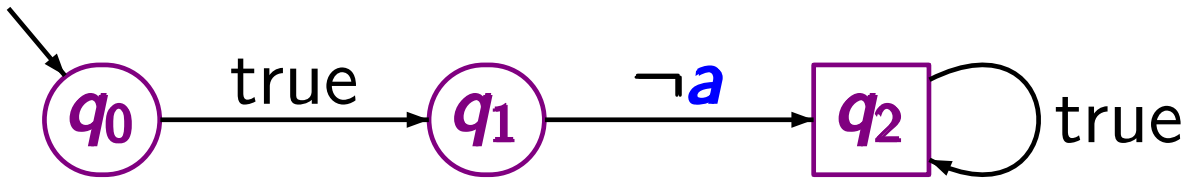
NBA for LT properties



$$\mathcal{L}_\omega(\mathcal{A}) \;=\; ?$$

set of atomic propositions $AP = \{a, b\}$

NBA for LT properties



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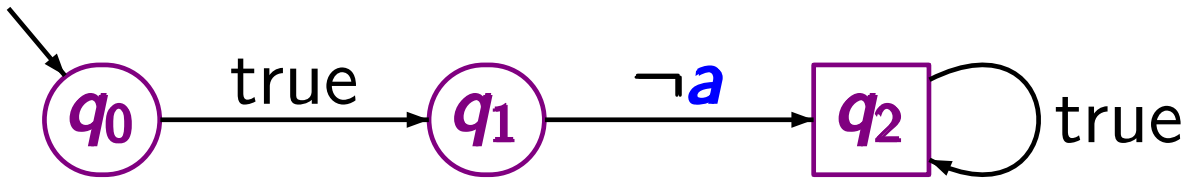

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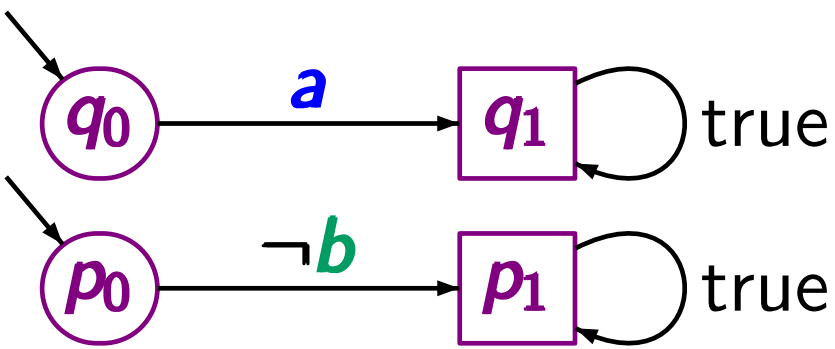

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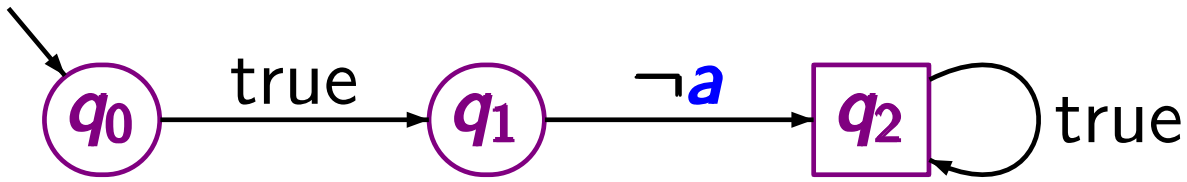

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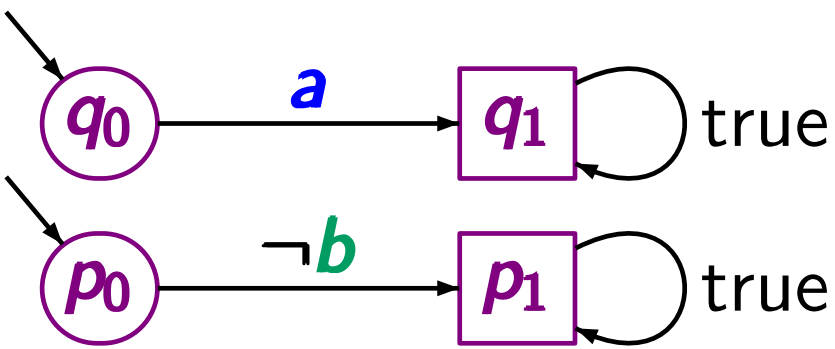

$$(a \vee \neg b).\text{true}^\omega$$

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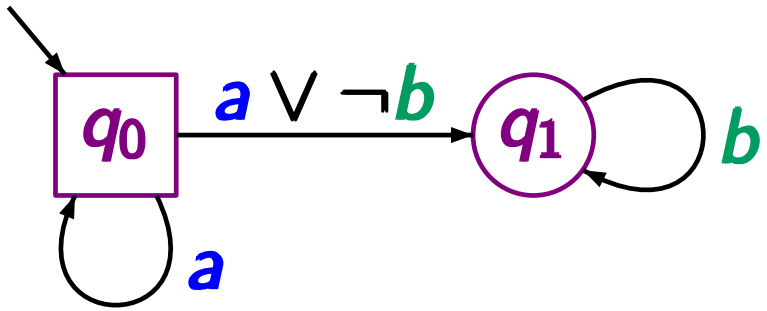

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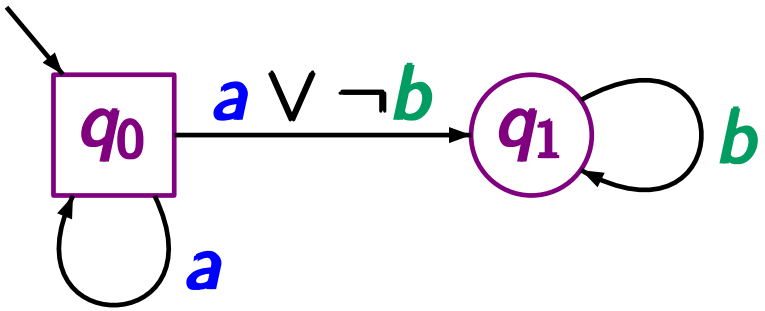

"always a" $\widehat{=}$ a^ω

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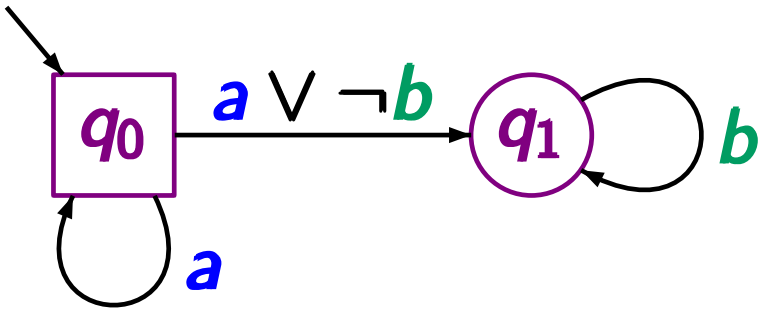

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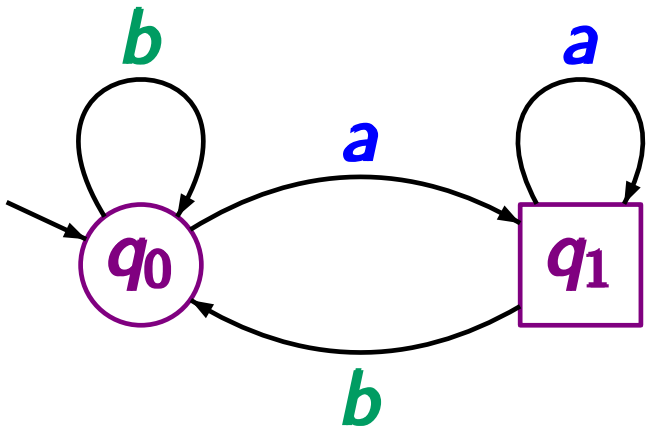

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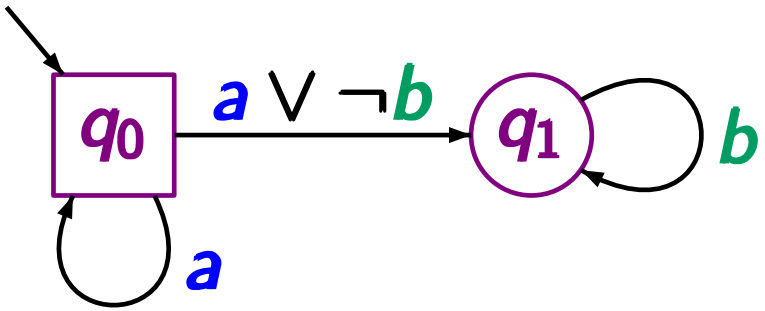

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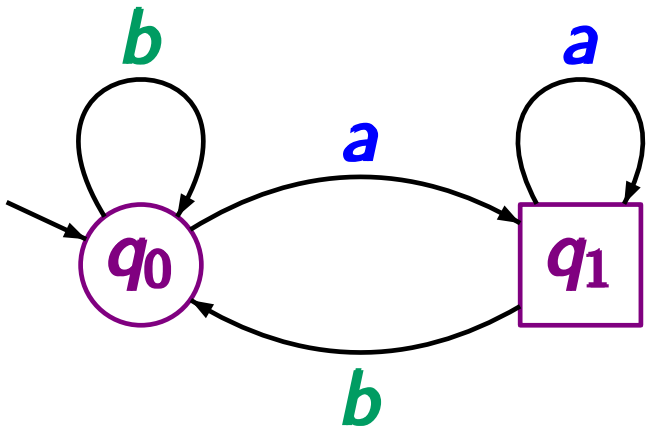

"infinitely often a and ..."

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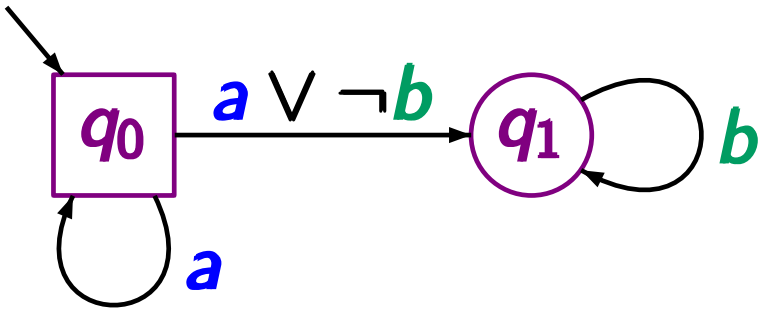

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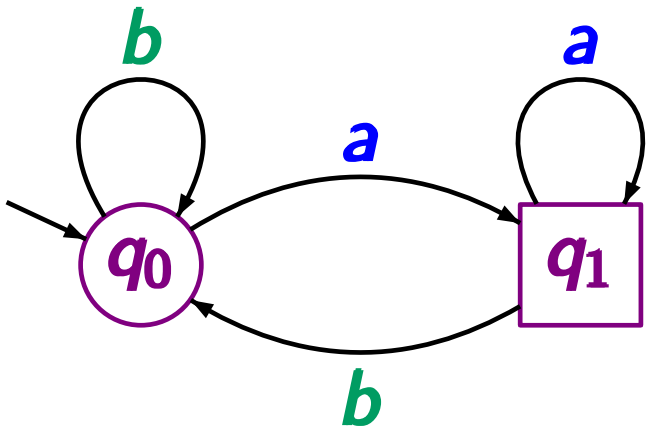

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$$\widehat{=} \left((a \vee b)^*.a \right)^\omega$$

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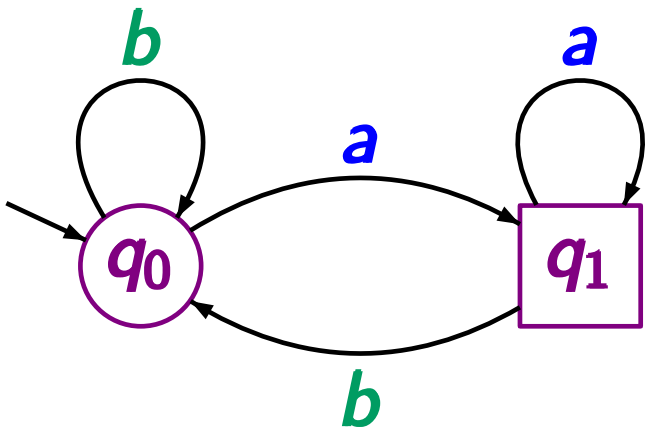

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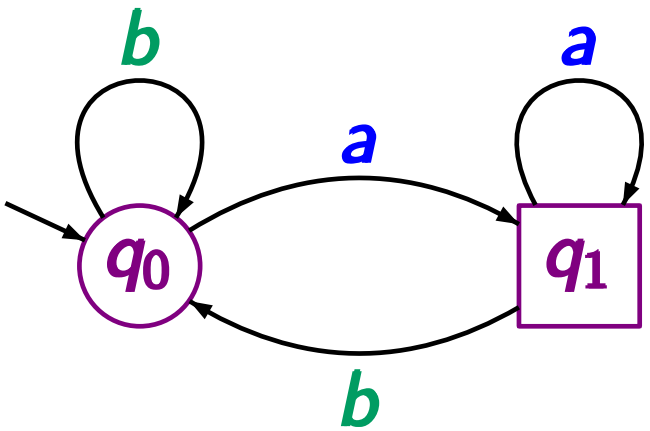

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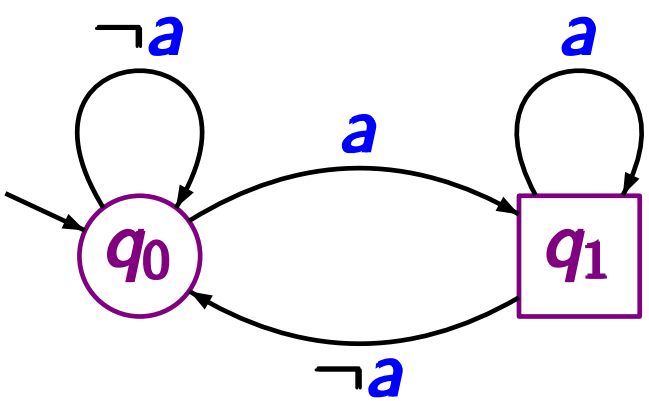

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From NBA to ω-regular expressions

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For each NBA $\mathcal{A}$ there is an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

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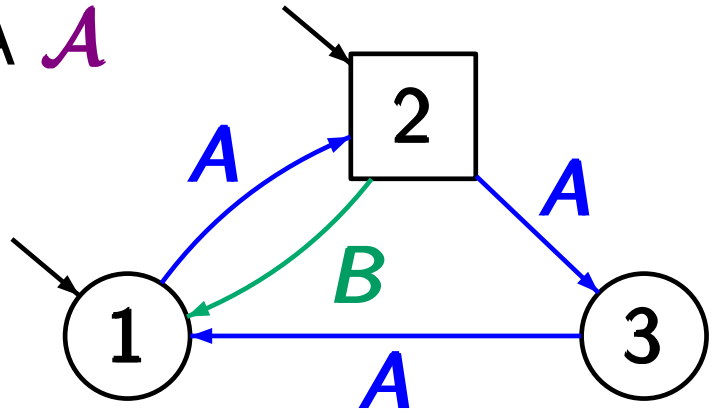
is ω-regular as $\mathcal{L}(\mathcal{A}_{q,p})$ and $\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\}$ are regular

Example: NBA $\rightsquigarrow$ ω-regular expression

NBA $\mathcal{A}$



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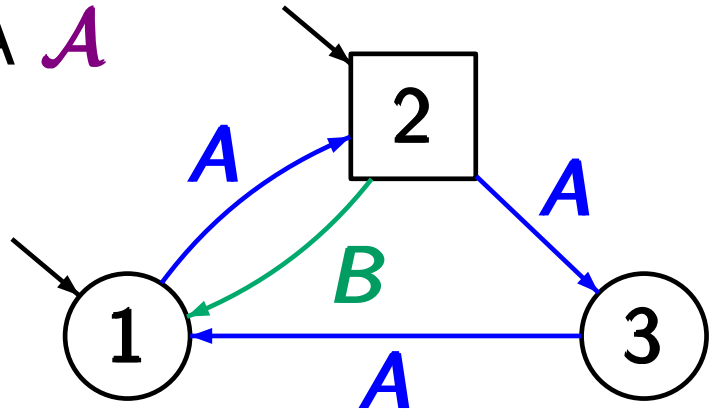

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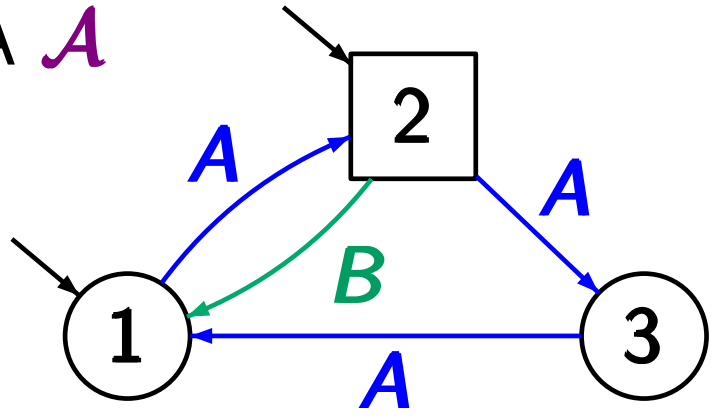

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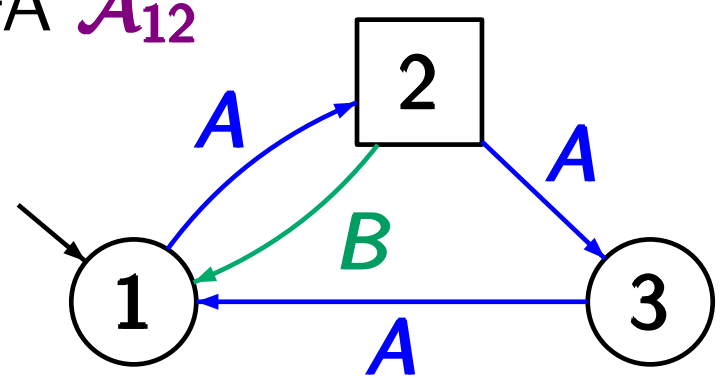
NFA $\mathcal{A}_{12}$



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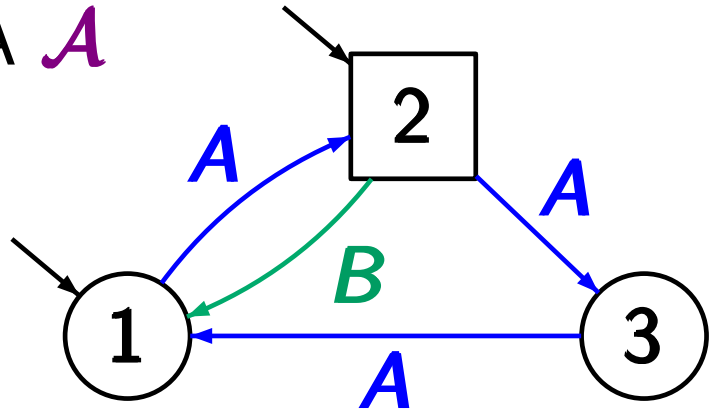

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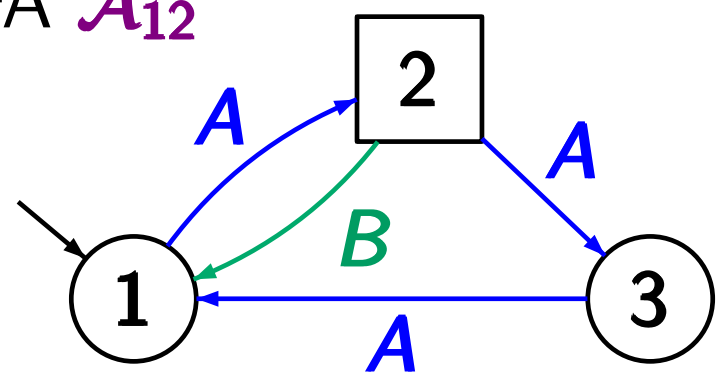

