Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL
- automata-based LTL model checking ⟵
- complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

LTL model checking problem

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(without terminal states)
LTL-formula φ over AP

question: does $\mathcal{T} \models \varphi$ hold **?**

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$$\pi \not\models \varphi, \text{ i.e., } \pi \models \neg\varphi$$

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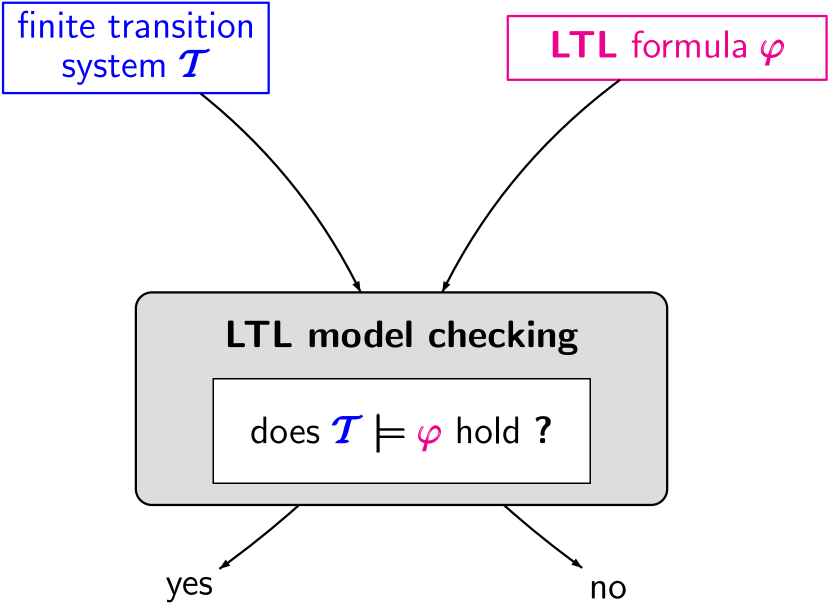
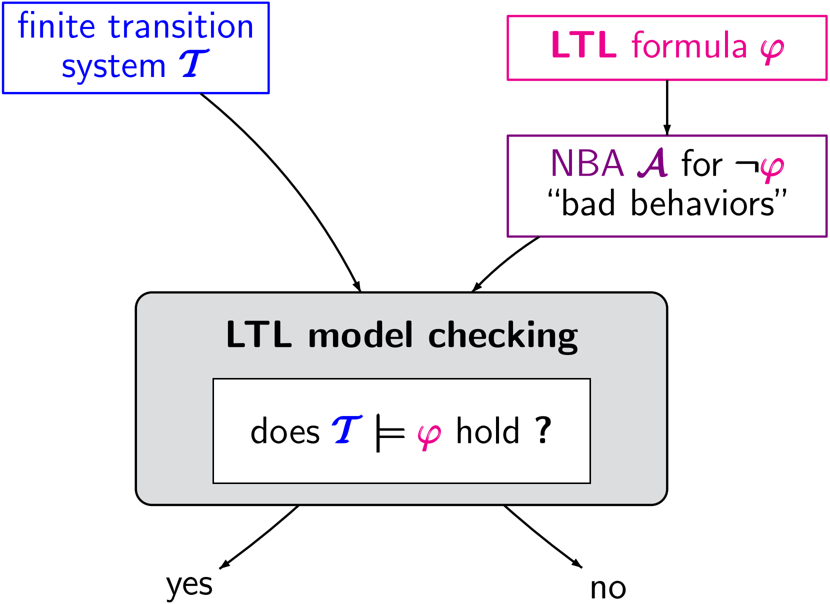
↑

construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets
the acceptance condition of $\mathcal{A}$

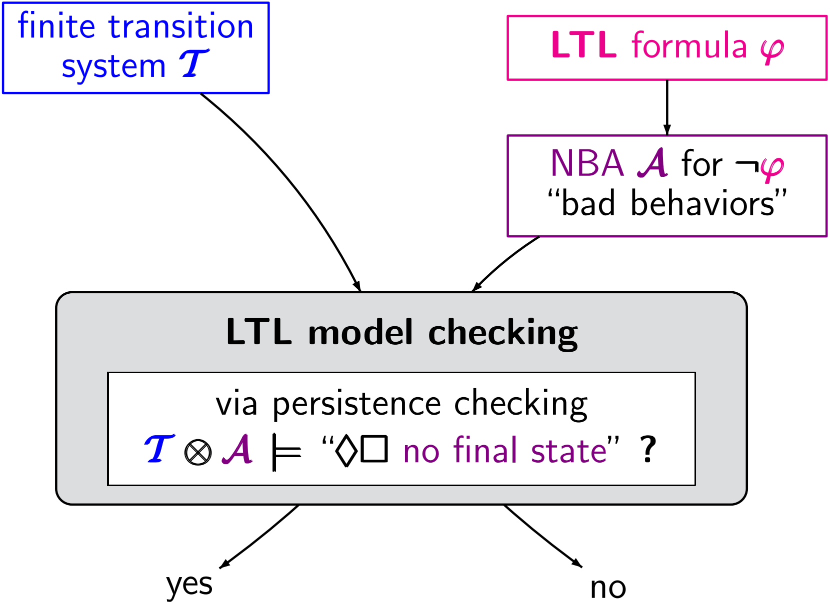
Automata-based LTL model checking





finite transition system $\mathcal{T}$
LTL formula φ
NBA $\mathcal{A}$ for $\neg\varphi$
"bad behaviors"
LTL model checking
does $\mathcal{T} \models \varphi$ hold ?
yes
no

Automata-based LTL model checking



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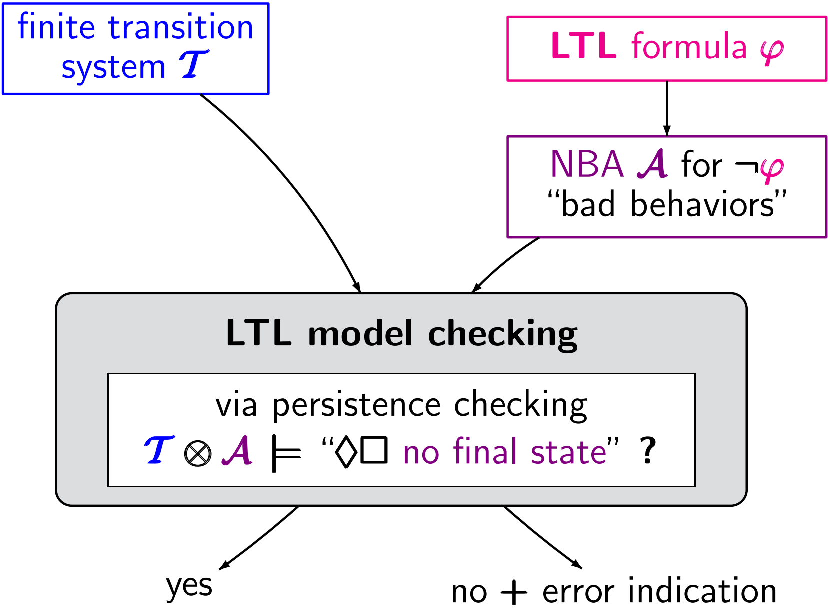

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$\mathcal{T} \models$ safety property E

iff $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}) = \varnothing$

where $\mathcal{A}$ is an NFA for the bad prefixes

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iff $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ ⟵ persistence checking

Recall: nondeterministic Büchi automata

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
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run for a word $A_0\, A_1\, A_2 \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0\, q_1\, q_2 \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run π is accepting if $\overset{\infty}{\exists}\, i \in \mathbb{N}.\ q_i \in F$

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From LTL to NBA

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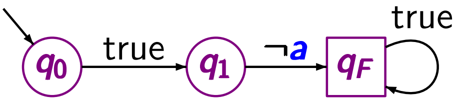
proof: ... later ...