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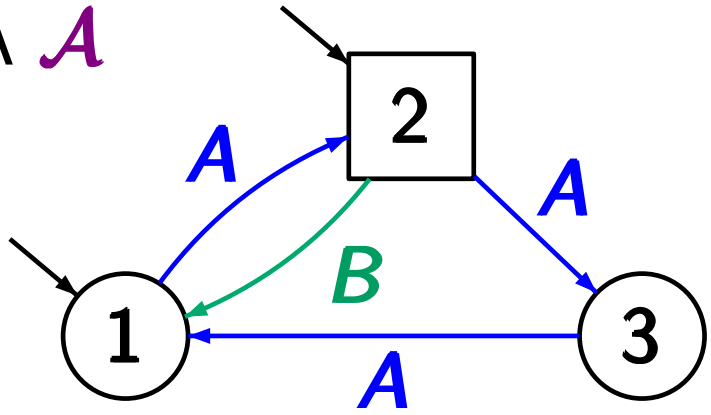

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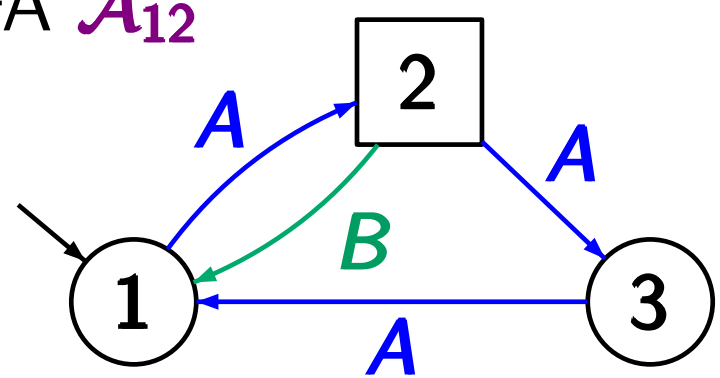

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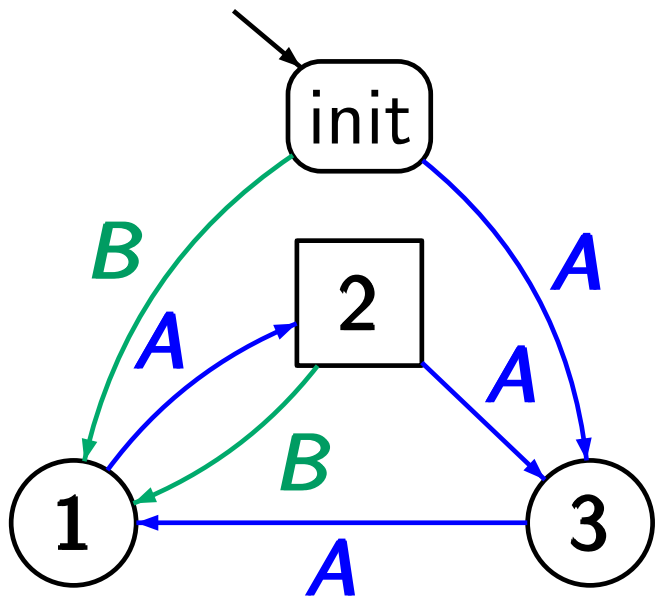

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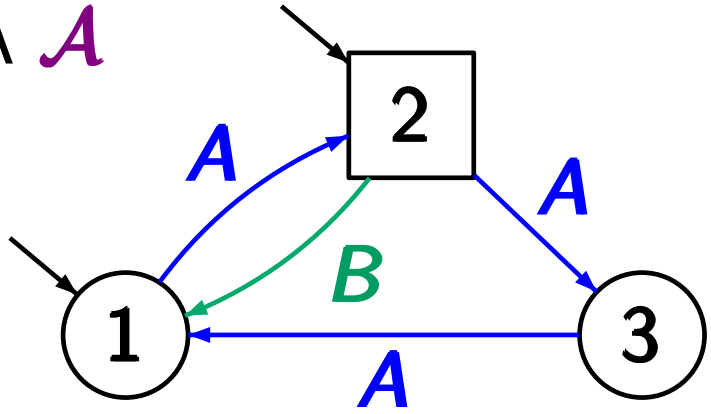

language of $\mathcal{A}$:

$$A.(B.A + A.A.A)^{\omega} + (B.A + A.A.A)^{\omega}$$

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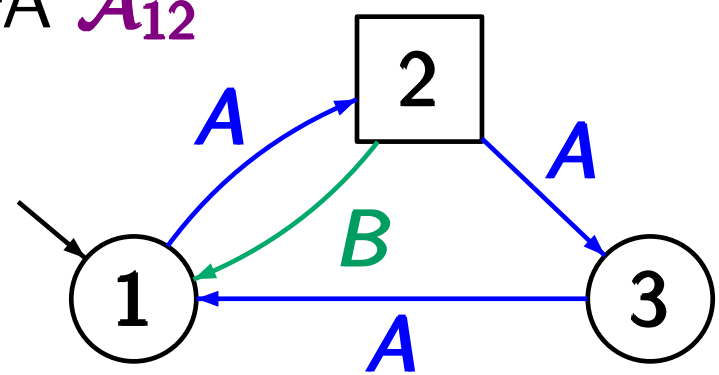

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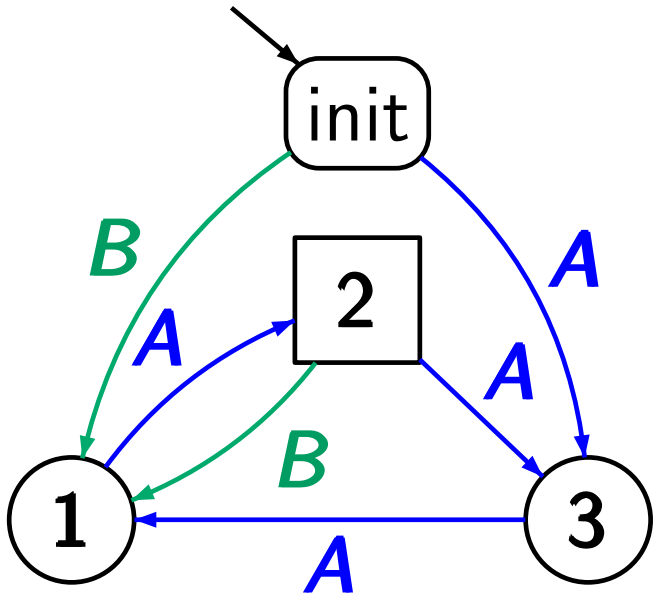

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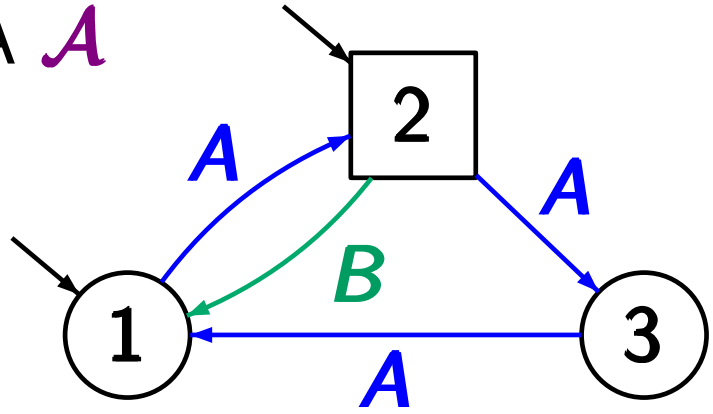

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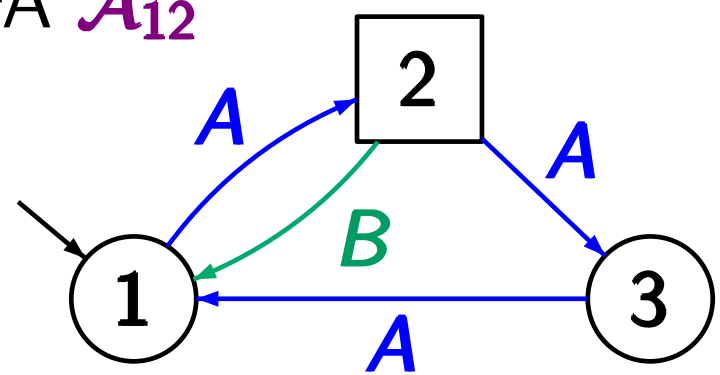
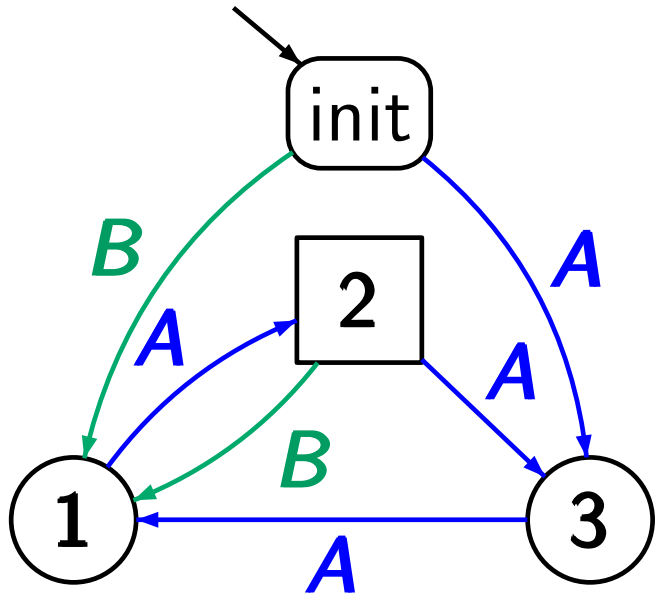
$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^*$

NFA $\mathcal{A}_{12}$



$L'_{22} \mathrel{\widehat{=}} (B.A + A.A.A)^+$



From ω-regular expressions to NBA

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For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an **NBA** $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

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NBA are closed under union

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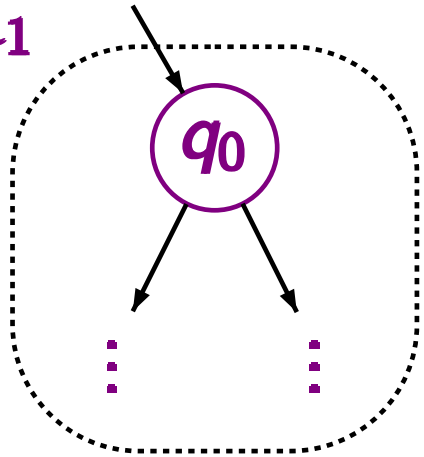
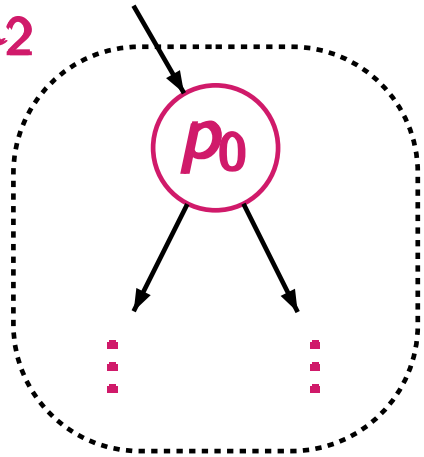
NBA $\mathcal{A}_1$



NBA $\mathcal{A}_2$



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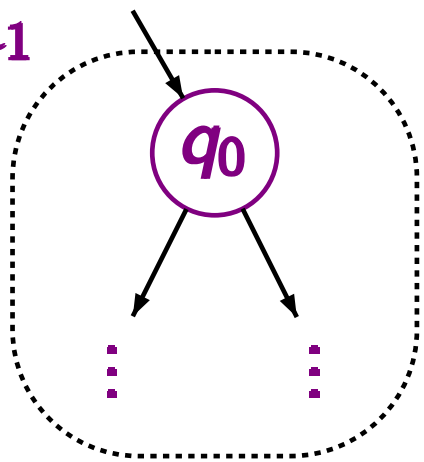
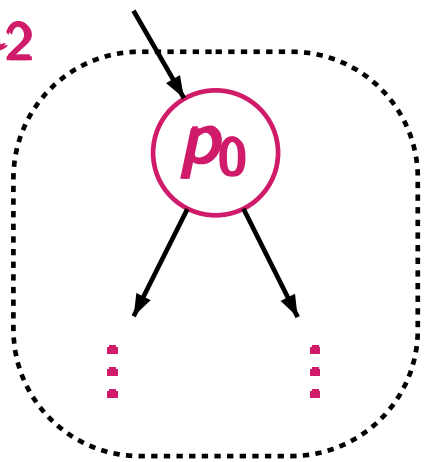
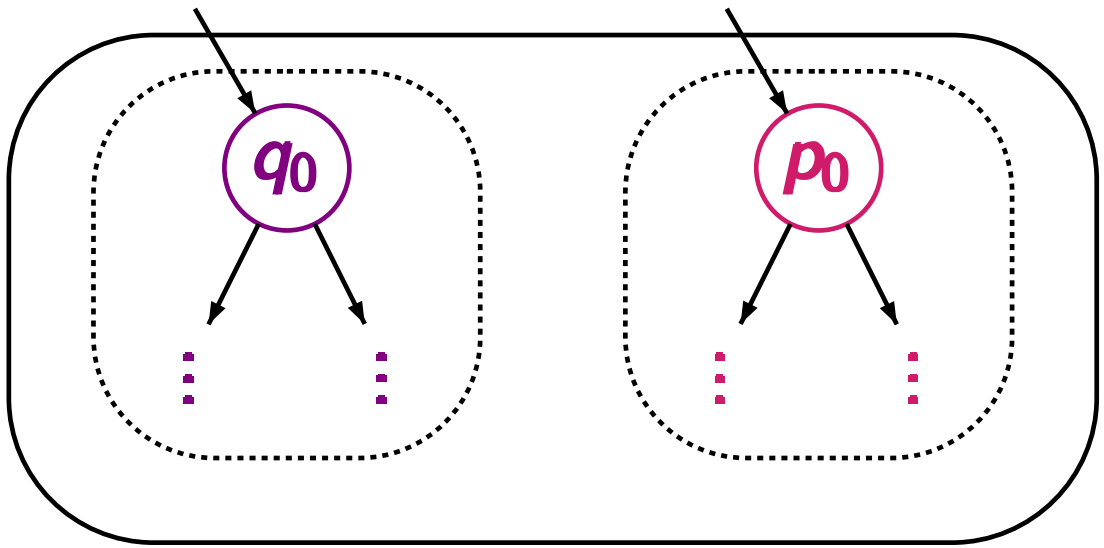


NBA $\mathcal{A}_2$



NBA for $\mathcal{L}_\omega(\mathcal{A}_1) \cup \mathcal{L}_\omega(\mathcal{A}_2)$



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Concatenation of an NFA and an NBA

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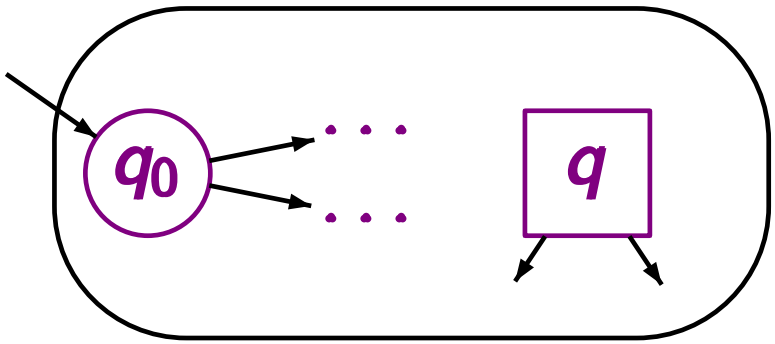
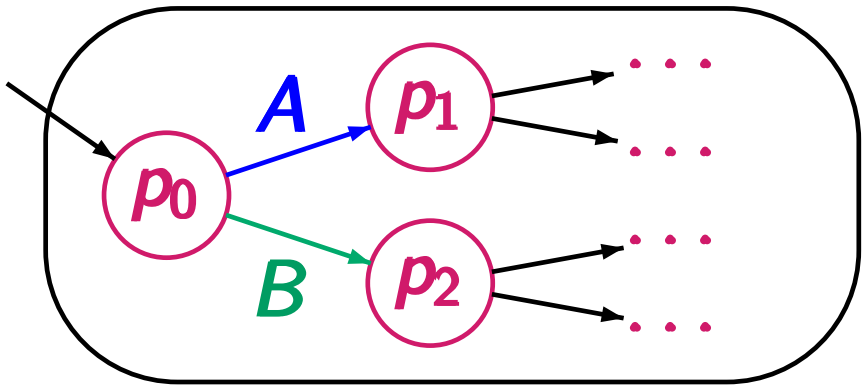
NFA $\mathcal{A}_1$



NBA $\mathcal{A}_2$



Concatenation of an NFA and an NBA

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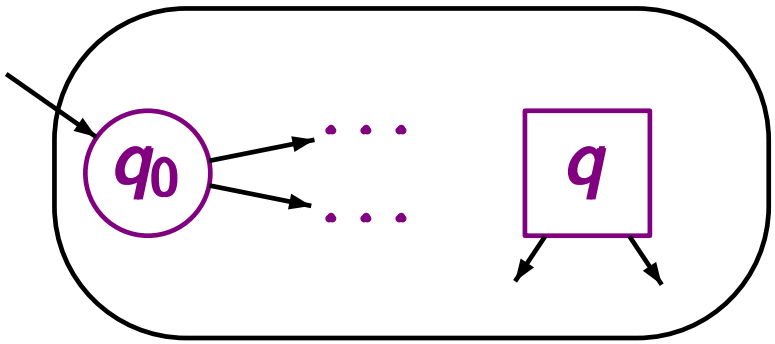
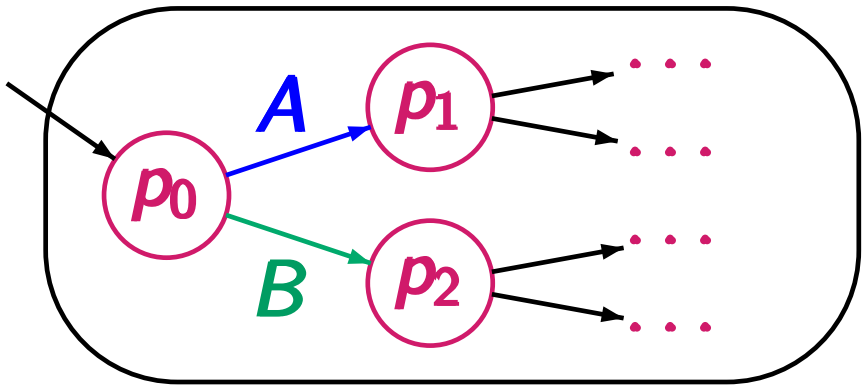
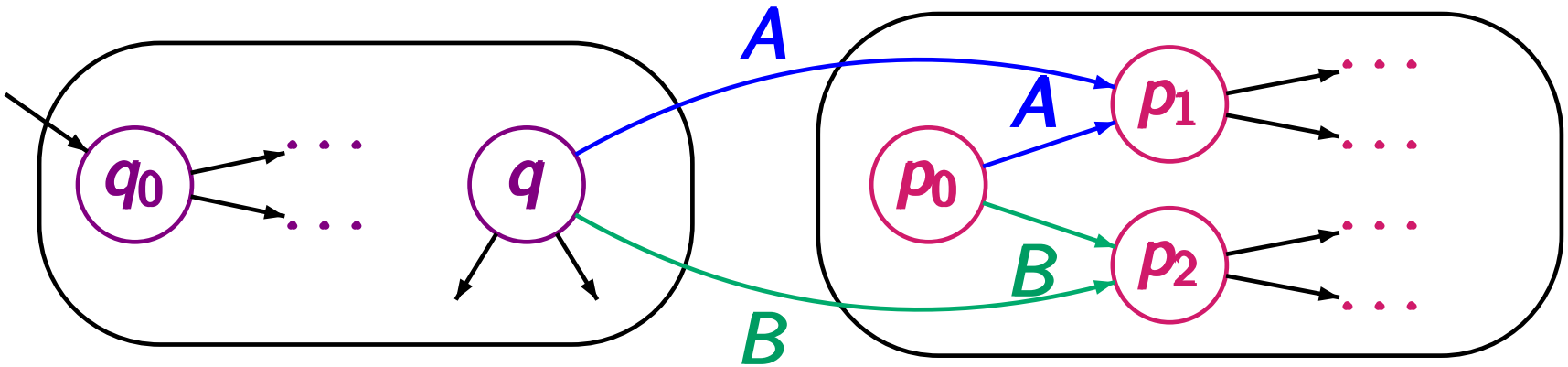


NBA $\mathcal{A}_2$



NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:



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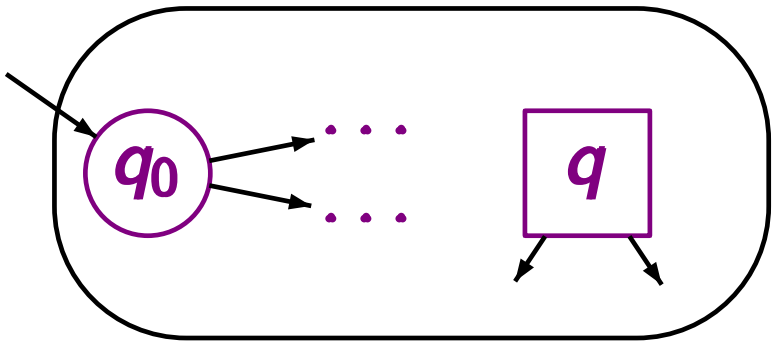
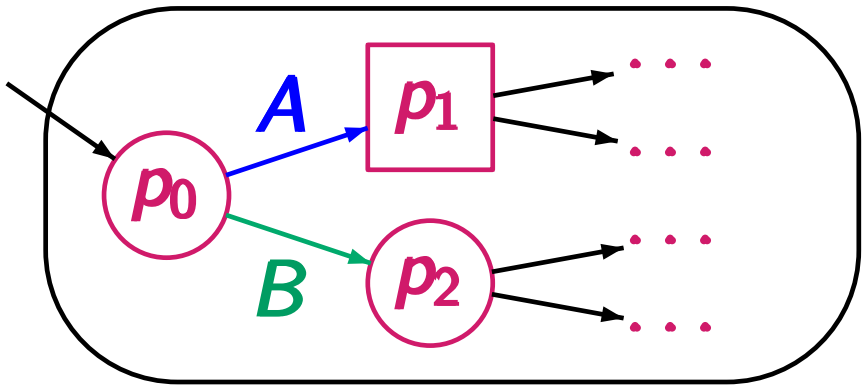
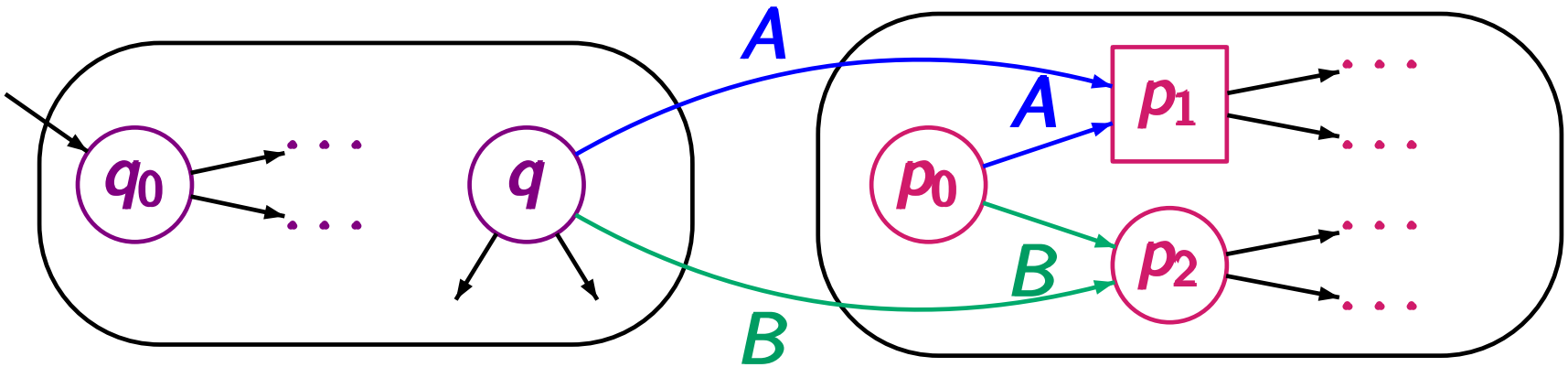


NBA $\mathcal{A}_2$



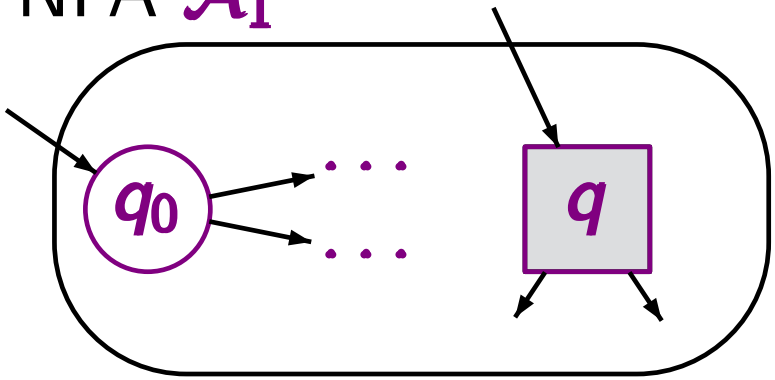
NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:



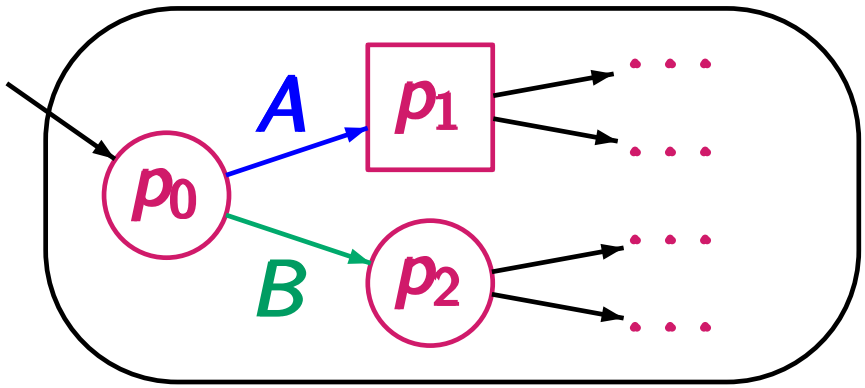
accept states as in $\mathcal{A}_2$

Concatenation of an NFA and an NBA

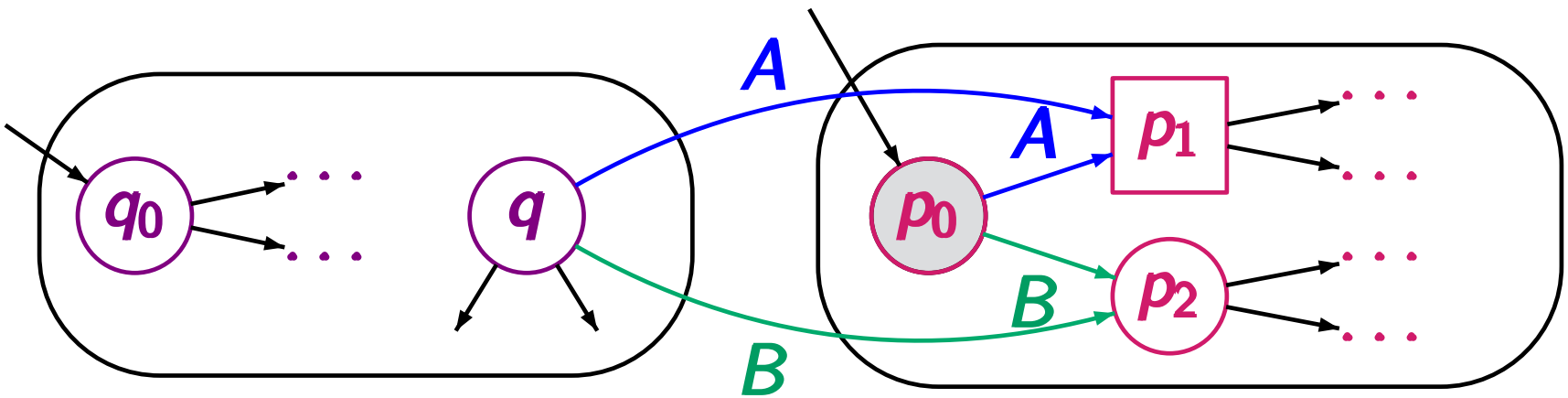
NFA $\mathcal{A}_1$



NBA $\mathcal{A}_2$



NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:



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From ω-regular expressions to NBA

For each ω-regular expression

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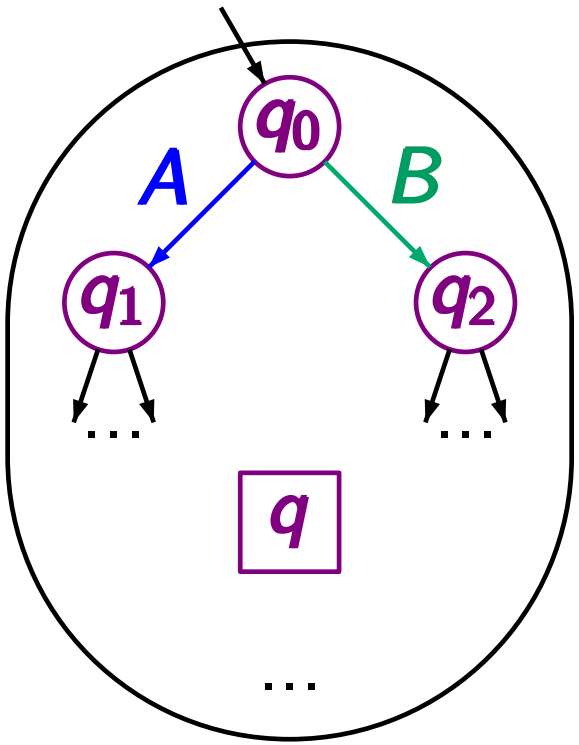
ω-operator for NFA

ω-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$

$\rightsquigarrow$

NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$



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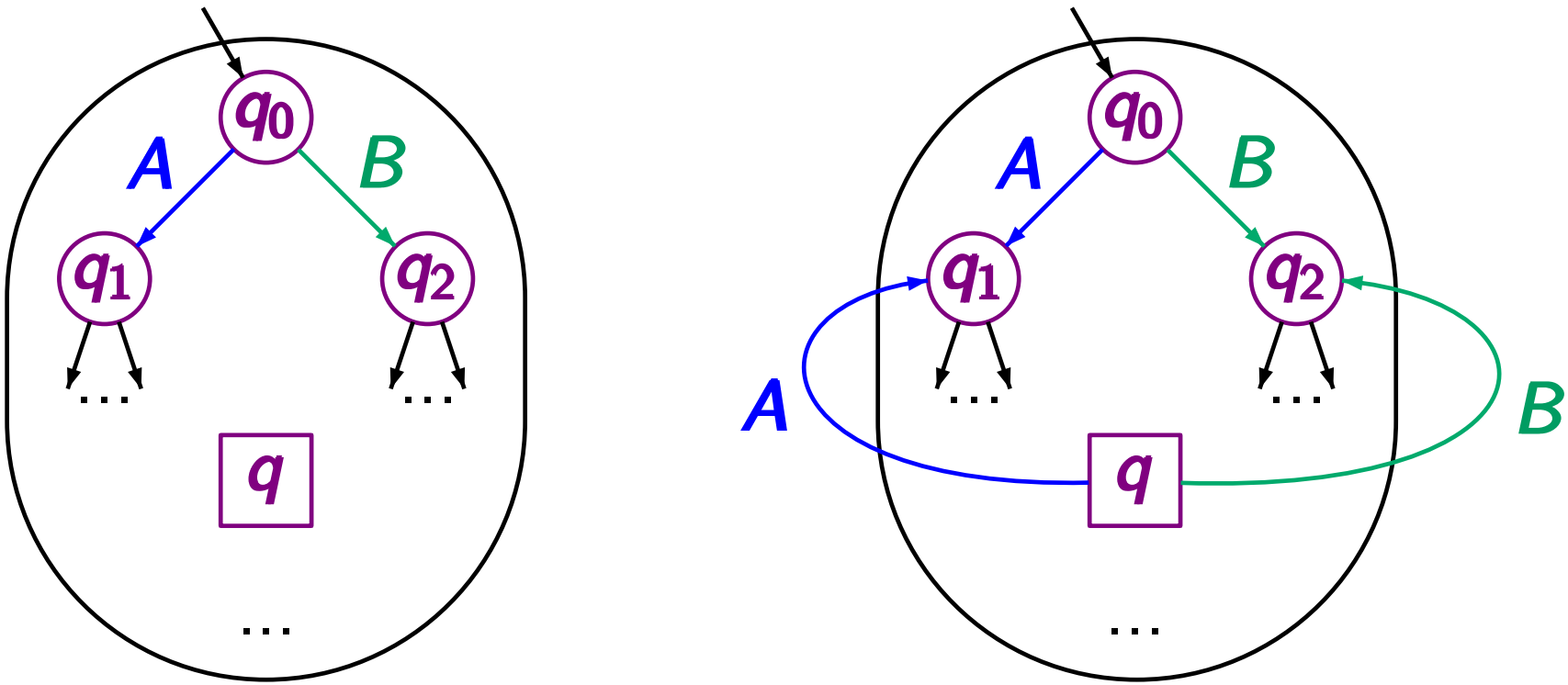
NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$



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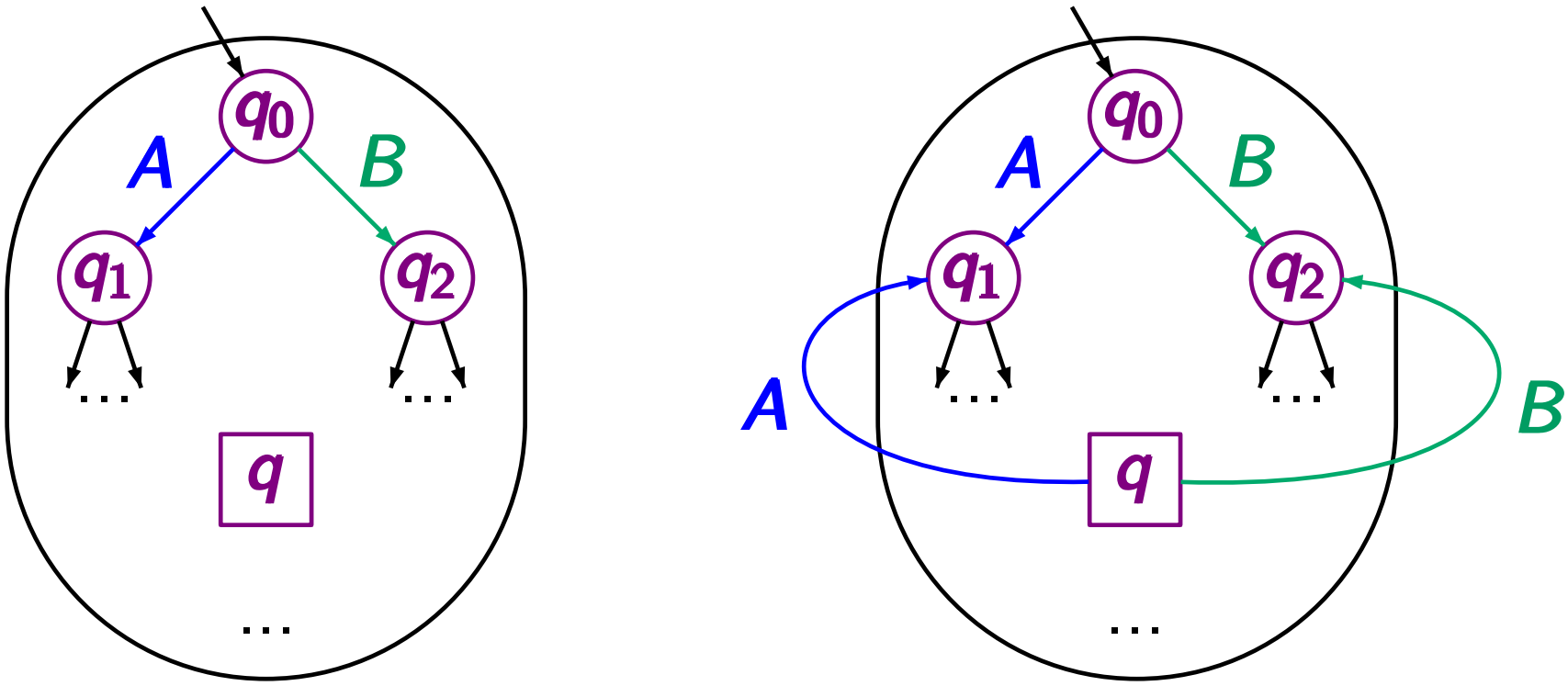

wrong !

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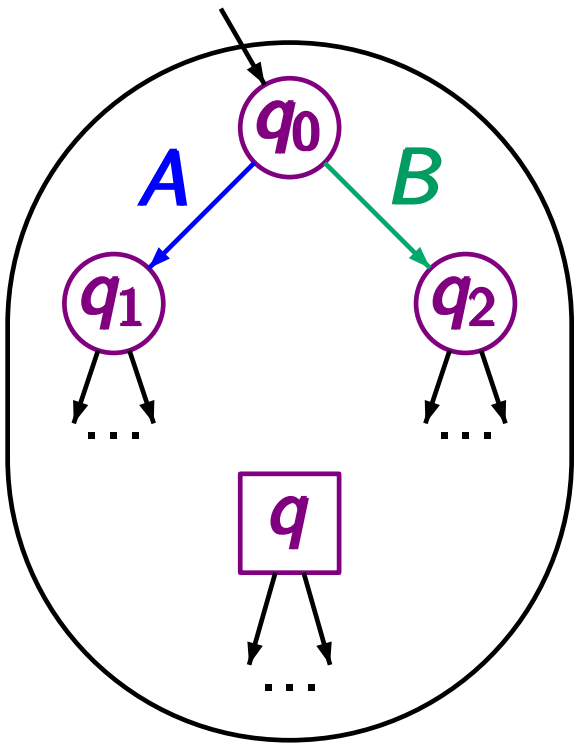
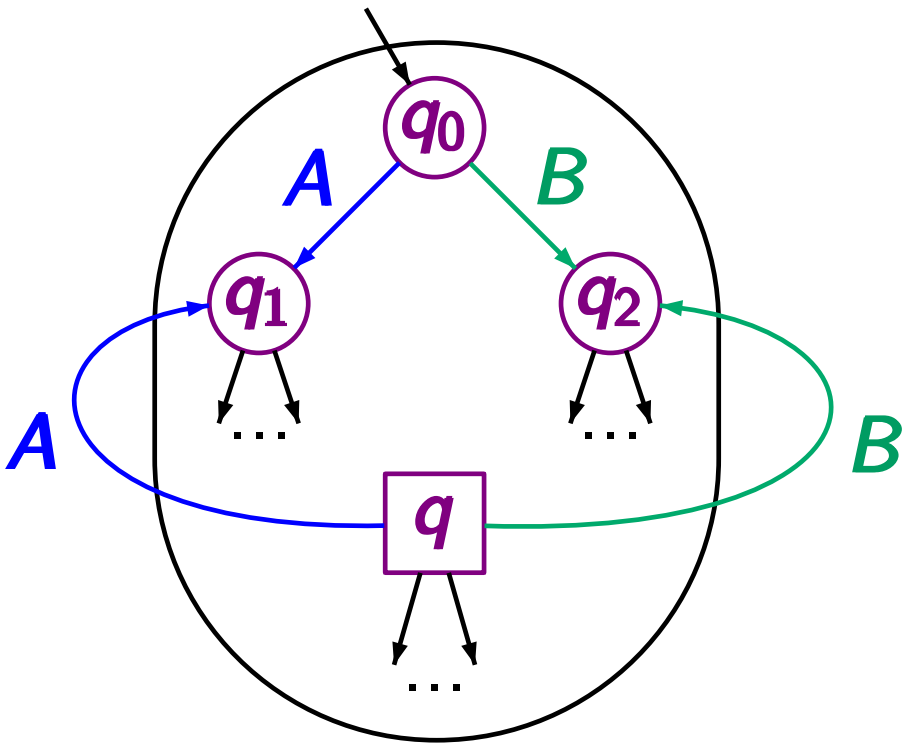
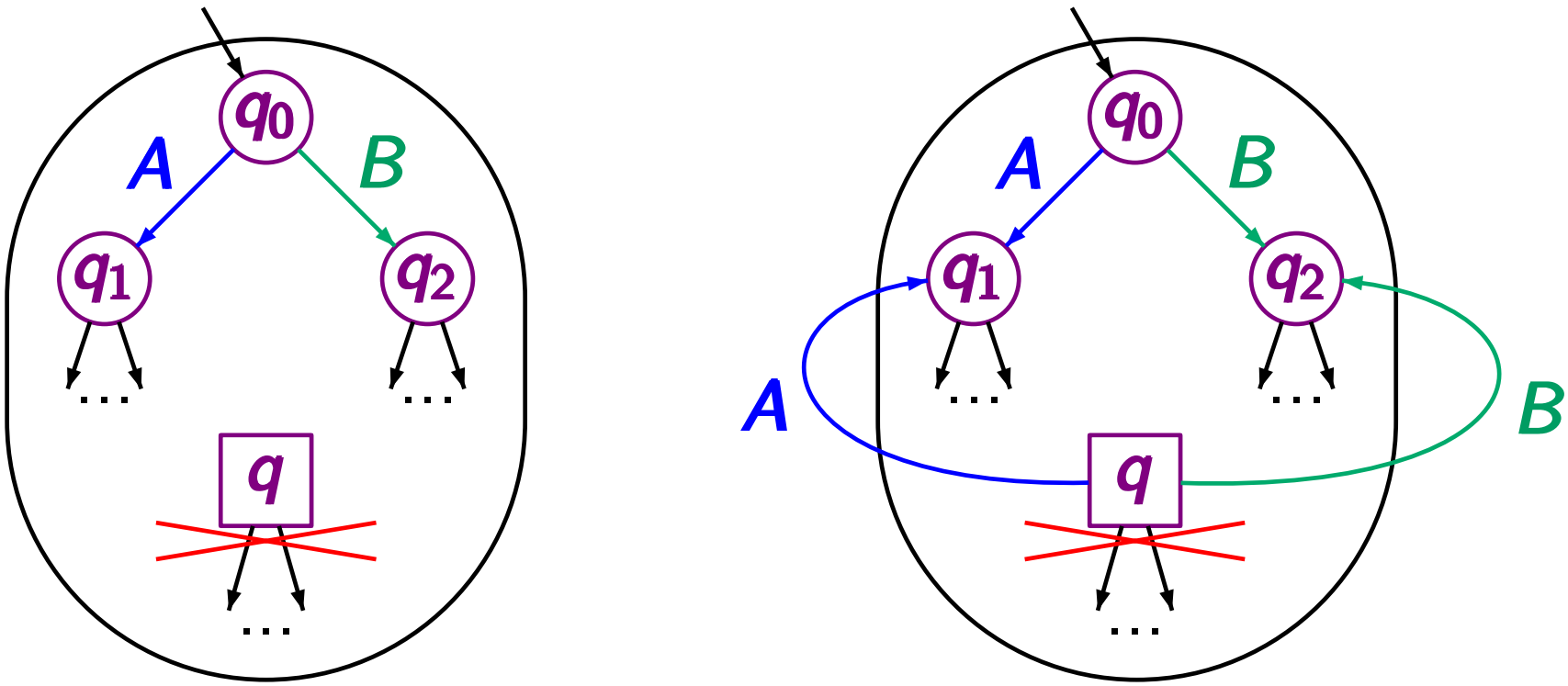