NBA for LTL formulas



$\mathcal{L}_\omega(\mathcal{A}) = \textbf{?}$

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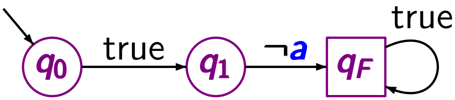

$$\mathcal{L}_\omega(\mathcal{A}) = Words(\bigcirc \neg a)$$

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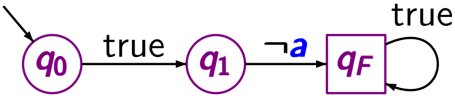

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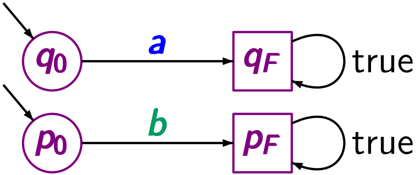

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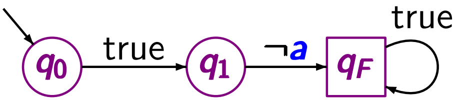

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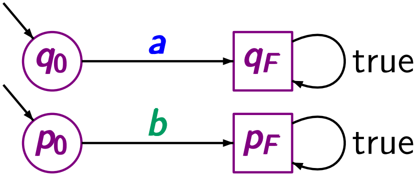

$\mathcal{L}_\omega(\mathcal{A}) = Words(a \vee b)$

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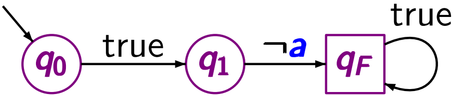

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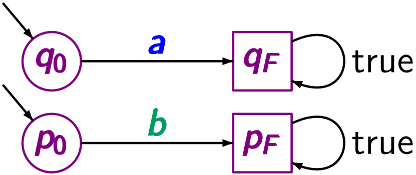

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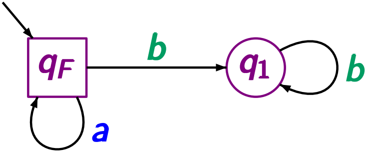

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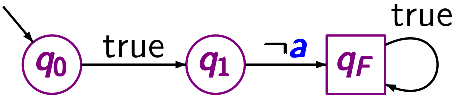

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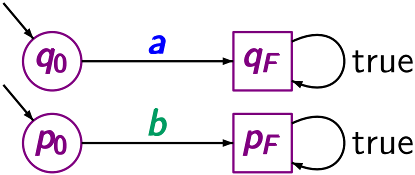

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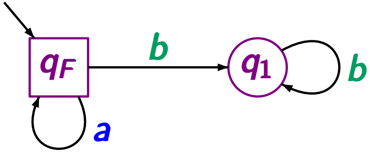

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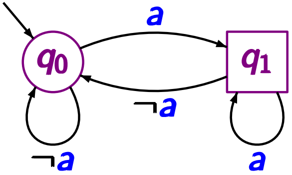
NBA for LTL formulas



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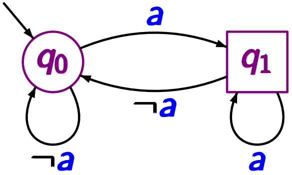

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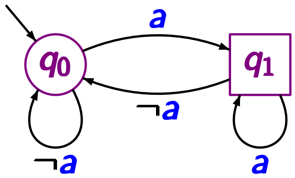

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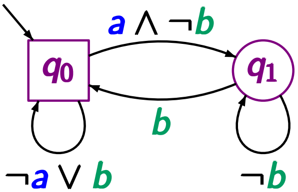

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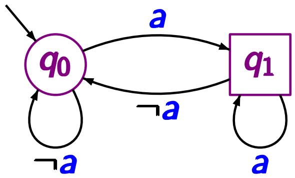

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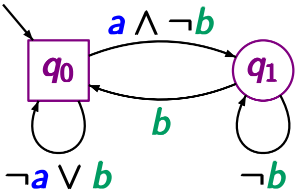

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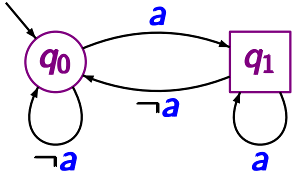
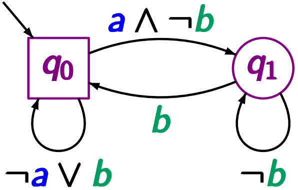
e.g., $\varnothing\,\varnothing\,\varnothing\,\varnothing \ldots = \varnothing^\omega$ and $(\{a\}\,\{b\})^\omega$ are accepted by $\mathcal{A}$

NBA for LTL formulas



$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\Box\Diamond a)$$



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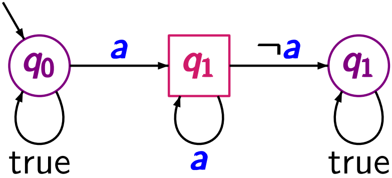
NBA for LTL formula



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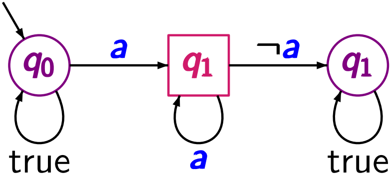

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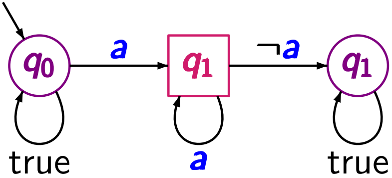

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possible runs for $\{a\}^\omega$

$q_0\ q_0\ q_0\ q_0\ q_0\ q_0\ \ldots$	not accepting
$q_0\ q_1\ q_1\ q_1\ q_1\ q_1\ \ldots$	accepting
$q_0\ q_0\ q_1\ q_1\ q_1\ q_1\ \ldots$	accepting
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$\vdots$

NFA and NBA for safety properties

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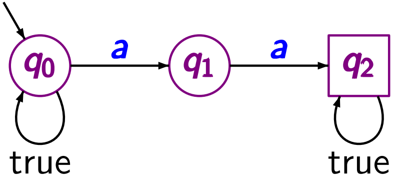
Example: $E \mathrel{\widehat{=}}$ "never a twice in a row"



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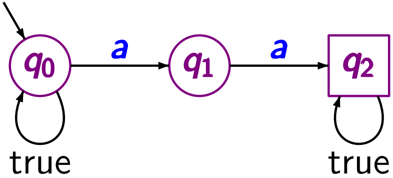
$$\mathcal{L}_\omega(\mathcal{A}) = \overline{E} = \left(2^{AP}\right)^\omega \setminus E = \mathit{Words}(\neg\varphi)$$

Example: $E \mathrel{\widehat{=}}$ "never a twice in a row"



$\varphi = \Box(a \rightarrow \bigcirc\neg a)$

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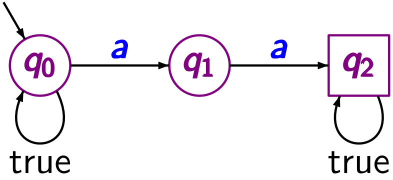

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even if $\mathcal{A}$ is a non-blocking DFA

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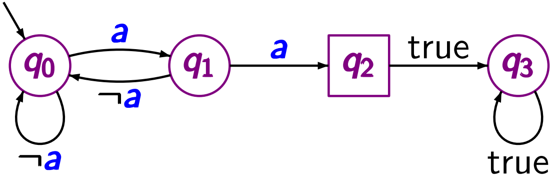
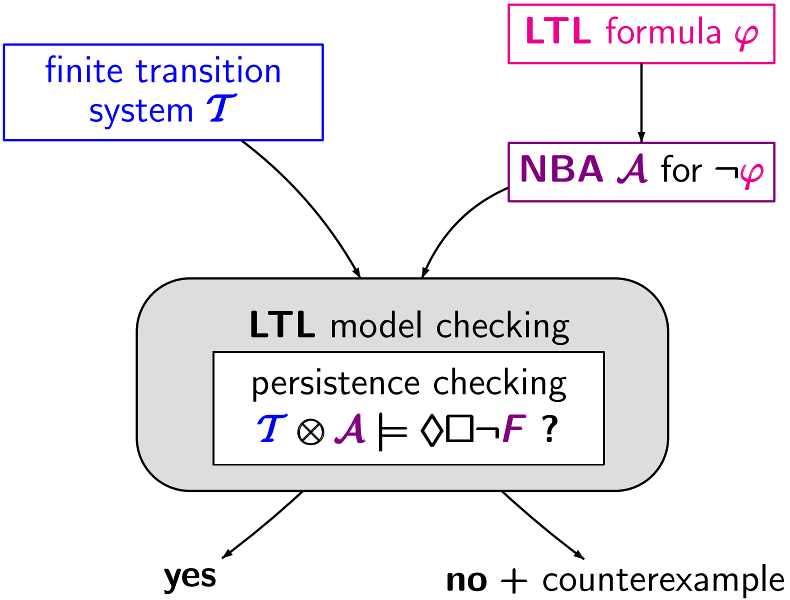


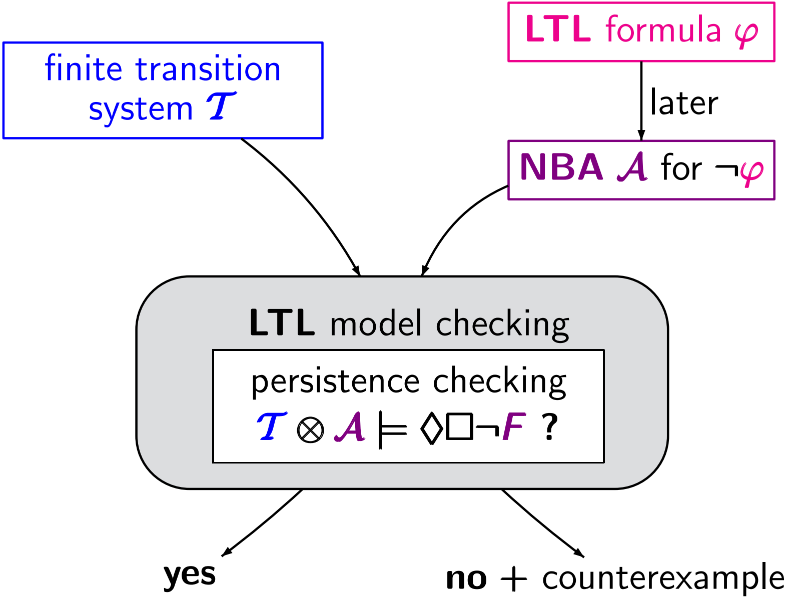
$\mathcal{L}_\omega(\mathcal{A}) = \varnothing$

Note: In the previous slides, you have to assume that the NFA has a self loop labeled true for each accept state (or else that the NFA is deterministic). If the NFA doesn't meet one of these conditions, then when considered as an NBA, it might not recognize E.

LTL model checking



LTL model checking



Recall: product transition system

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

$\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, F)$ NBA or NFA

non-blocking, $Q_0 \cap F = \varnothing$

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initial states: $S_0' = \{\langle s_0, q \rangle : s_0 \in S_0, q \in \delta(Q_0, L(s_0))\}$

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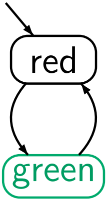
transition relation:

$$\frac{s \xrightarrow{\alpha} s' \ \wedge \ q' \in \delta(q, L(s'))}{\langle s, q \rangle \xrightarrow{\alpha}{}' \langle s', q' \rangle}$$

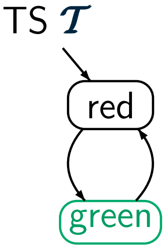
Example: LTL model checking

TS $\mathcal{T}$



LTL formula $\varphi = \Box\Diamond green$

Example: LTL model checking



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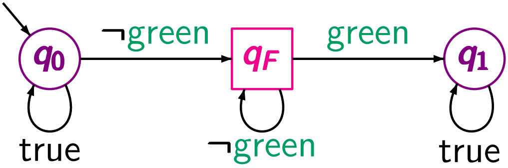
NBA $\mathcal{A}$ for the complement

$\neg\varphi \equiv \Diamond\Box\neg$green



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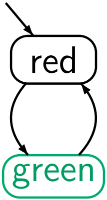

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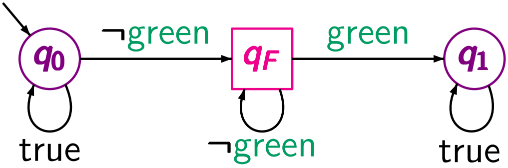
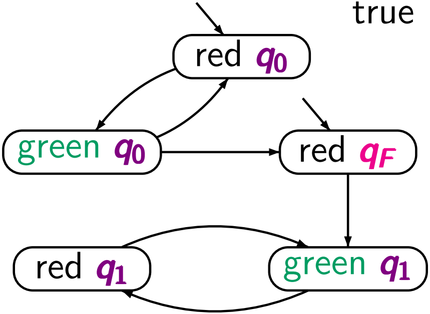
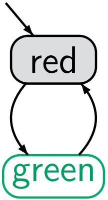




reachable fragment of the product TS $\mathcal{T} \otimes \mathcal{A}$

Example: LTL model checking

TS $\mathcal{T}$



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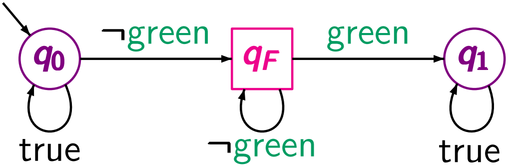
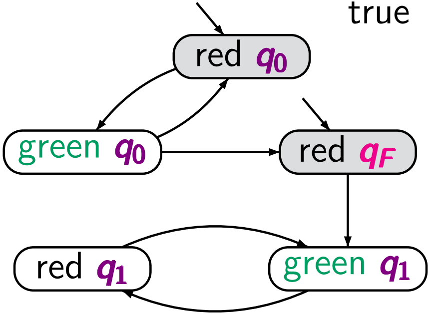