wrong !

ω-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\rightsquigarrow$ NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$



wrong !

... correct, if $\delta(q, x) = \varnothing \quad \forall q \in F \; \forall x \in \Sigma$

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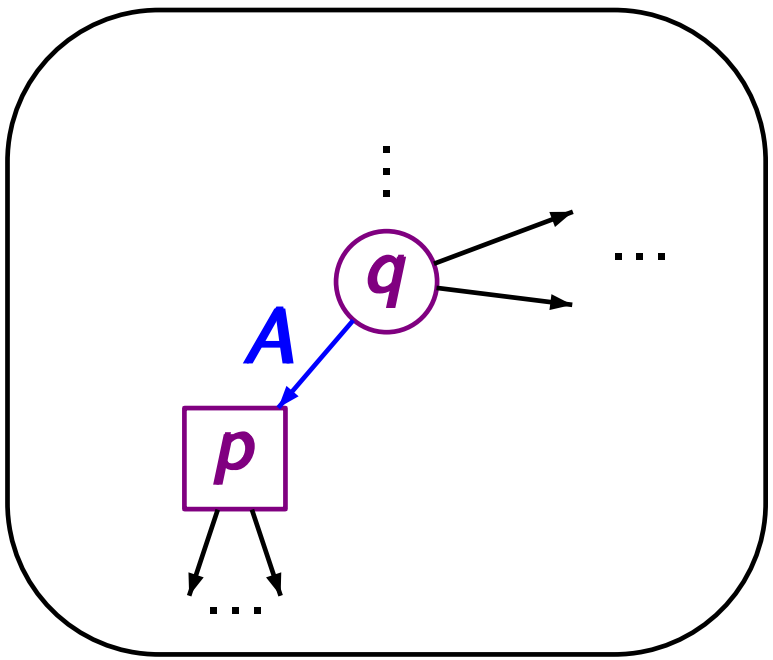
NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\Longrightarrow$ **NFA $\mathcal{B}$** for L s.t. all final states are terminal



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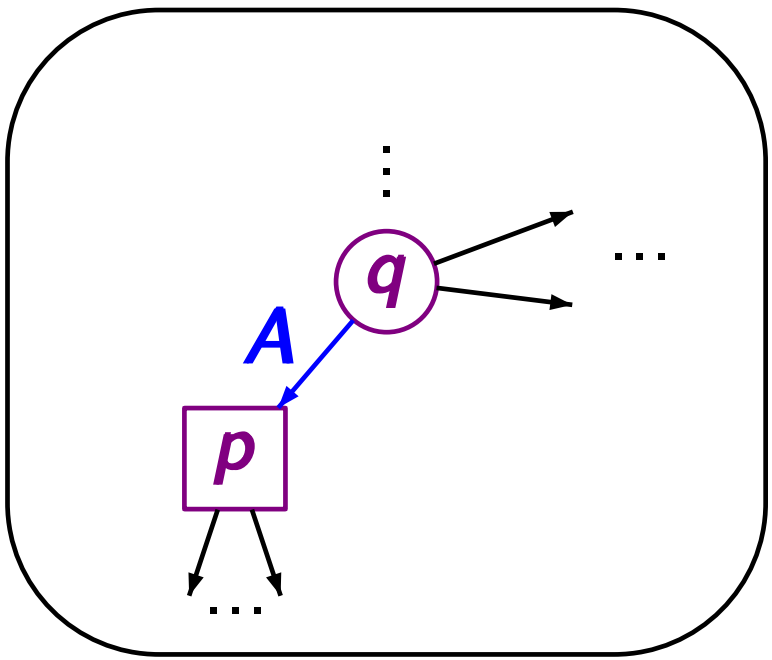

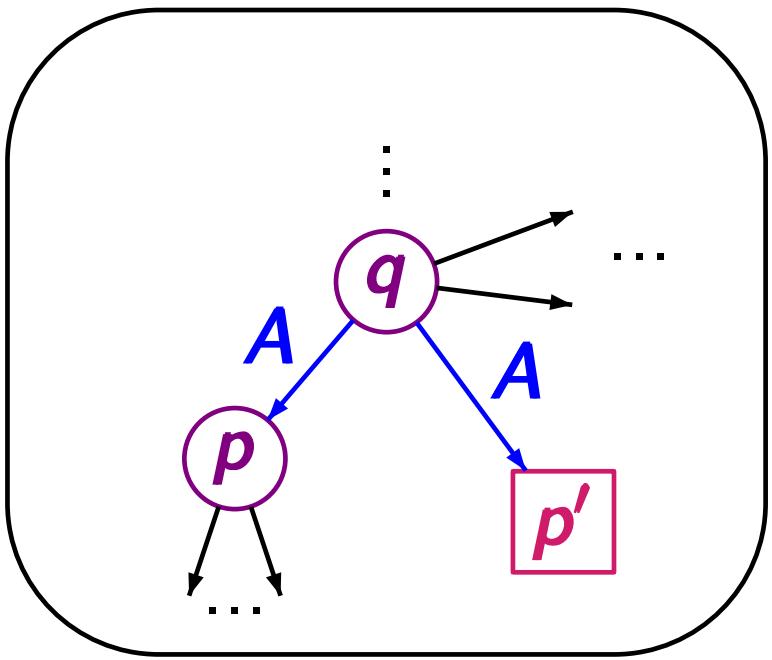

... add a new final state p' ...

ω-operator for NFA

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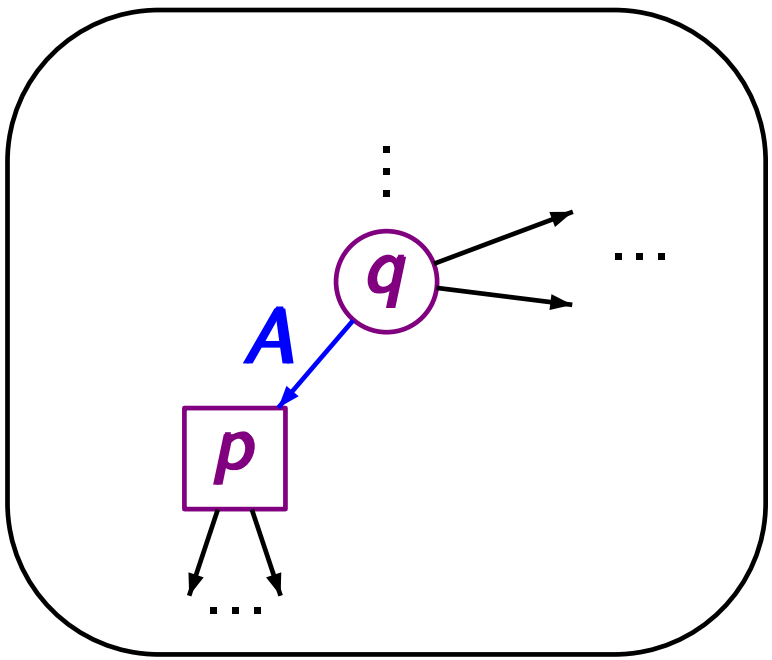
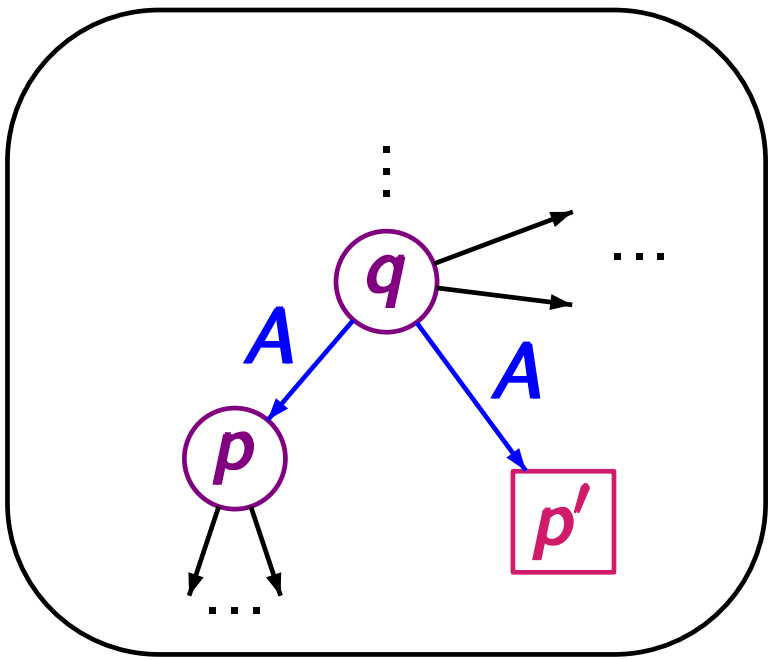
$\Downarrow$

NBA $\mathcal{B}^\omega$





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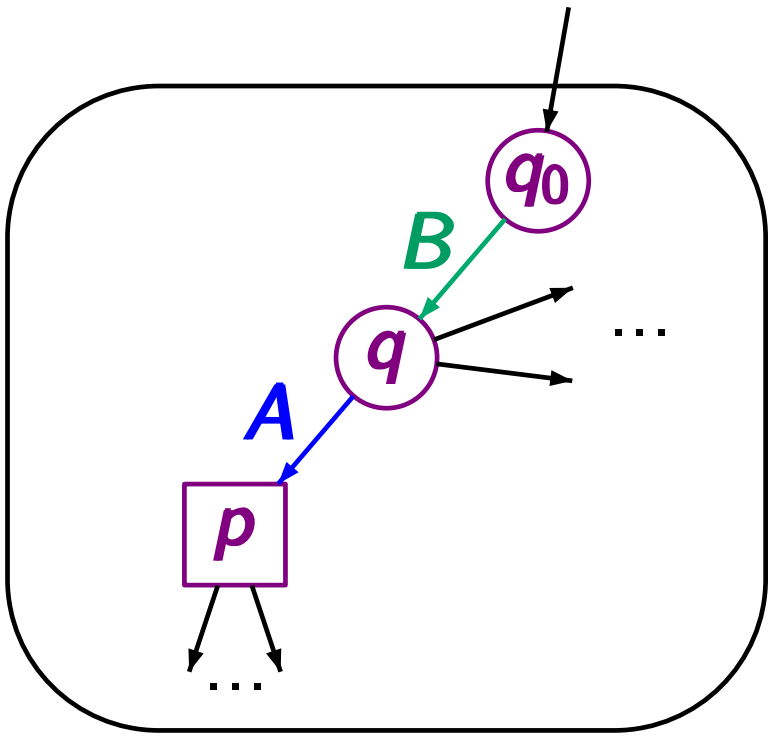

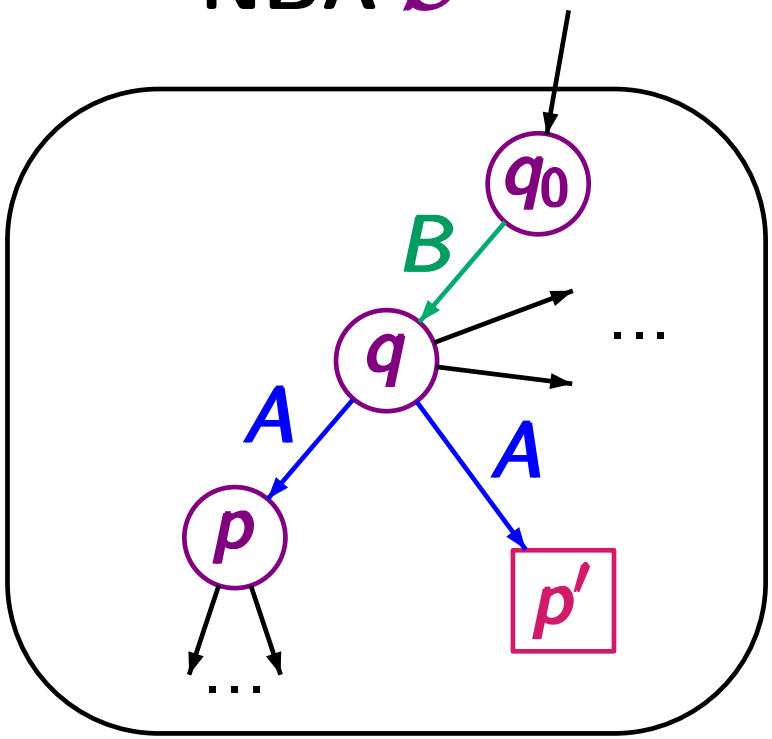

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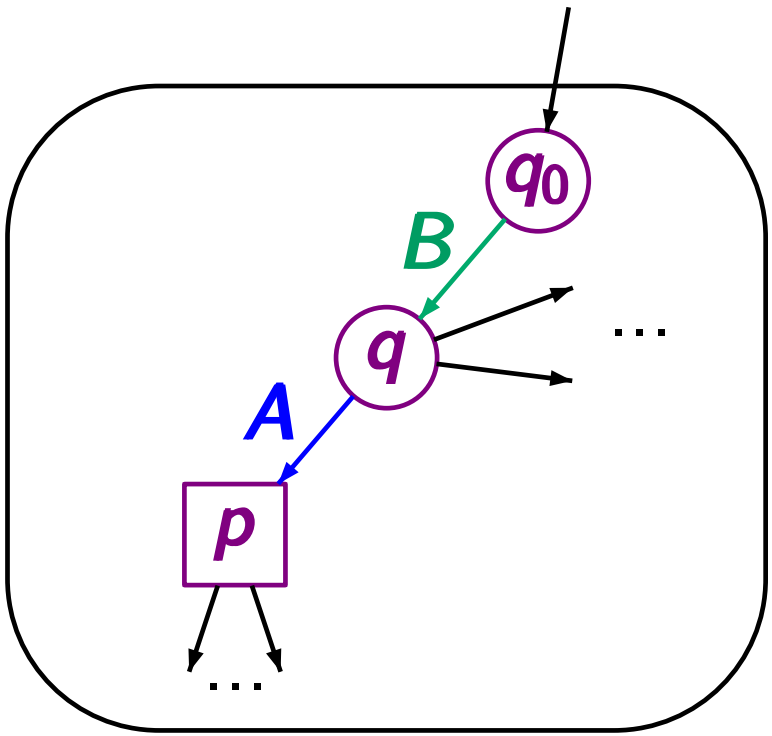

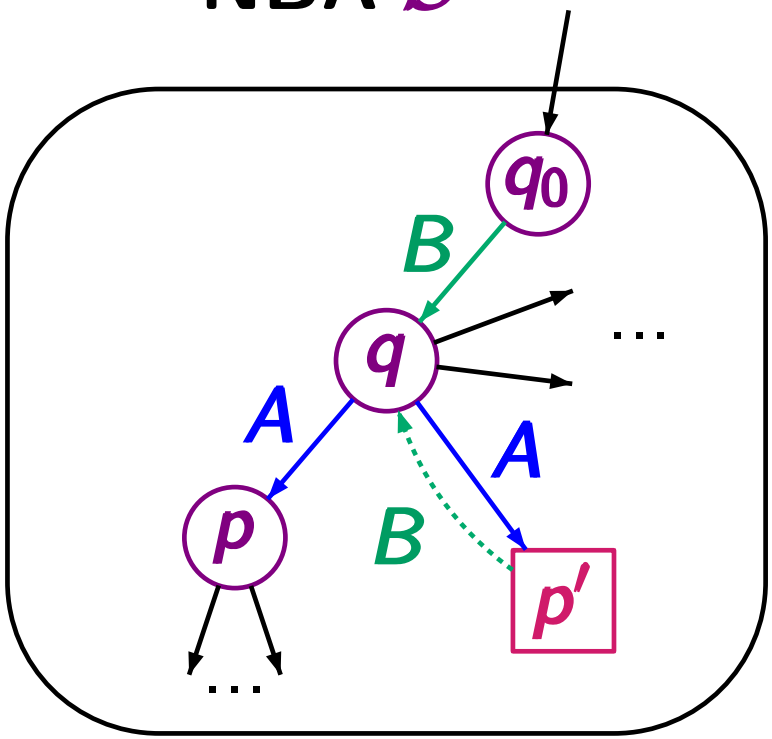

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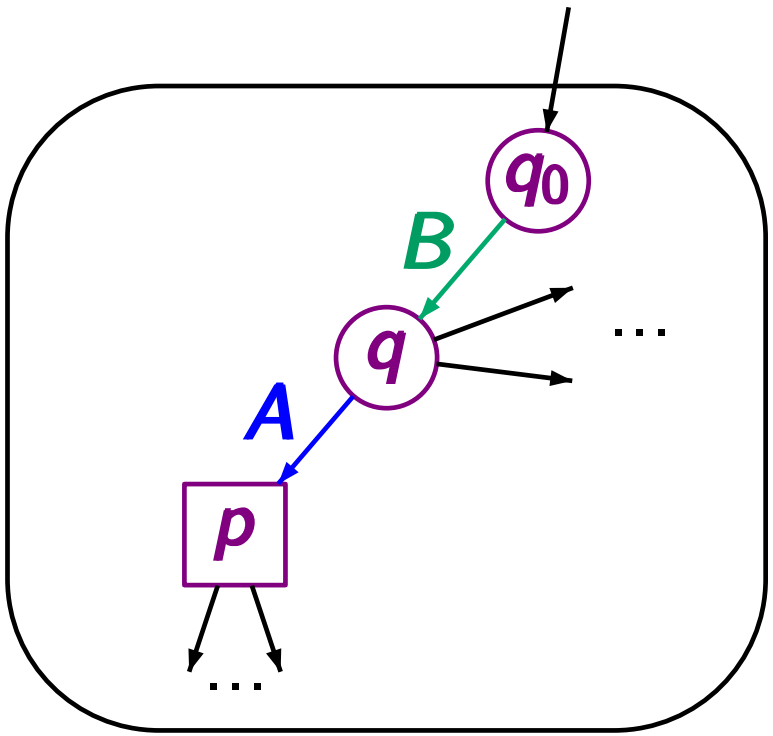
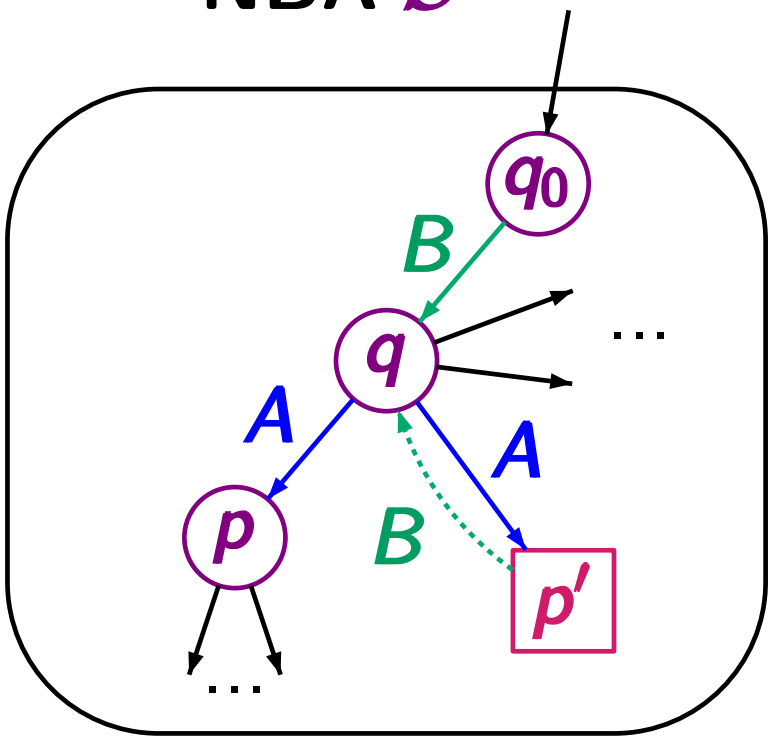
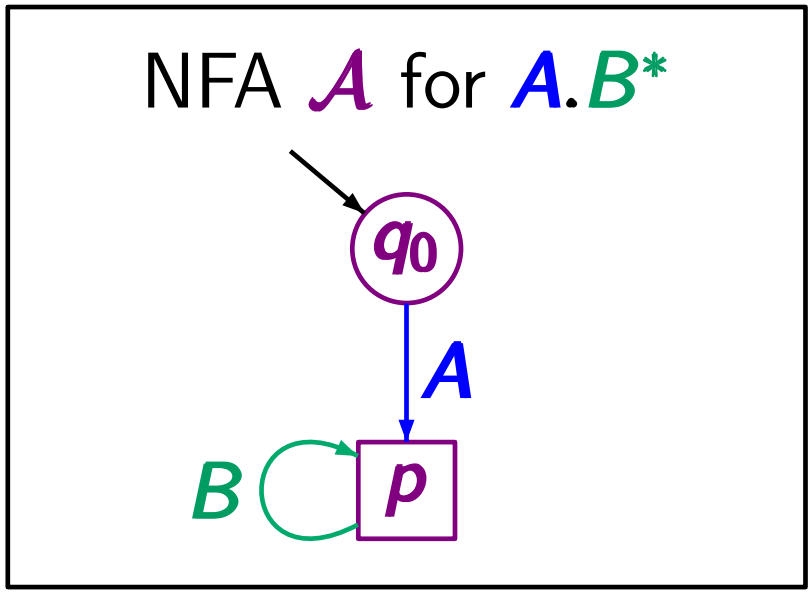




$$\mathcal{L}(\mathcal{A})^\omega = \mathcal{L}_\omega(\mathcal{B}^\omega)$$

Example: ω-operator for NFA



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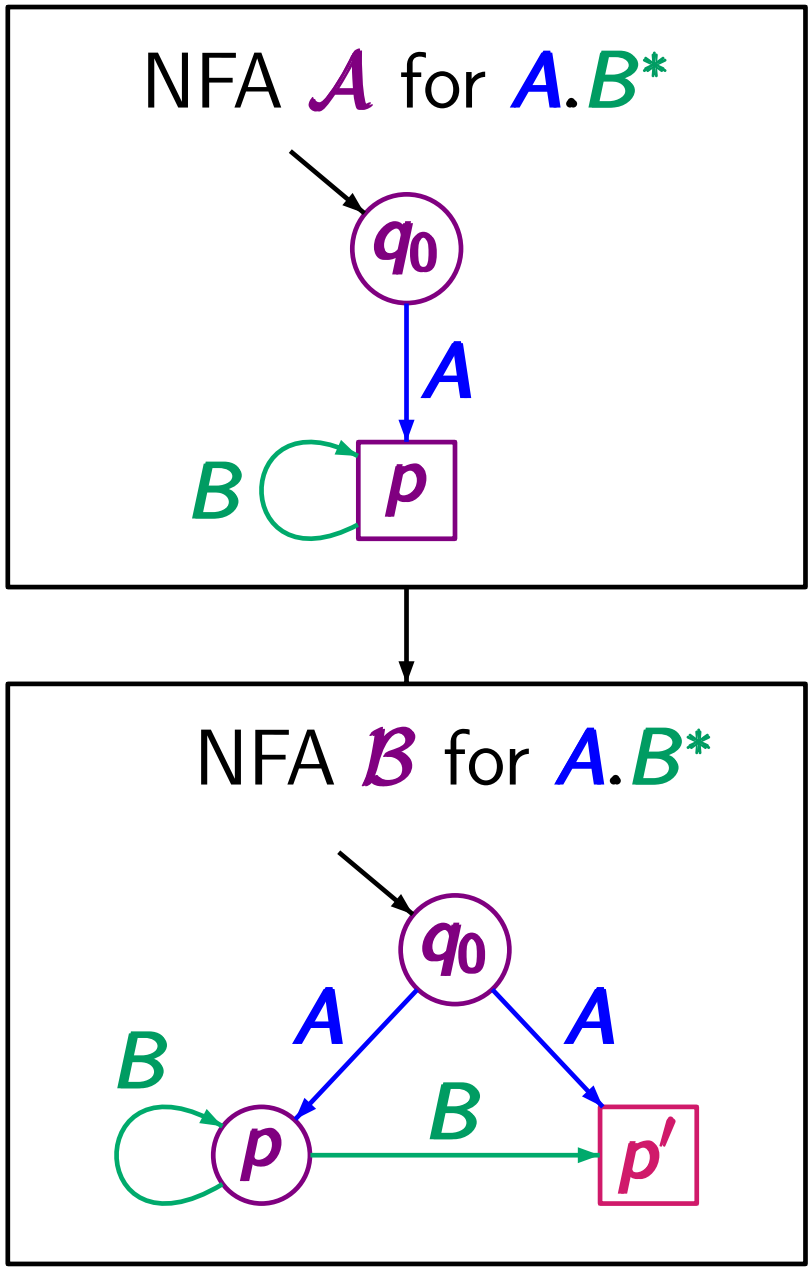

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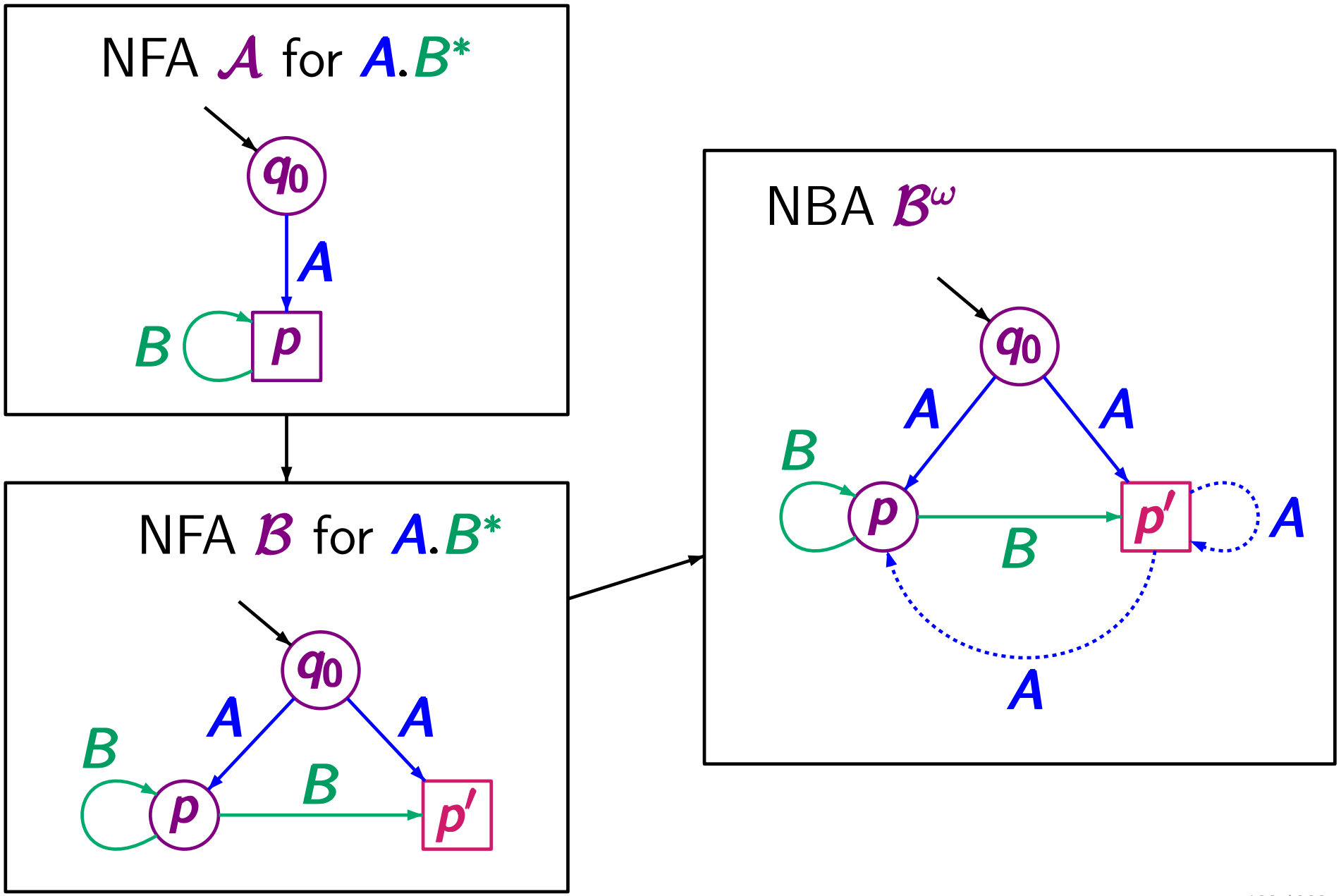
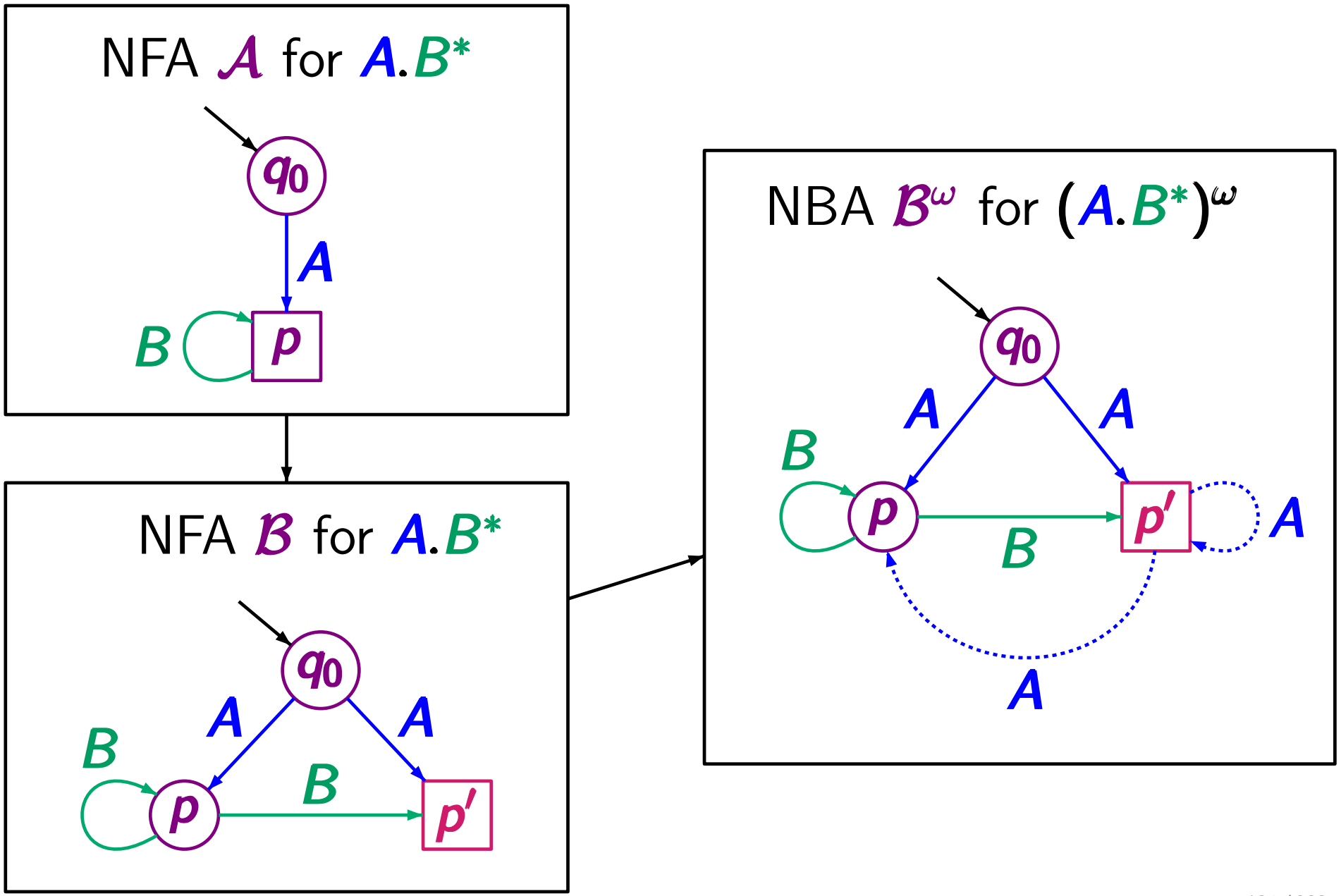


Example: ω-operator for NFA



Equivalence of ω-regular expressions and NBA

(1) For each NBA $\mathcal{A}$ there exists an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

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If E be an LT property then:

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Corollary:

If E be an LT property, i.e., $E \subseteq \left(2^{AP}\right)^\omega$, then:

E is ω-regular iff $E = \mathcal{L}_\omega(\mathcal{A})$ for some **NBA** $\mathcal{A}$ over the alphabet 2^{AP}

Closure properties of ω-regular properties

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remind: Kleene's theorem for regular languages:

The class of regular languages is closed under

- union, intersection, complementation
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  much more difficult than for NFA,
  via other types of ω-automata

Nonemptiness for NBA

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given: NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

question: does $\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing$ hold ?

Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing \quad \text{iff} \quad \exists q_0 \in Q_0 \; \exists p \in F \; \exists x \in \Sigma^* \; \exists y \in \Sigma^+ . \; p \in \delta(q_0, x) \cap \delta(p, y)$$

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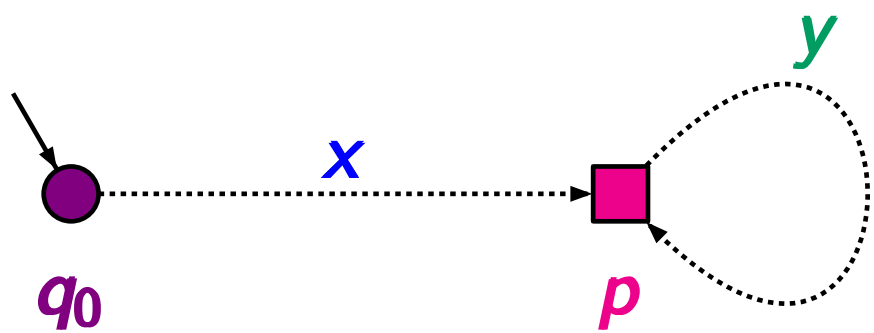
$\uparrow$

there exists a reachable accept state $p \in F$ that belongs to a cycle



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iff there exist finite words $x, y \in \Sigma^*$
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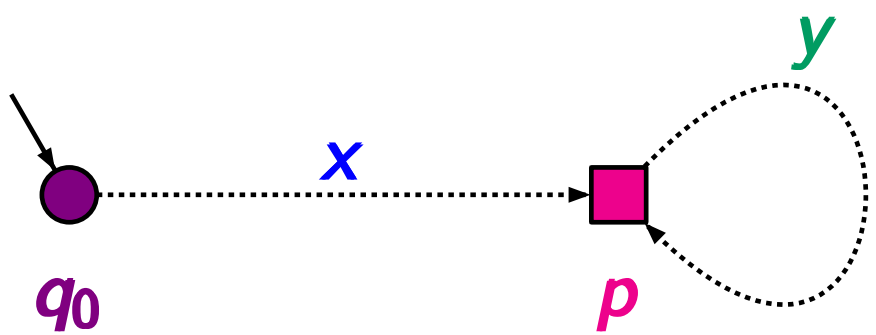

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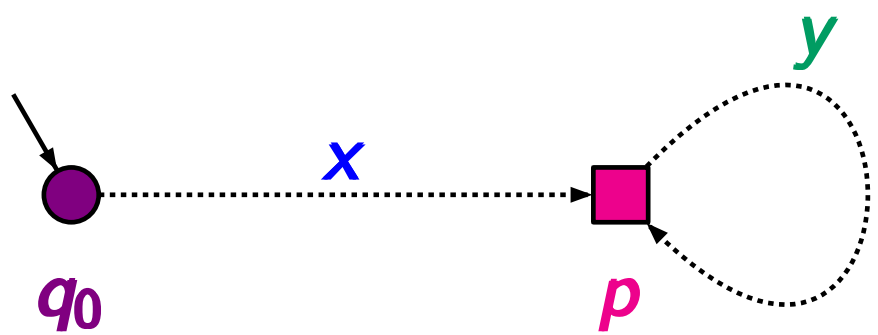
↑

"ultimatively periodic words"



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The emptiness problem for NBA is solvable by means of graph algorithms in time $\mathcal{O}(\mathit{poly}(\mathcal{A}))$

Deterministic Büchi automata (DBA)

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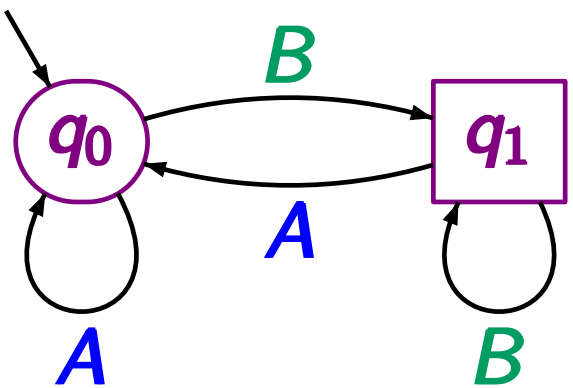

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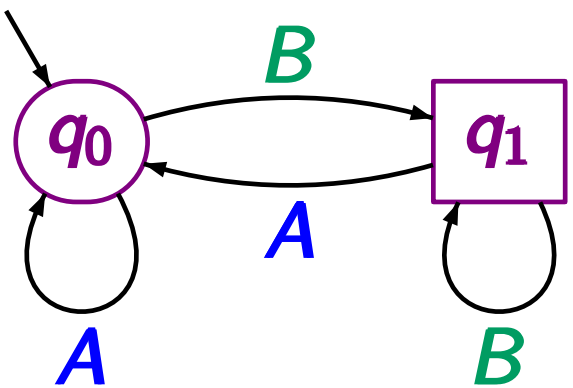