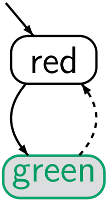
initial states:

$\langle \text{red}, q \rangle$ where

$$q \in \delta(q_0, L(\text{red})) = \delta(q_0, \varnothing) = \{q_0, q_F\}$$

Example: LTL model checking

TS $\mathcal{T}$



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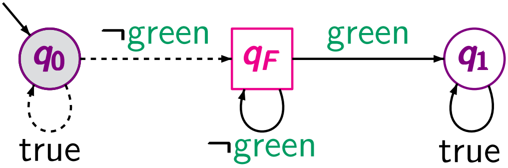

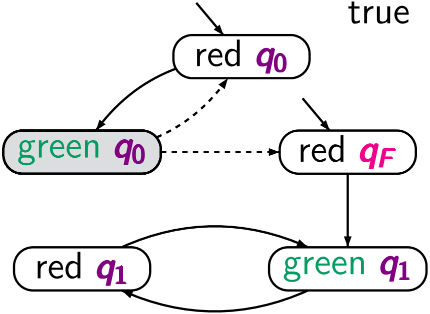

transition

$\langle \text{green}, q_0 \rangle \rightarrow \langle \text{red}, q \rangle$

$$q \in \delta(q_0, L(\text{red})) = \delta(q_0, \varnothing) = \{q_0, q_F\}$$

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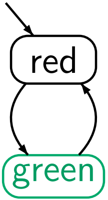

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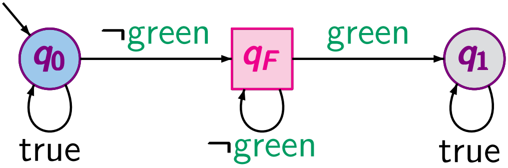
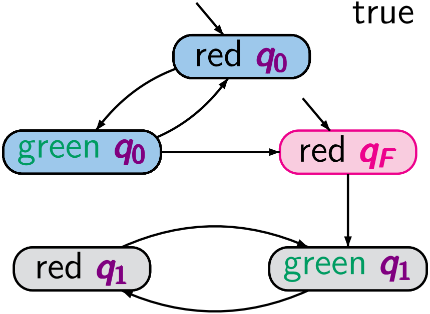




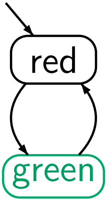
atomic propositions

$AP' = \{q_0, q_F, q_1\}$

obvious labeling function

Example: LTL model checking

TS $\mathcal{T}$



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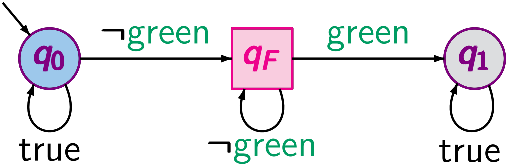
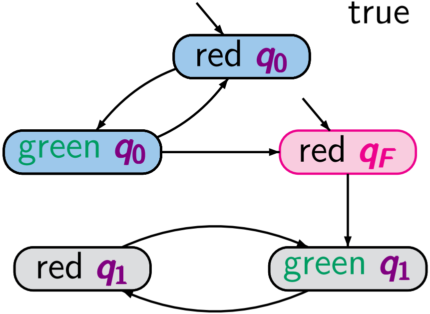
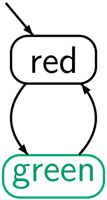




$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

Example: LTL model checking

TS $\mathcal{T}$



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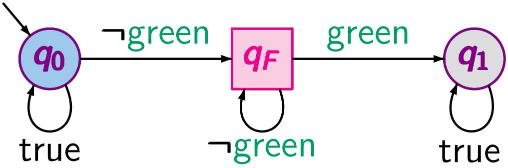

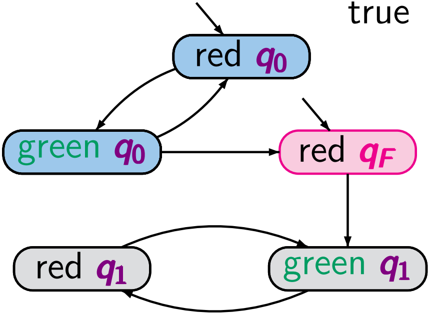

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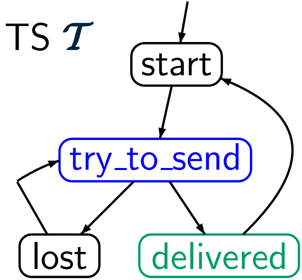
hence: $\mathcal{T} \models \varphi$

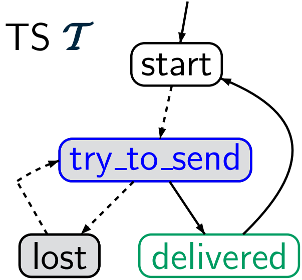
Example: LTL model checking



LTL formula $\varphi = \Box(\textit{try} \rightarrow \Diamond \textit{del})$

"each (repeatedly) sent message will eventually be delivered"

Example: LTL model checking



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Example: LTL model checking

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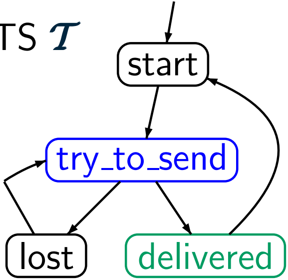
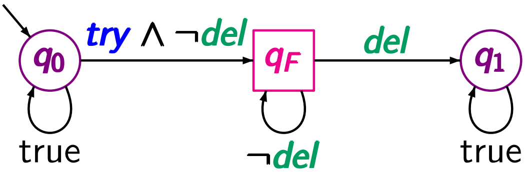


NBA $\mathcal{A}$ for $\neg\varphi \equiv \Diamond(try \wedge \Box\neg del)$



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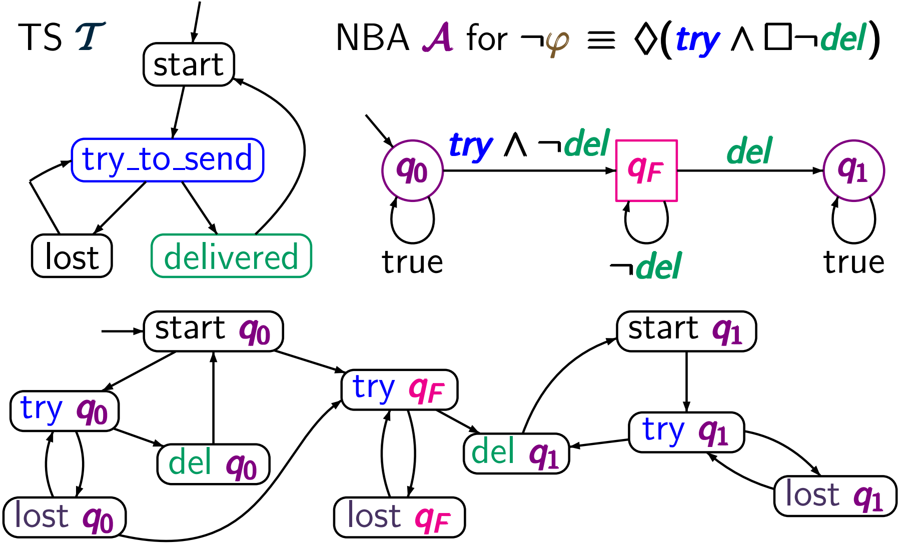
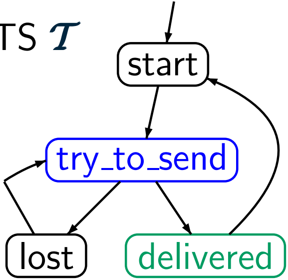


reachable fragment of the product-TS

Example: LTL model checking

TS $\mathcal{T}$



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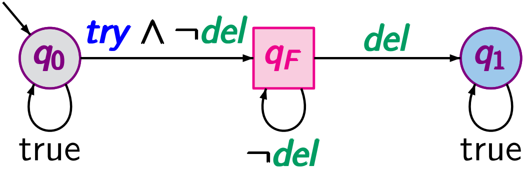
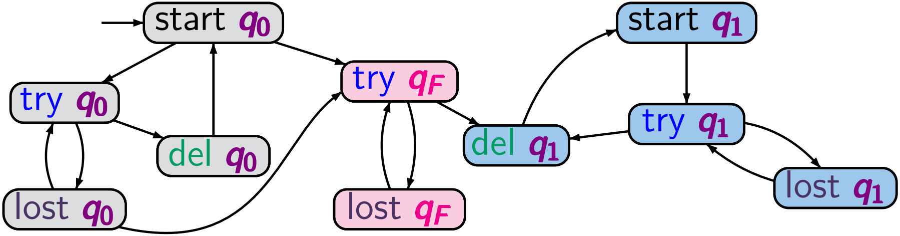
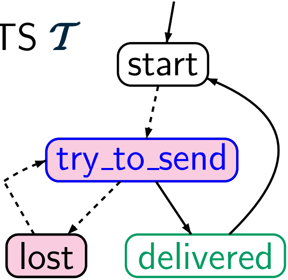




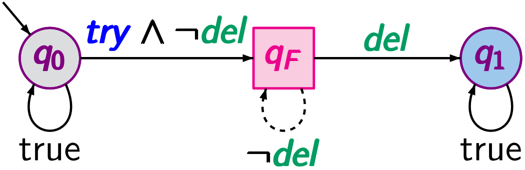
set of atomic propositions $AP' = \{q_0, q_1, q_F\}$

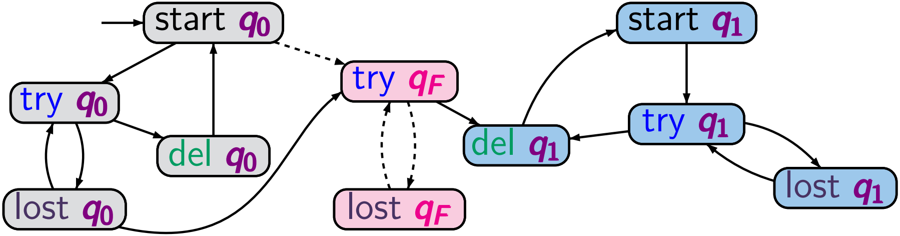
Example: LTL model checking

TS $\mathcal{T}$



NBA $\mathcal{A}$ for $\neg\varphi \equiv \lozenge(try \wedge \square\neg del)$

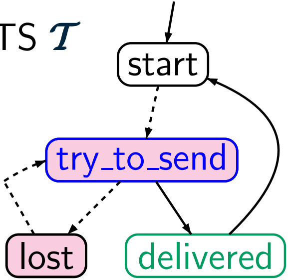




$$\mathcal{T} \otimes \mathcal{A} \not\models \lozenge\square\neg F$$

Example: LTL model checking

TS $\mathcal{T}$



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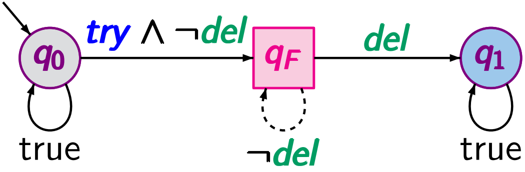

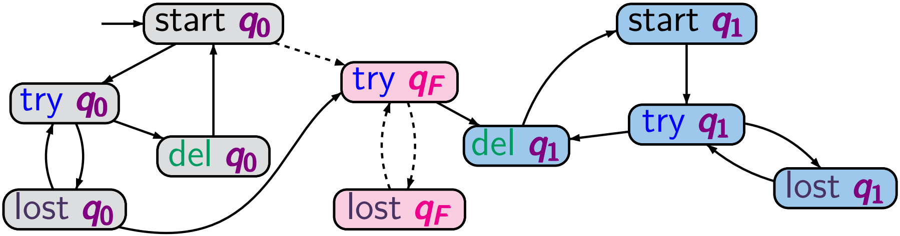

$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$ hence: $\mathcal{T} \not\models \varphi$

LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

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construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

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IF $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

THEN return "**yes**"

ELSE compute a counterexample

$$\langle s_0, p_0\rangle \ldots \langle s_n, p_n\rangle \ldots \langle s_n, p_n\rangle$$

for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$

return "**no**" and $s_0 \ldots s_n \ldots s_n$

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

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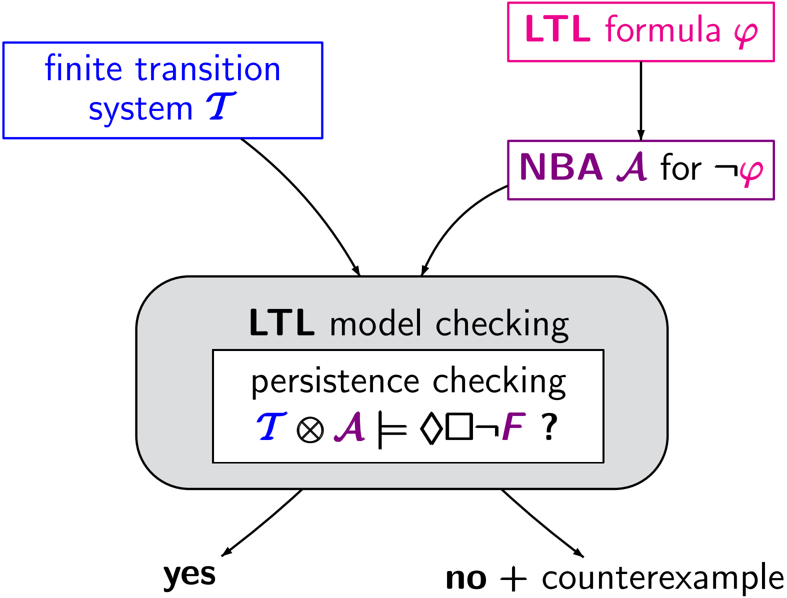
for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$