DBA for "infinitely often B"

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Determinization by powerset construction

well-known:

the powerset construction for the determinization (and complementation) of finite automata **(NFA)**

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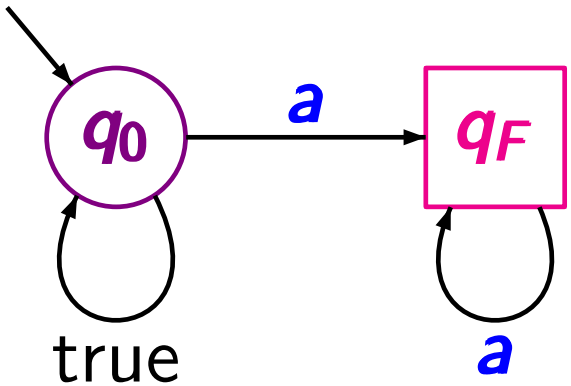
question:

does the powerset construction also work for Büchi automata **(NBA)** **?**

Determinization by powerset construction

NBA for "eventually forever a"



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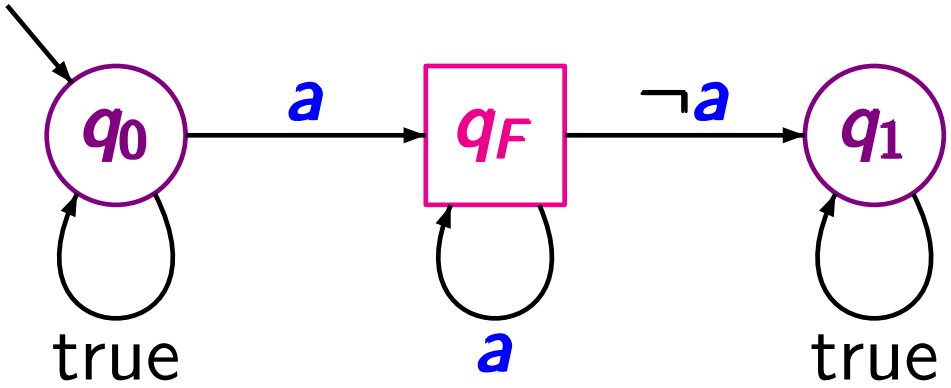

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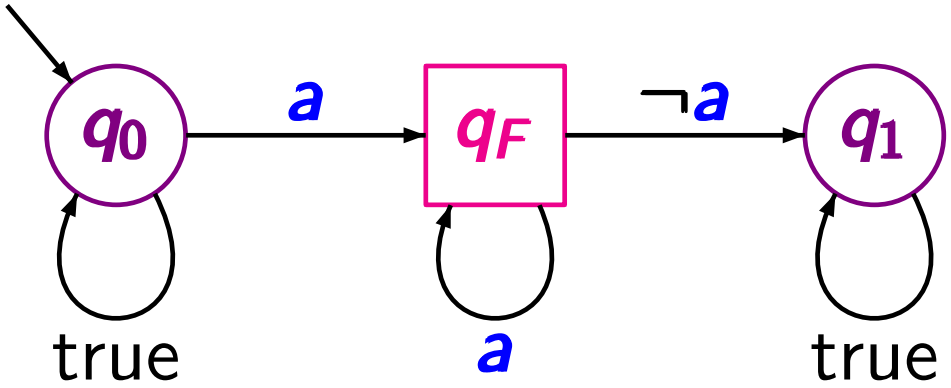

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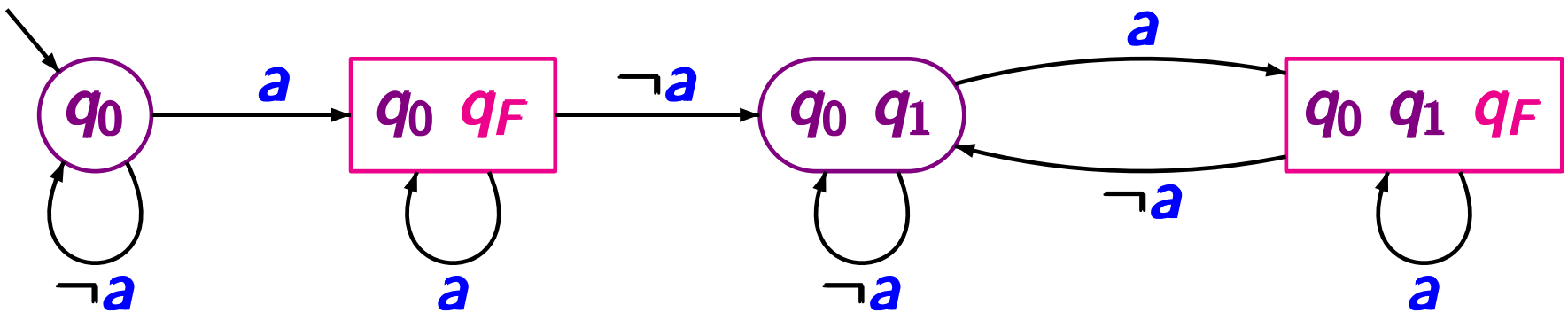

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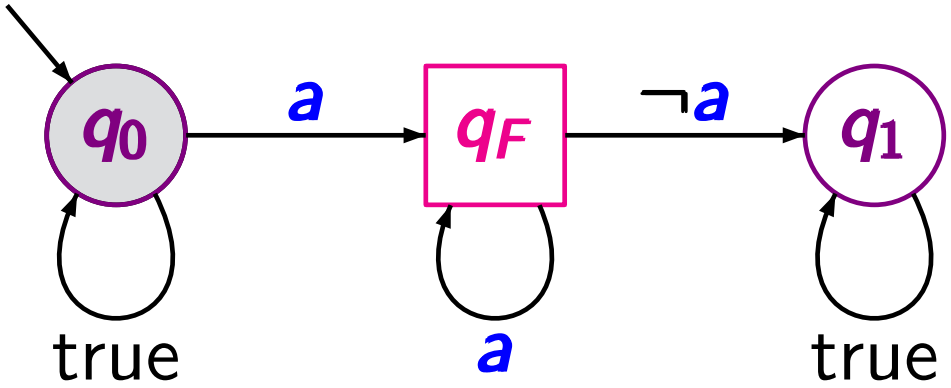

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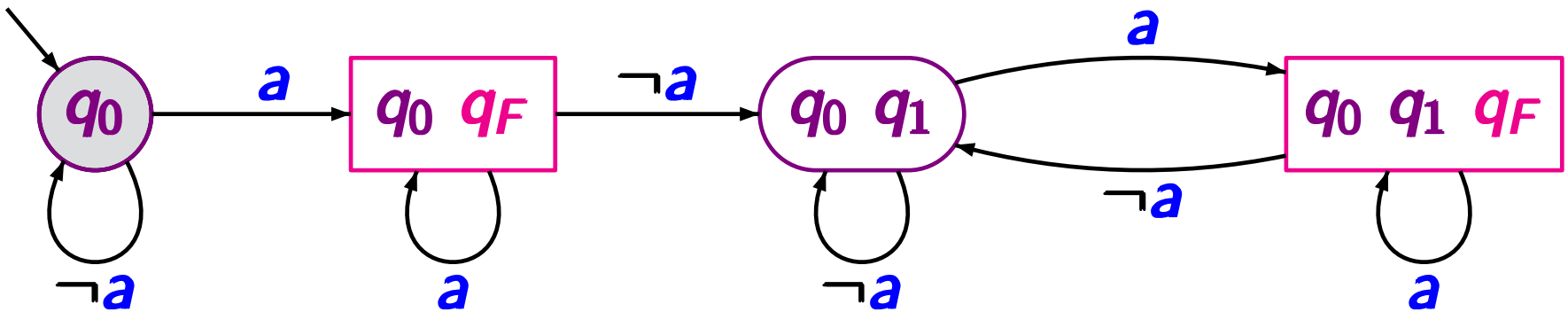

e.g., $\delta(q_0, a) = \{q_0, q_F\}$ and $\delta(q_0, \neg a) = \{q_0\}$

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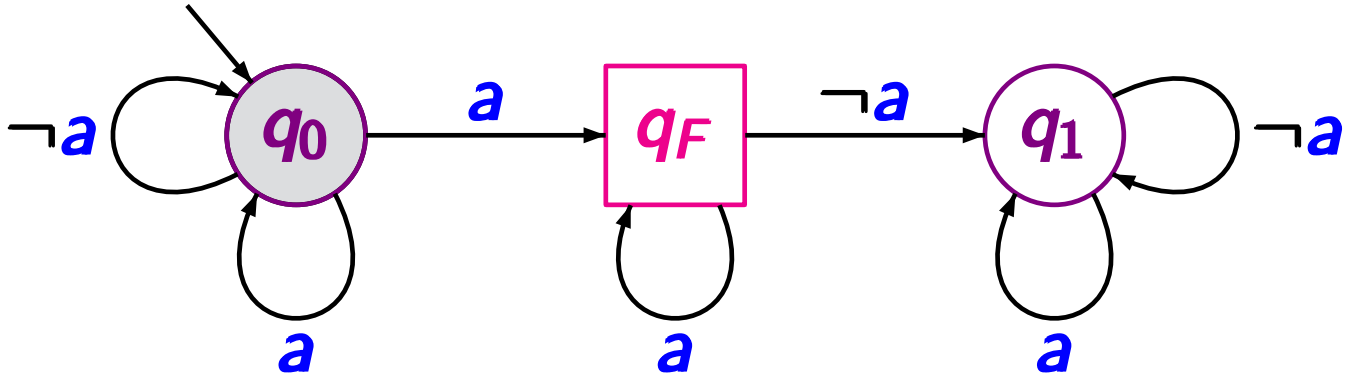

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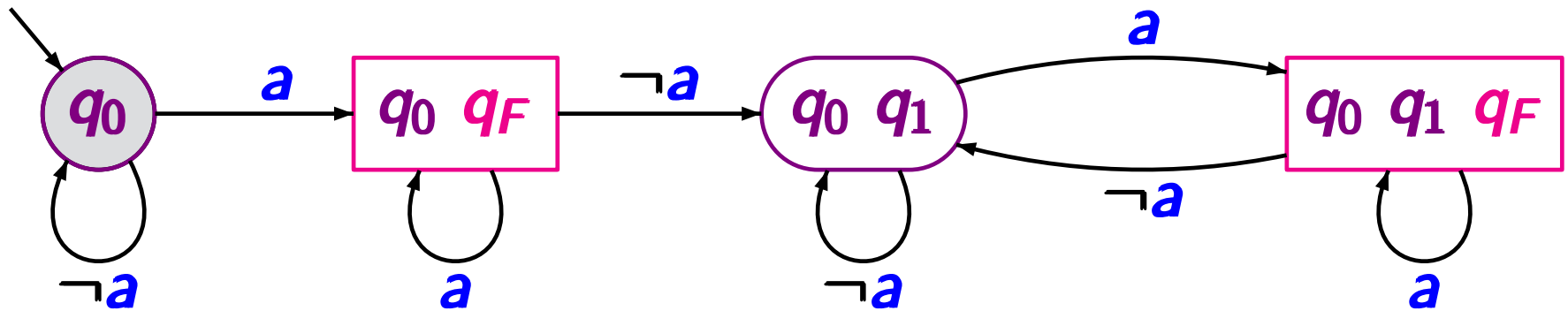

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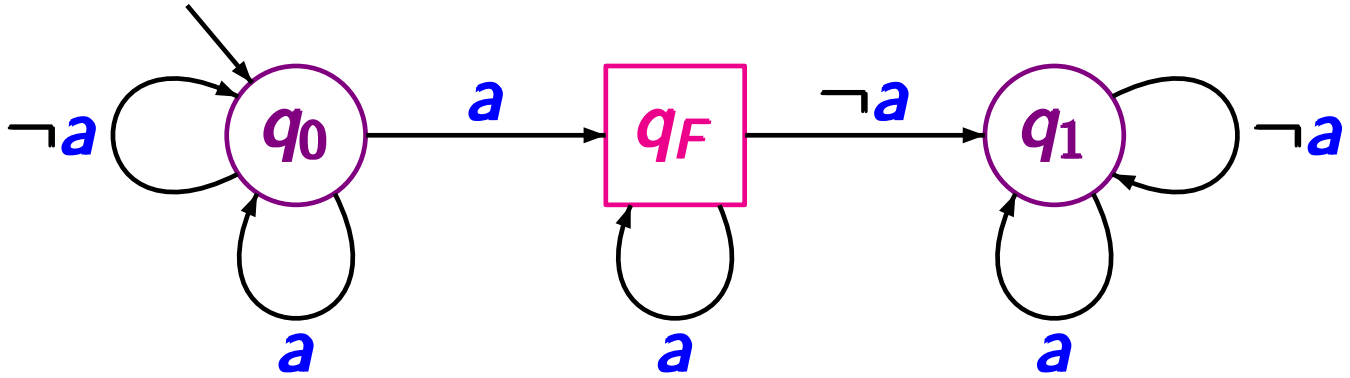

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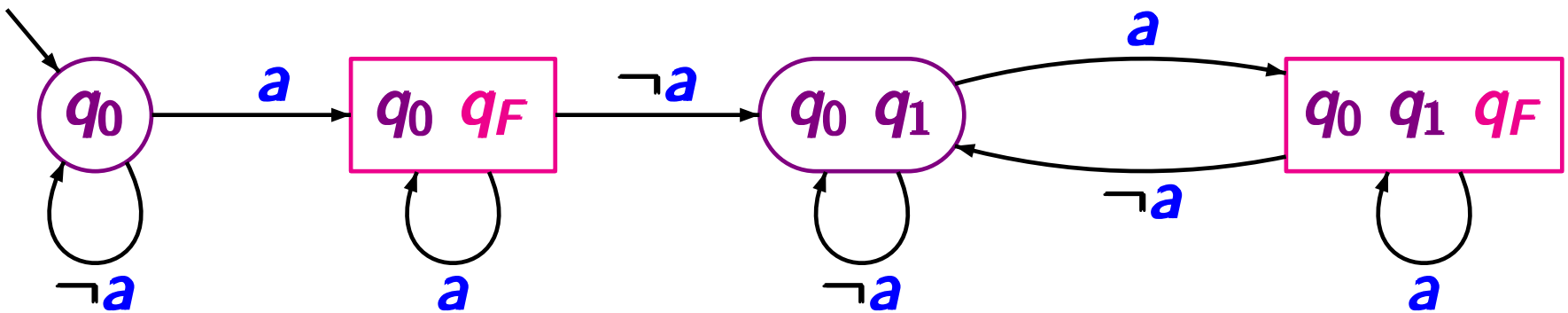
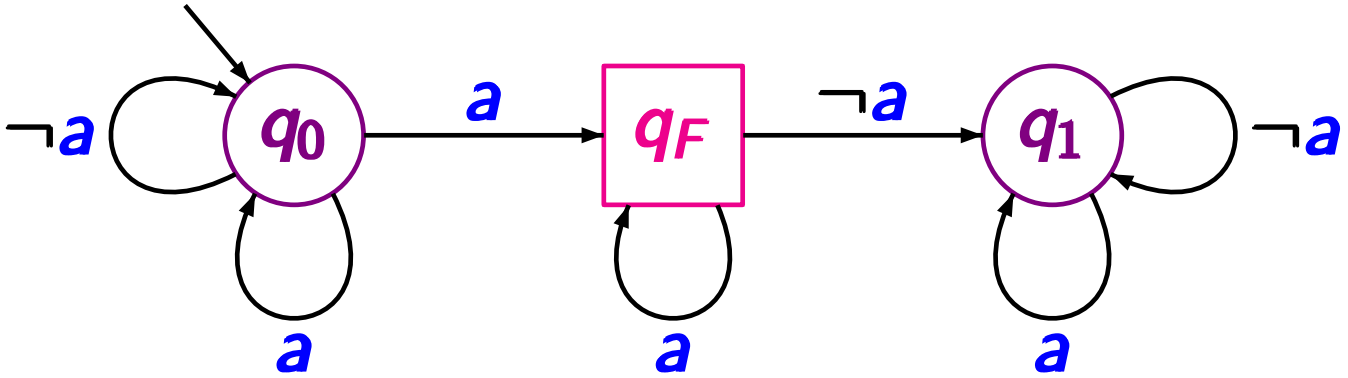


DBA for "infinitely often a"

Powerset construction ← fails for NBA

NBA for "eventually forever a"



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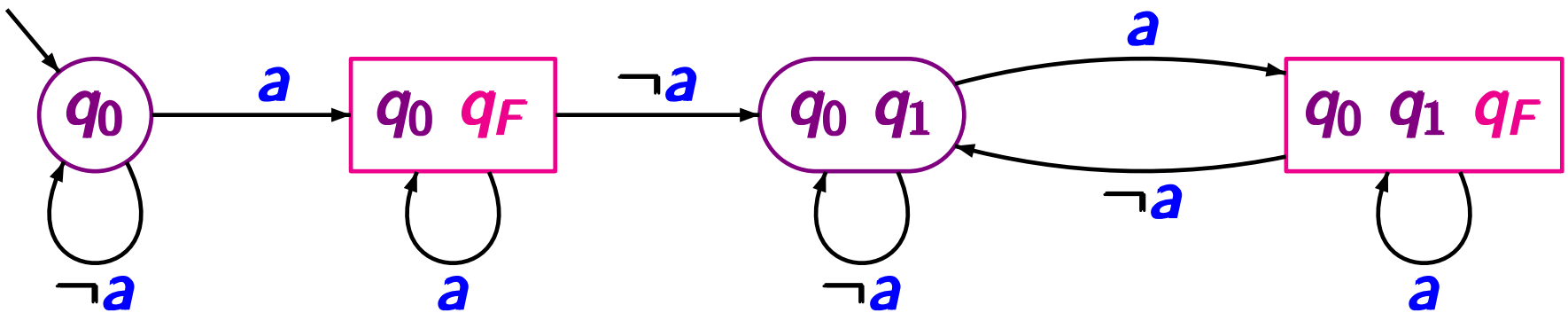

DBA for "infinitely often a"

Complementation of DBA

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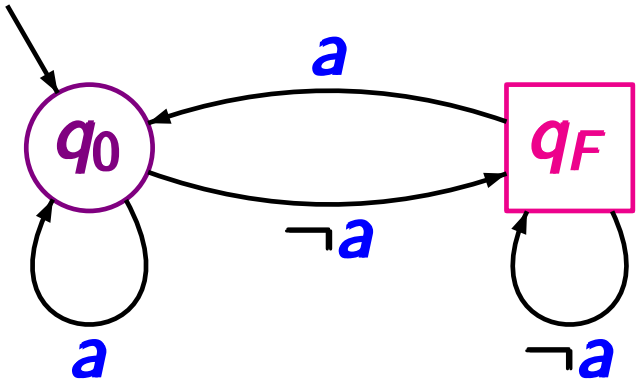
well-known:

DFA can be complemented by complementation of the acceptance set

question:

does this also work for **DBA** ?