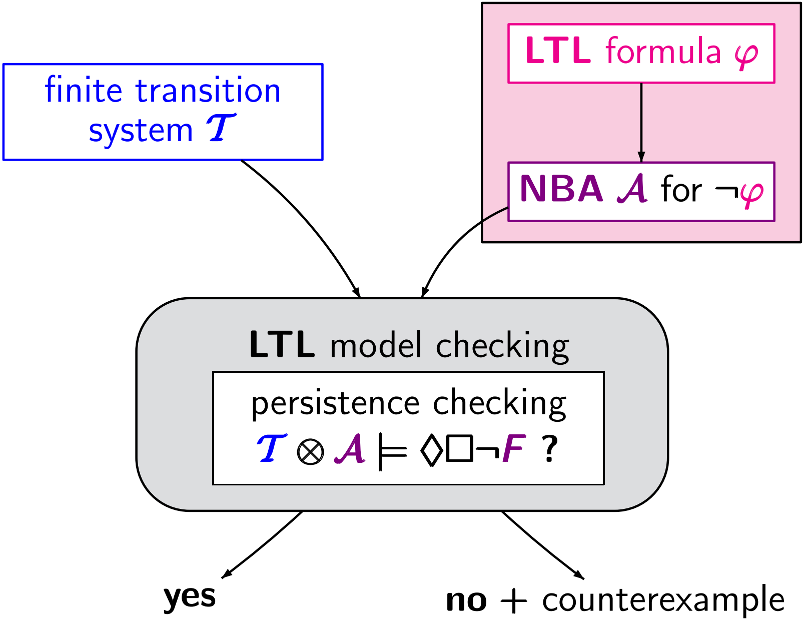
return "**no**" and $s_0 \ldots s_n \ldots s_n$

time complexity: $\mathcal{O}(size(\mathcal{T}) \cdot size(\mathcal{A}))$

LTL model checking



LTL model checking



From LTL to NBA

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For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$$

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LTL formula φ

↓

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nondeterministic
Büchi automaton

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LTL formula φ

↓

GNBA $\mathcal{G}$ s.t. $\mathcal{L}_\omega(\mathcal{G}) = Words(\varphi)$ — generalized NBA, several acceptance sets

↓

NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$ — nondeterministic Büchi automaton, **1** acceptance set

From LTL to NBA

For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) = Words(\varphi)$$

LTL formula φ

↓

GNBA $\mathcal{G}$ s.t. $\mathcal{L}_\omega(\mathcal{G}) = Words(\varphi)$

generalized NBA
k acceptance sets

↓ (k copies of $\mathcal{G}$)

NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

nondeterministic
Büchi automaton
1 acceptance set

Encoding of LTL semantics in a GNBA

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idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

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semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	
next $\bigcirc$	
until **U**	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic *true*, $\neg$, $\wedge$	in the states
next $\bigcirc$	
until $\mathbf{U}$	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until **U**	

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until U	via expansion law

$$\psi_1 \, \mathsf{U} \, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \, \mathsf{U} \, \psi_2))$$

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propositional logic *true*, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until **U**	via expansion law

$$\psi_1 \, \mathsf{U} \, \psi_2 \equiv \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \, \mathsf{U} \, \psi_2))$$

encoded in the states

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic *true*, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until U	via expansion law

$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

encoded in the states (ψ_2, ψ_1)

encoded in the transition relation ($\bigcirc$)

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, ¬, ∧	in the states
next $\bigcirc$	in the transition relation
until **U**	expansion law, **least fixed point**

$$\psi_1 \, \mathsf{U} \, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \, \mathsf{U} \, \psi_2))$$

- ψ_2, ψ_1: encoded in the states
- $\bigcirc$: encoded in the transition relation
- least fixed point: acceptance condition

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for *Words*(φ)

states of $\mathcal{G}$ $\quad \widehat{=} \quad$ (certain) sets of subformulas of φ

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0 A_1 A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0 B_1 B_2 \ldots$ in $\mathcal{G}$

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$$A_0 \quad A_1 \quad A_2 \quad A_3 \quad \ldots \quad \in Words(\varphi)$$

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$$\begin{array}{cccccl} A_0 & A_1 & A_2 & A_3 & \ldots & \in Words(\varphi) \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run} \end{array}$$

where $B_i = \{\psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\}$

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$$\begin{array}{cccccl} A_0 & A_1 & A_2 & A_3 & \ldots & \in Words(\varphi) \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run} \end{array}$$

where $B_i = \{\psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\}$

$cl(\varphi)$: set of subformulas of φ and their negations

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Example: $\varphi = a \, \mathsf{U} (\neg a \wedge b)$

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Example: $\varphi = a \mathsf{U} (\neg a \wedge b)$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G} \;\widehat{=}\;$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\,A_1\,A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0\,B_1\,B_2 \ldots$ in $\mathcal{G}$

Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b)$

$$
\begin{array}{ccccccc}
\{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
B_0 & B_1 & B_2 & B_3 & B_4 & B_5 &
\end{array}
$$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

Example: $\varphi = a \, \mathsf{U}(\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

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Example: $\varphi = a\, \mathsf{U} (\neg a \wedge b)$ $\psi = \neg a \wedge b$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$\downarrow$

a
$\neg b$
$\neg \psi$
φ

just for better readability: tuple rather than set notation

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Example: $\varphi = a\,\mathsf{U}(\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$

$\{a\}$	$\{a\}$
$\downarrow$	$\downarrow$
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ

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Example: $\varphi = a\,\mathsf{U}(\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$

$\{a\}$	$\{a\}$	$\{a, b\}$
$\downarrow$	$\downarrow$	$\downarrow$
a	a	a
$\neg b$	$\neg b$	b
$\neg \psi$	$\neg \psi$	$\neg \psi$
φ	φ	φ

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Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
a	a	a	$\neg a$
$\neg b$	$\neg b$	b	b
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ
φ	φ	φ	φ

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Example: $\varphi = a\, \mathsf{U} (\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
a	a	a	$\neg a$	$\neg a$			
$\neg b$	$\neg b$	b	b	$\neg b$			
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ	$\neg \psi$			
φ	φ	φ	φ	$\neg \varphi$			

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ	a, b, $\neg\psi$, φ	$\neg a$, b, ψ, φ	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\ldots$	

Closure of LTL formulas

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Let φ be an LTL formula. Then:

$$\mathit{subf}(\varphi) \stackrel{\text{def}}{=} \text{set of all subformulas of } \varphi$$

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where ψ and $\neg\neg\psi$ are identified

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Example: if $\varphi = a \,\mathsf{U}(\neg a \wedge b)$ then

$$cl(\varphi) = \{a, b, \neg a \wedge b, \varphi\} \cup \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$$

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Example: if $\varphi' = \Box a$

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Example: if $\varphi = a\,\mathsf{U}(\neg a \wedge b)$ then

$$cl(\varphi) = \{a, b, \neg a \wedge b, \varphi\} \cup \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$$

Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(true\,\mathsf{U}\,\neg a)$ then

$$cl(\varphi') = \{a, \neg a, true, \neg true, \Box a, \neg\Box a\}$$

Elementary formula-sets

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Let $B \subseteq cl(\varphi)$. B is called elementary if:

Elementary formula-sets

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(1) B is consistent w.r.t. propositional logic

(2) B is maximal consistent

(3) B is locally consistent with respect to until **U**:

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

if $\psi \in B$ then $\neg\psi \notin B$

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if $\psi \in B$ then $\neg\psi \notin B$

if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$

(2) B is maximal consistent

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if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$
if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$

(2) B is maximal consistent

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if $\psi \in B$ then $\neg\psi \notin B$
if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$
if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$
if $false \in cl(\varphi)$ then $false \notin B$

(2) B is maximal consistent

(3) B is locally consistent with respect to until $\mathbf{U}$:

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

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if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$

if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$

if $\mathit{false} \in cl(\varphi)$ then $\mathit{false} \notin B$

(2) B is maximal consistent

if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

(3) B is locally consistent with respect to until $\mathbf{U}$:

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

if $\psi \in B$ then $\neg\psi \notin B$
if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$
if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$
if $\mathit{false} \in cl(\varphi)$ then $\mathit{false} \notin B$

(2) B is maximal consistent

if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

(3) B is locally consistent with respect to until $\mathbf{U}$:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

if $\psi \in B$ then $\neg\psi \notin B$
if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$
if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$
if $\mathit{false} \in cl(\varphi)$ then $\mathit{false} \notin B$

(2) B is maximal consistent

if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

(3) B is locally consistent with respect to until U:

if $\psi_1 \,\mathsf{U}\, \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

if $\psi_2 \in B$ and $\psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)$ then $\neg(\psi_1 \,\mathsf{U}\, \psi_2) \notin B$

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$\psi \notin B$	iff	$\neg\psi \in B$
$\psi_1 \wedge \psi_2 \in B$	iff	$\psi_1 \in B$ and $\psi_2 \in B$
$\mathit{true} \in cl(\varphi)$	implies	$\mathit{true} \in B$

(ii) B is locally consistent with respect to until $\mathbf{U}$, i.e., if $\psi_1 \,\mathbf{U}\, \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \,\mathbf{U}\, \psi_2 \in B$

Elementary or not?

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$

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$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary

propositional inconsistent

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$

Elementary or not?

Let $\varphi = a \,\mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary

propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ not elementary, not maximal

as $\neg a \wedge b \notin B_2$

$\neg(\neg a \wedge b) \notin B_2$

Elementary or not?

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ — not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ — not elementary, not maximal
as $\neg a \wedge b \notin B_2$
$\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$

Elementary or not?

Let $\varphi = a\,\mathbf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ — not elementary, propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ — not elementary, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ — not elementary, not locally consistent for **U**

Elementary or not?

Let $\varphi = a \, \mathsf{U} (\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ not elementary, not maximal
as $\neg a \wedge b \notin B_2$
$\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ not elementary
not locally consistent for **U**

$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Elementary or not?

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ — not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ — not elementary, not maximal
as $\neg a \wedge b \notin B_2$
$\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ — not elementary
not locally consistent for **U**

$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$ — elementary

Example: elementary formula-sets

closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. U

For $\varphi = a\,\mathsf{U}(\neg a \wedge b)$, the elementary sets are:

$\{\ a,\ b, \neg(\neg a \wedge b), \varphi\}$	$\{\ a,\ b, \neg(\neg a \wedge b), \neg\varphi\}$
$\{\ a, \neg b, \neg(\neg a \wedge b), \varphi\}$	$\{\ a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$
$\{\neg a,\ b,\ \neg a \wedge b, \varphi\}$	$\{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$:

semantics of ...	encoding
propositional logic *true*, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until **U**	expansion law, least fixed point

$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

- ψ_2, ψ_1: encoded in the states
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Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$:

semantics of ...	encoding
propositional logic *true*, $\neg$, $\wedge$	in the states ⟵ elementary formula sets
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- ψ_2, ψ_1: elementary formula sets
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- least fixed point: acceptance condition

GNBA for LTL-formula φ

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$$\psi_1 \, \mathsf{U} \, \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \, \mathsf{U} \, \psi_2 \in B')$$

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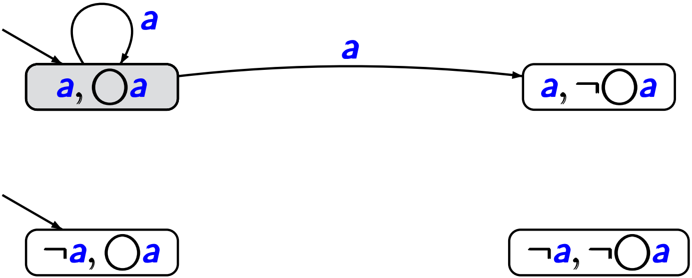

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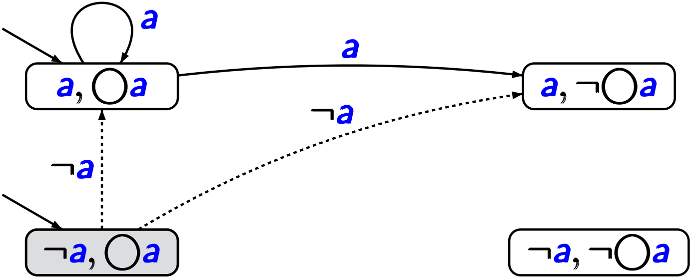

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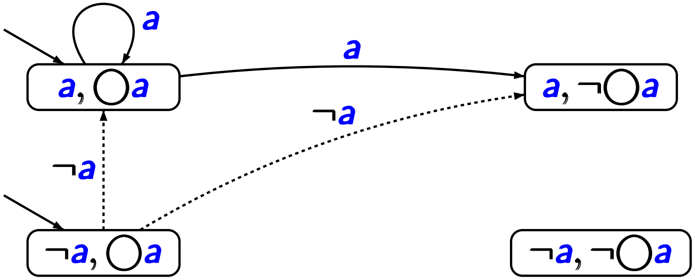

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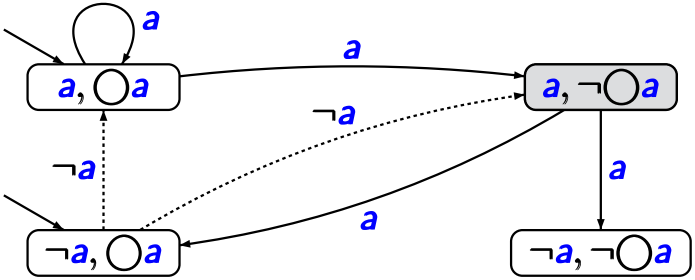