Complementation of DBA



DBA for
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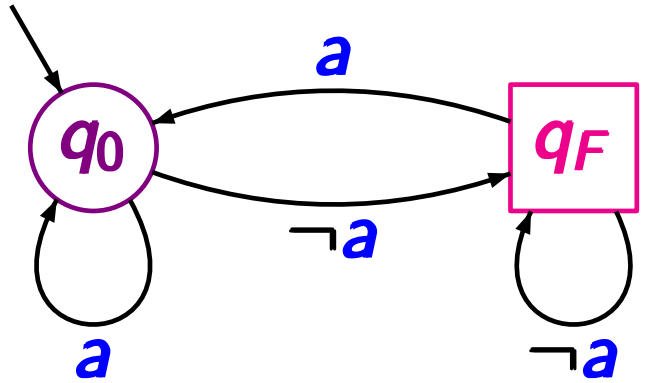

DBA for
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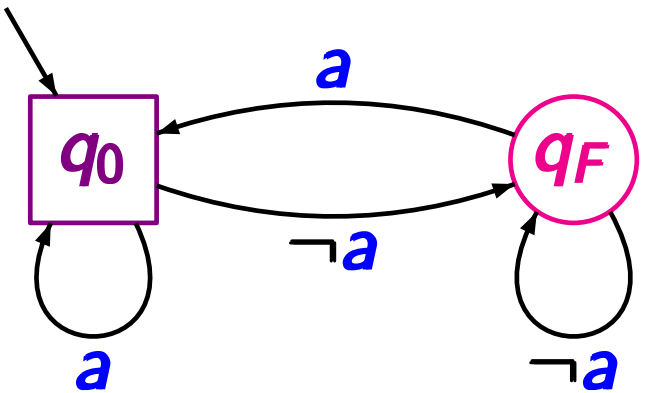
complement automaton



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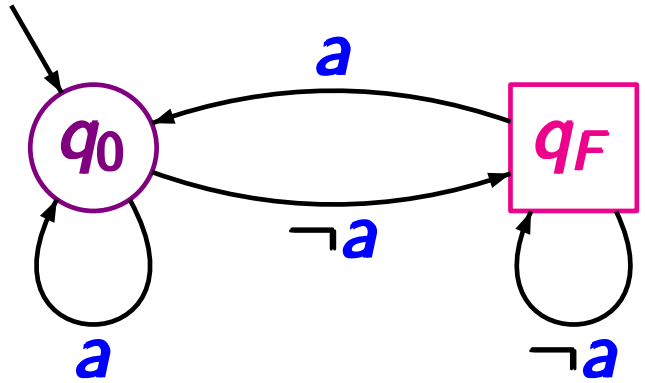

DBA for
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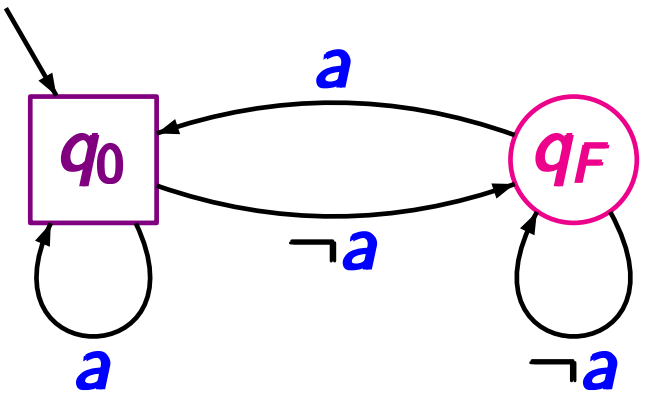

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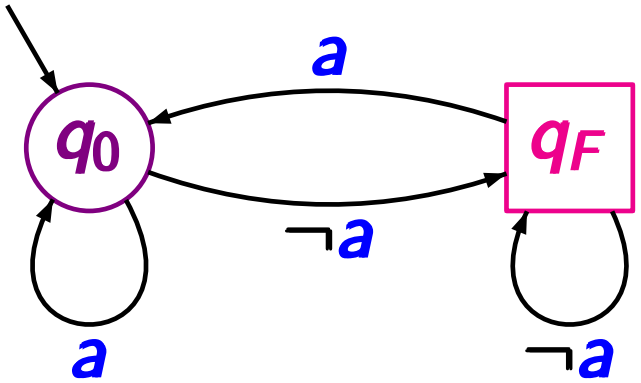
Complementation ⟵ fails for DBA



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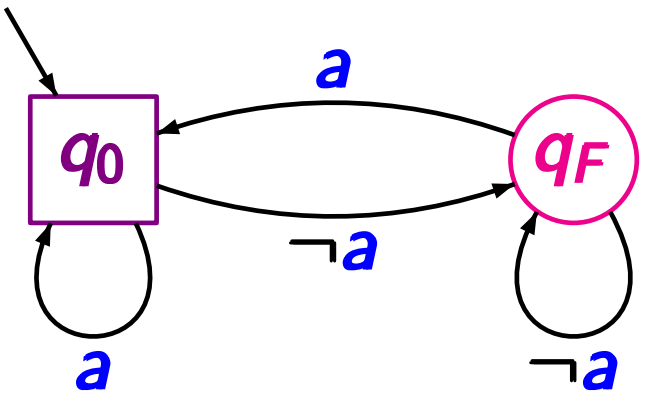

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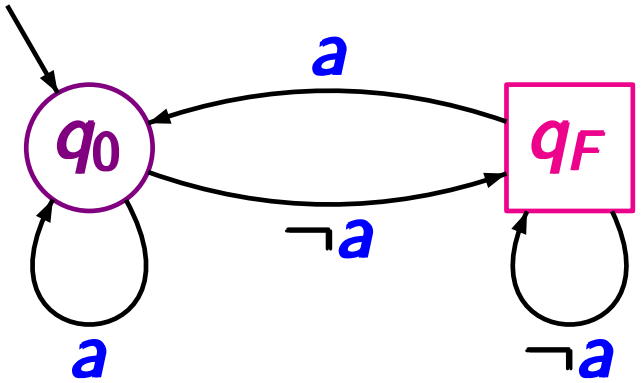

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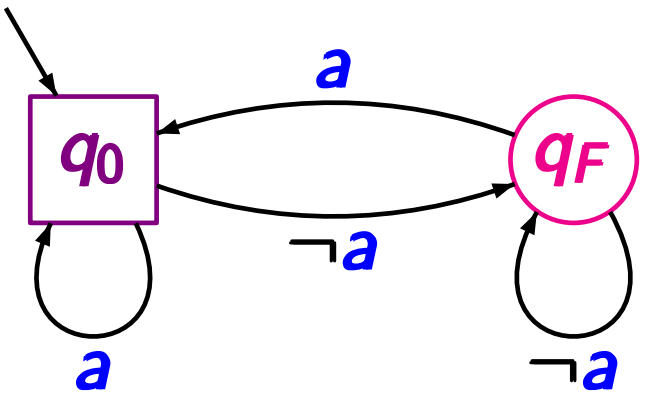

DBA for
"infinitely often *a*"

There is **no DBA** for the LT-property
"eventually forever *a*"

On the power of DBA

There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega\big((A+B)^*.A^\omega\big)$

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$(A^*.B)^\omega$	"infinitely many B's"	DBA-recognizable
$(A+B)^*.A^\omega$	"only finitely many B's"	not DBA-recognizable

Generalized NBA (GNBA)

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A generalized nondeterministic Büchi automaton is a tuple

$$\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$$

where Q, Σ, δ, Q_0 are as in NBA, but $\mathcal{F}$ is a set of accept sets, i.e., $\mathcal{F} \subseteq 2^Q$.

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$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, i \in \mathbb{N} \ \text{ s.t. } \ q_i \in F$$

Accepted language of a GNBA

GNBA $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ as NBA, but $\mathcal{F} \subseteq 2^Q$

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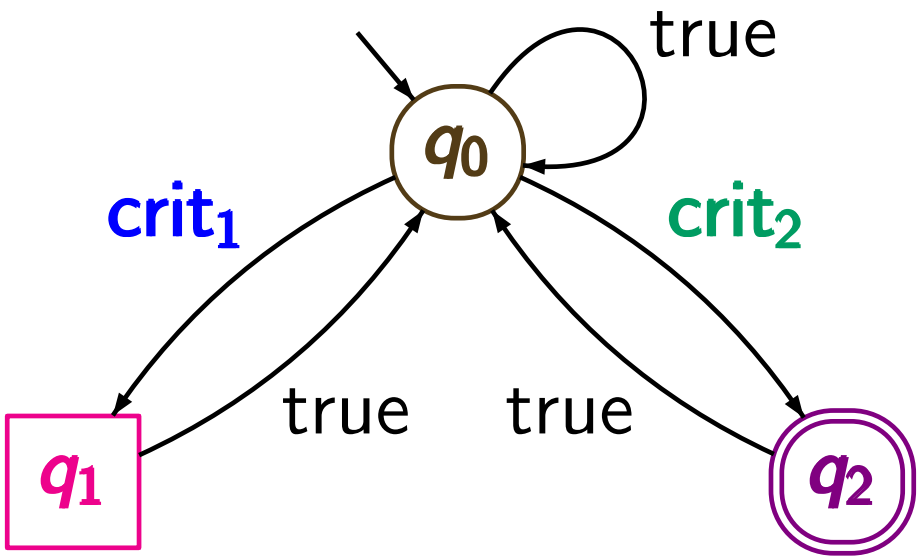
$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, i \in \mathbb{N} \text{ s.t. } q_i \in F$$

accepted language:

$$\mathcal{L}_\omega(\mathcal{G}) \overset{\text{def}}{=} \{\sigma \in \Sigma^\omega : \sigma \text{ has an accepting run in } \mathcal{G}\}$$

Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$



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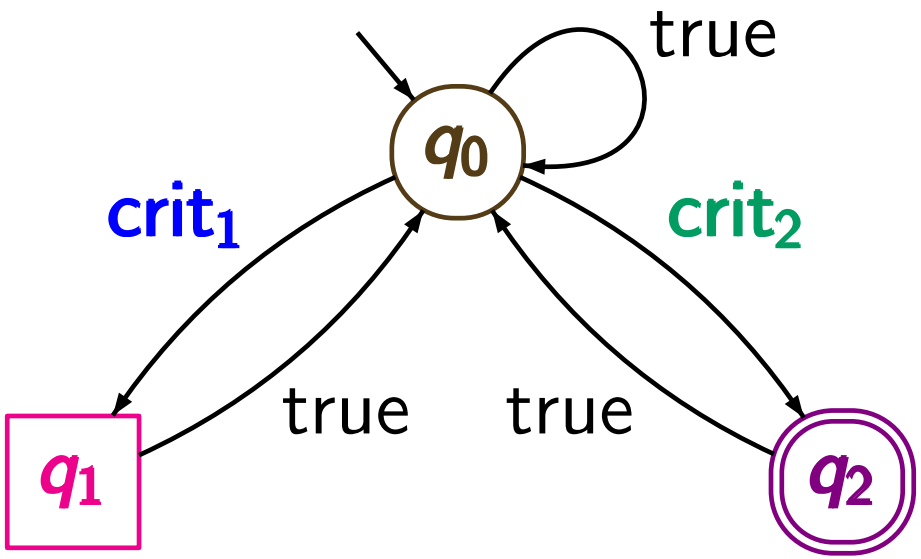

$$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$$

specifies the LT-property

"infinitely often crit_1 and infinitely often crit_2"

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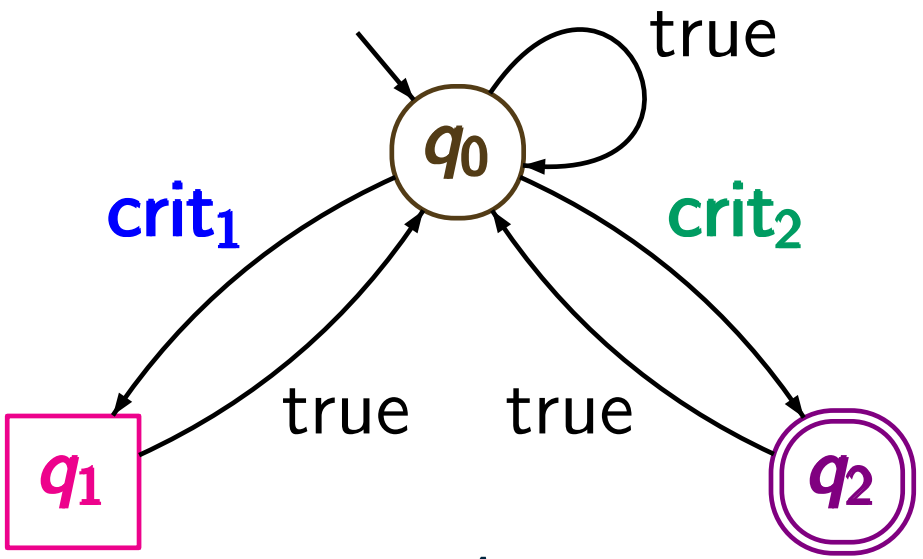

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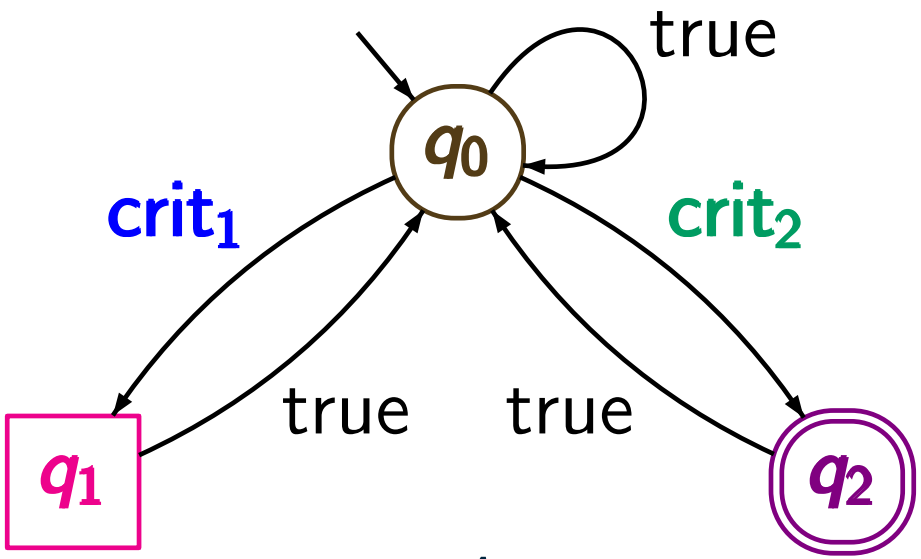

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hence: if $A_0 A_1 A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$ then

$$\overset{\infty}{\exists}\, i \geq 0.\ \text{crit}_1 \in A_i \ \wedge \overset{\infty}{\exists}\, i \geq 0.\ \text{crit}_2 \in A_i$$

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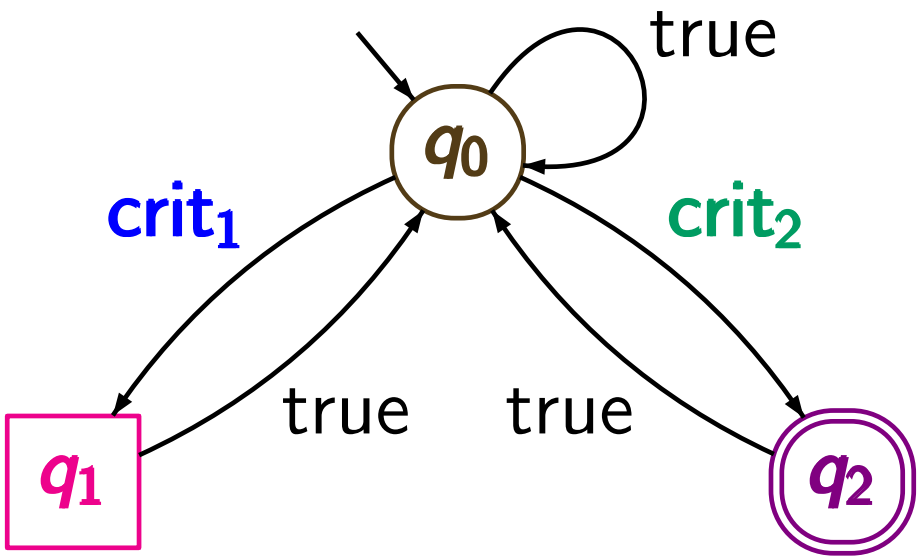

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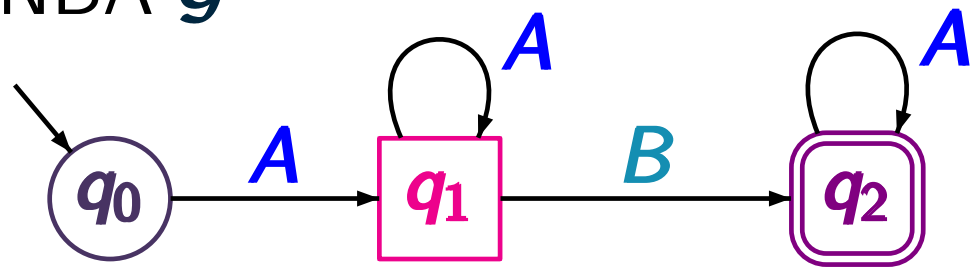
all words $A_0\, A_1\, A_2 \ldots \in \Sigma^\omega$ s.t. $\overset{\infty}{\exists}\, i \geq 0.\ \text{crit}_1 \in A_i$ and $\overset{\infty}{\exists}\, i \geq 0.\ \text{crit}_2 \in A_i$ have an accepting run of the form:

$$q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2\, q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2 \ldots$$

Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $\mathcal{G}$



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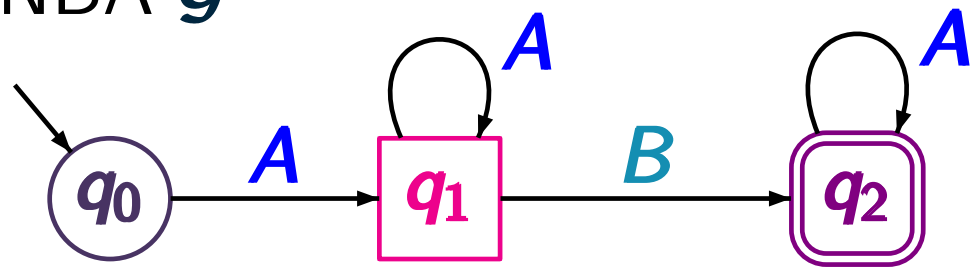

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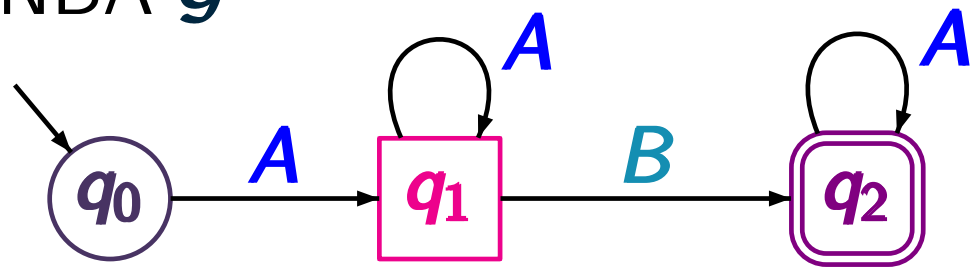

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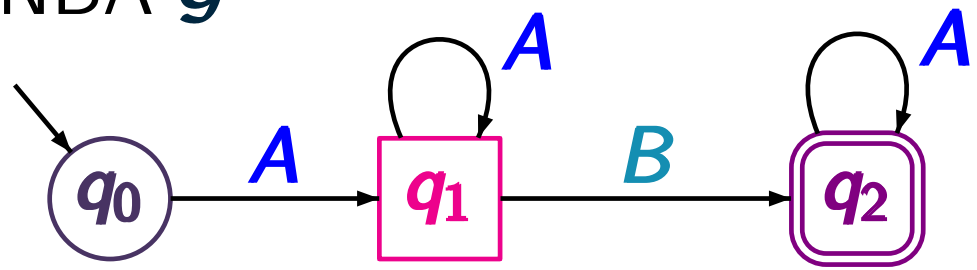

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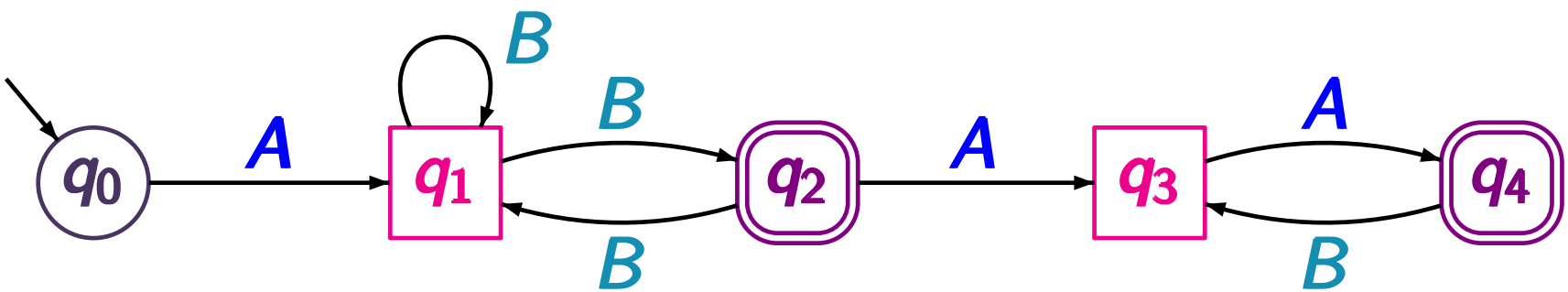
GNBA $\mathcal{G}'$ with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$



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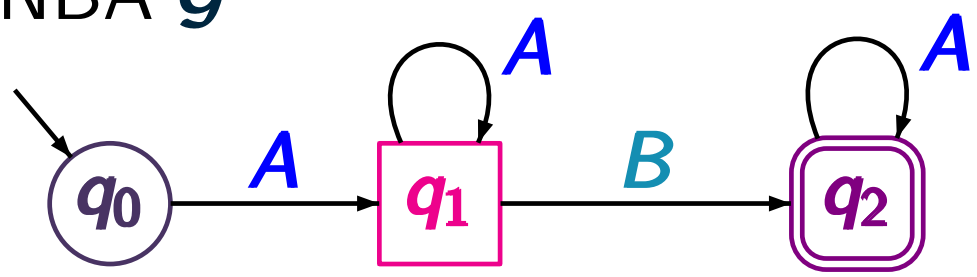

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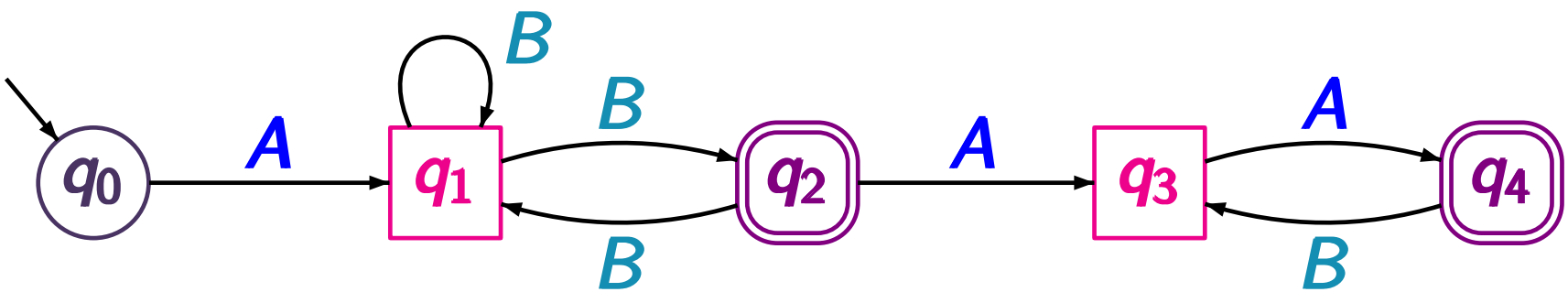

accepted language: **?**

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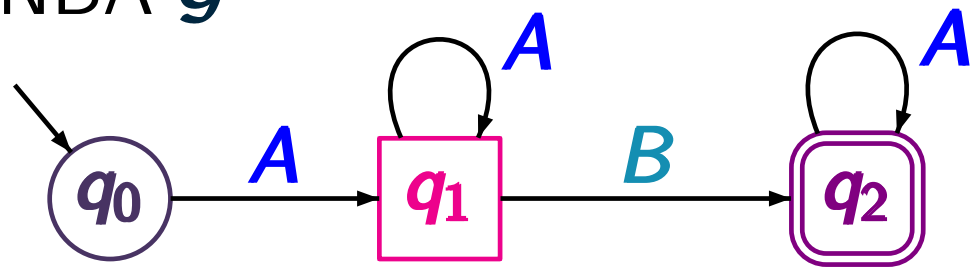

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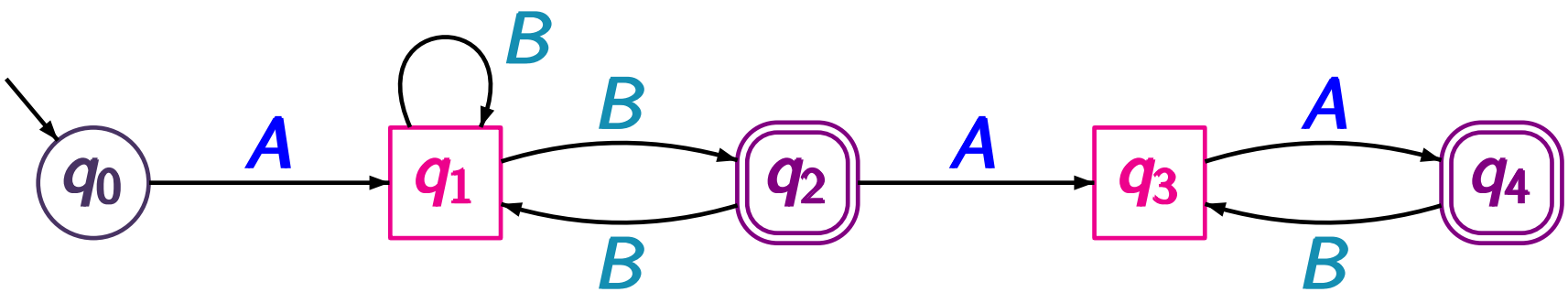

accepted language: $A.B^\omega + A.B^+.A.(A.B)^\omega$

Empty acceptance condition

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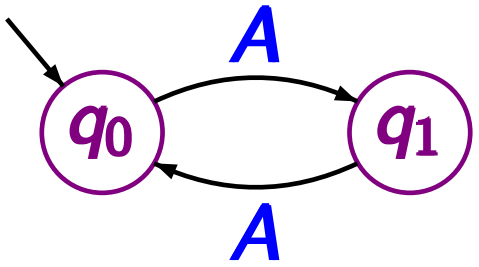

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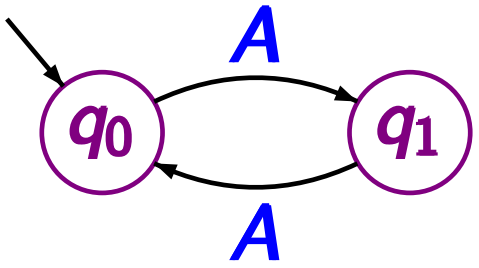

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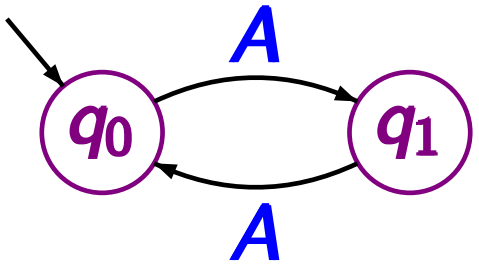

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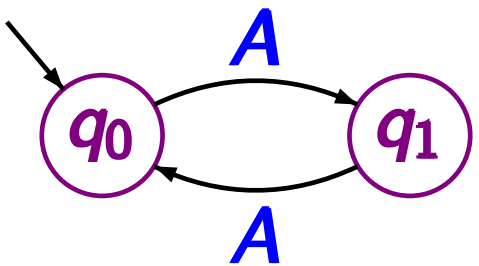

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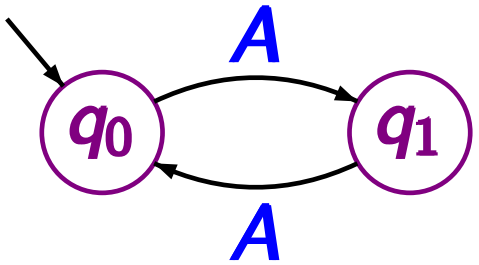

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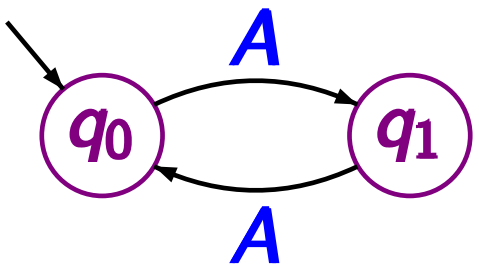

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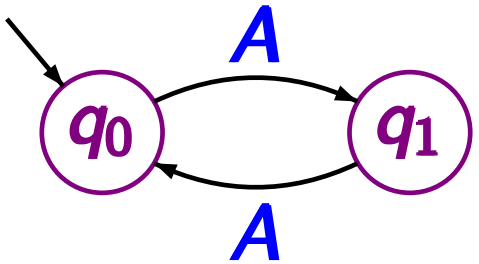

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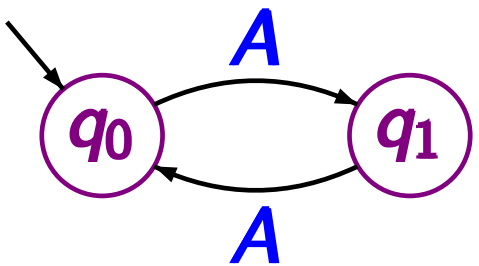

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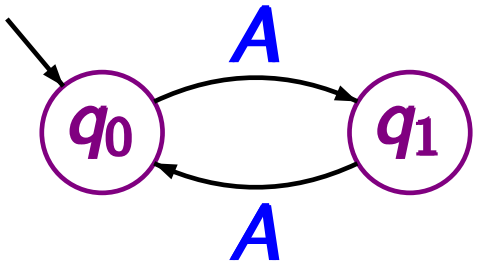

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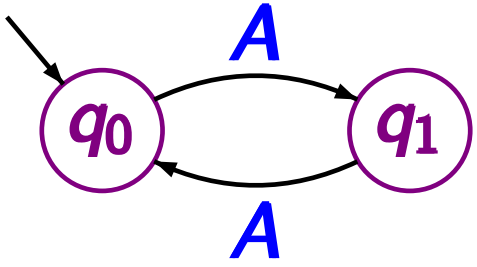

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$$\mathcal{L}_\omega(\mathcal{G}) = \left\{ \begin{array}{c} \text{set of all infinite words} \\ \text{that have an infinite run} \end{array} \right.$$

Correct or wrong?

For every GNBA $\mathcal{G}$ there exists a GNBA $\mathcal{G}'$ such that

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$$\begin{aligned} \text{GNBA } \mathcal{G} &= (Q, \Sigma, \delta, Q_0, \varnothing) \\ &\Downarrow \\ \text{GNBA } \mathcal{G}' &= (Q, \Sigma, \delta, Q_0, \{Q\}) \end{aligned}$$

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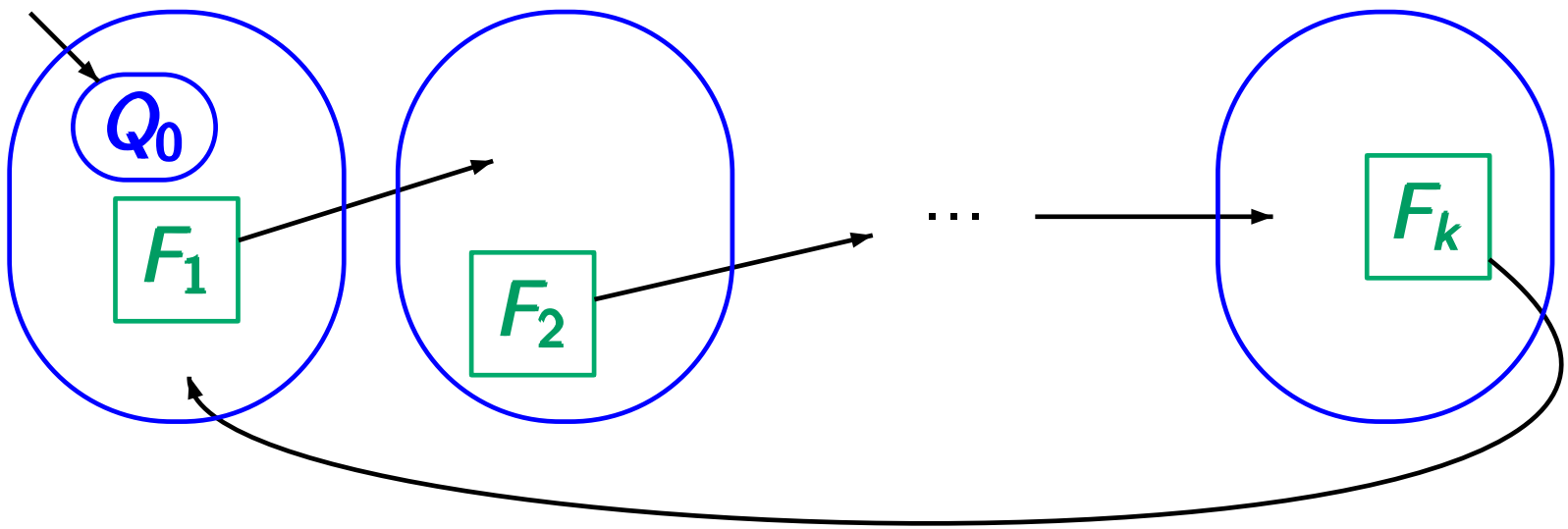

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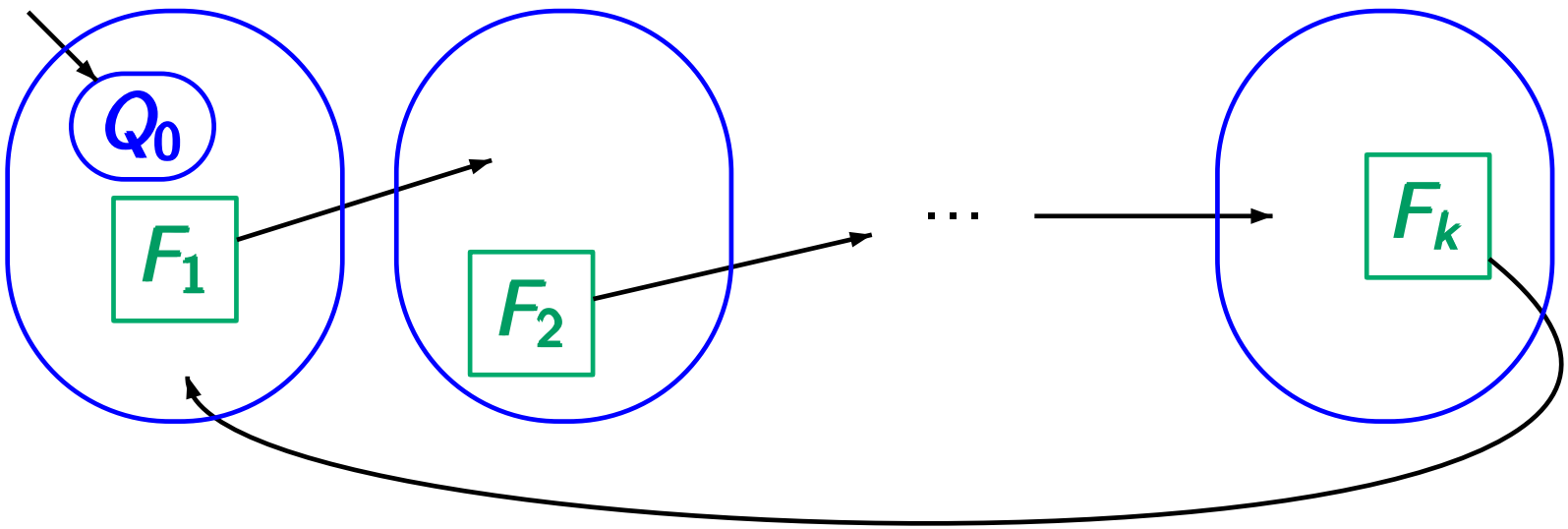
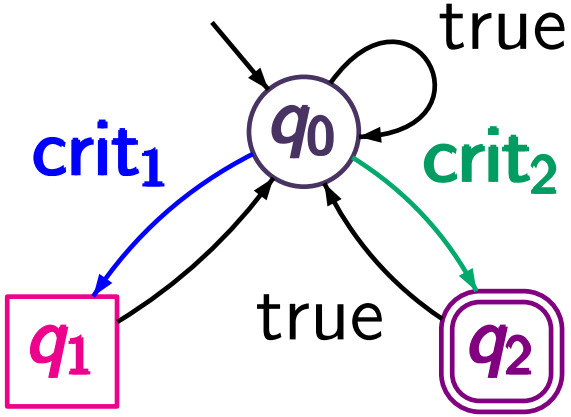


size of the NBA: $size(\mathcal{A}) = \mathcal{O}(size(\mathcal{G}) \cdot |\mathcal{F}|)$

Example: from GNBA to NBA

GNBA $\mathcal{G}$



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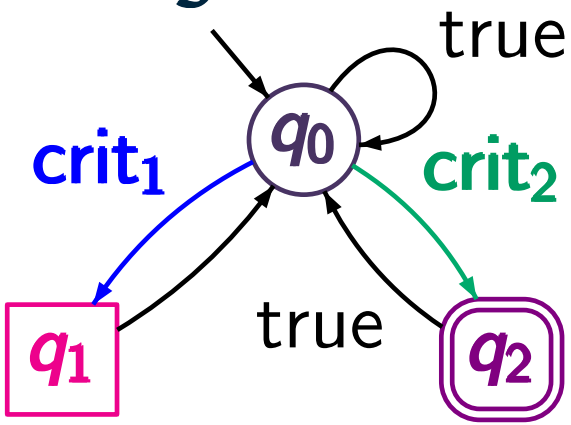

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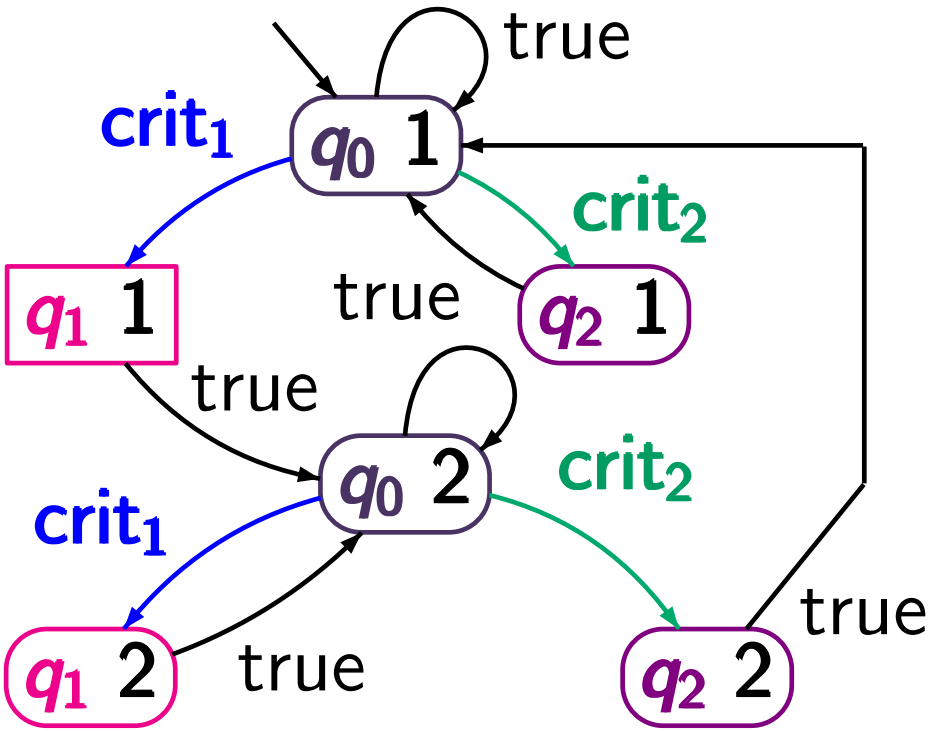

NBA $\mathcal{A}$

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Intersection for NBA

$$\left.\begin{array}{l} \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{array}\right\} \text{ two } \textbf{NBA}$$

goal: define an **NBA** $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

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recall:

intersection for finite automata **NFA** $\mathcal{A}_1$ and $\mathcal{A}_2$ is realized by a product construction that

- runs $\mathcal{A}_1$ and $\mathcal{A}_2$ in parallel (synchronously)
- checks whether both end in a final state

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i.e., both F_1 and F_2 are visited infinitely often

$\rightsquigarrow$ product of $\mathcal{A}_1$ and $\mathcal{A}_2$ yields a **GNBA**

Intersection for NBA

$$\left.\begin{array}{l} \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{array}\right\} \text{ two NBA}$$

goal: define an NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

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GNBA $\mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2$ $\rightsquigarrow$ equivalent NBA $\mathcal{A}$

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- the class of languages given by ω-regular expressions
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The class of ω-regular languages is closed under union, intersection and complementation.