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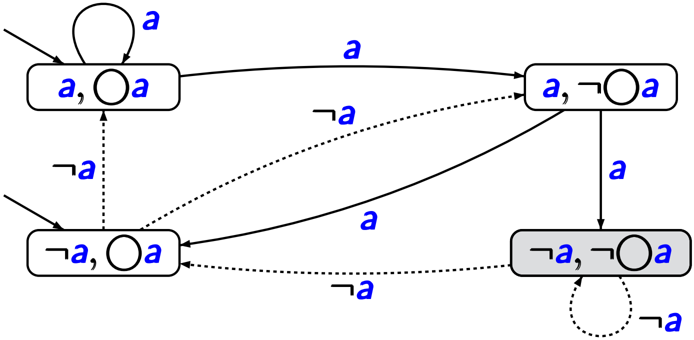

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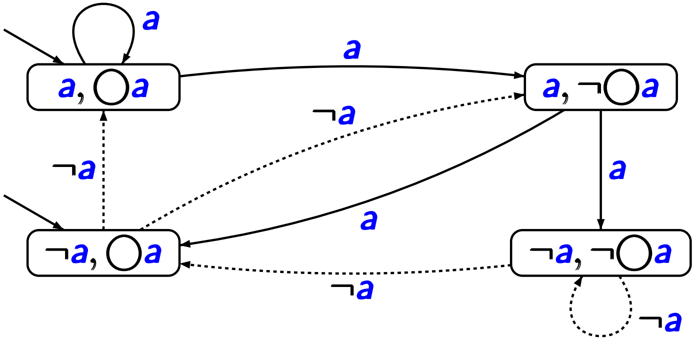
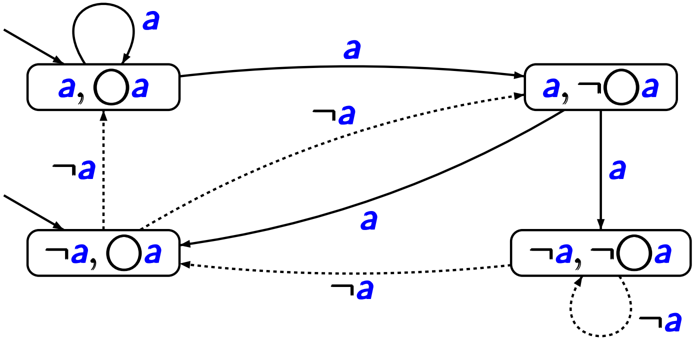


set of acceptance sets:

Example: GNBA for $\varphi = \bigcirc a$



set of acceptance sets: $\mathcal{F} = \varnothing$

hence: all words having an infinite run are accepted

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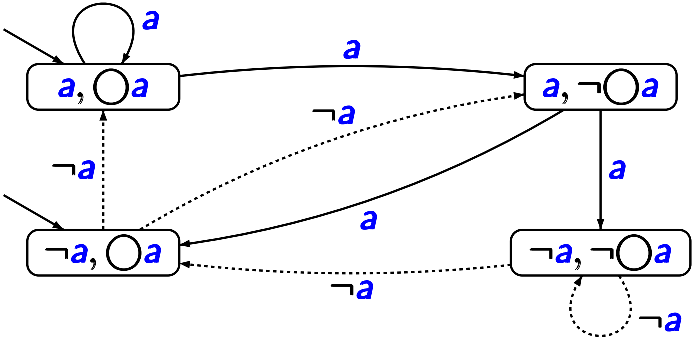

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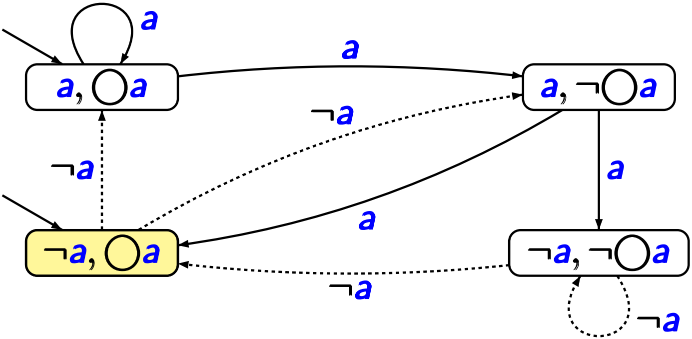

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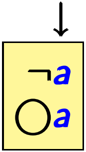

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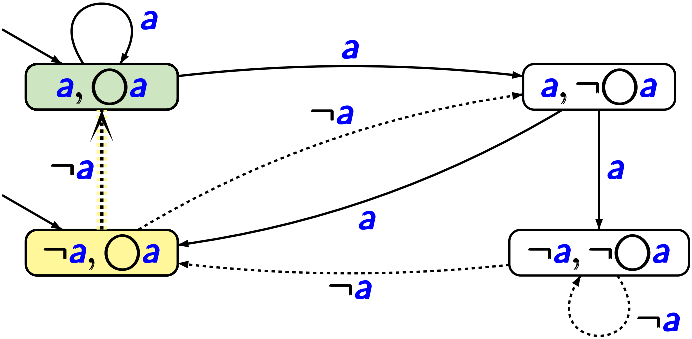

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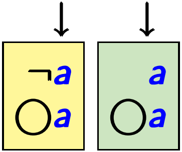

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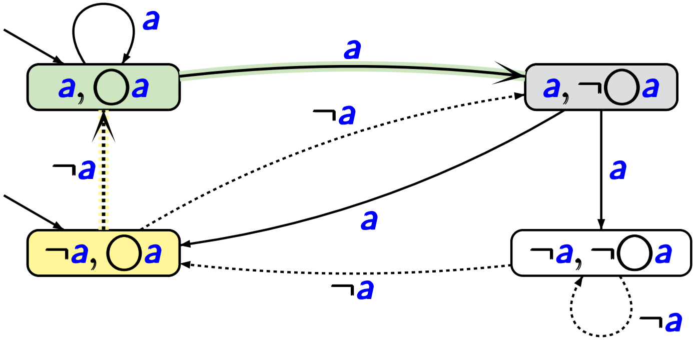

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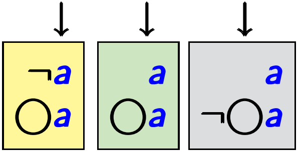

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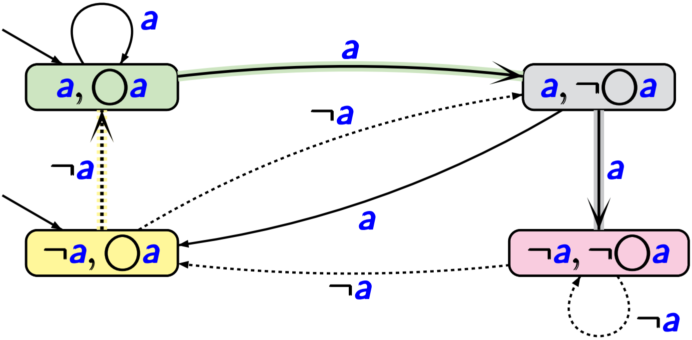

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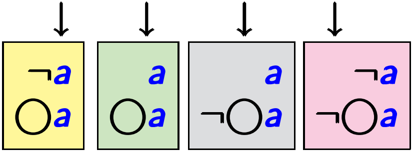

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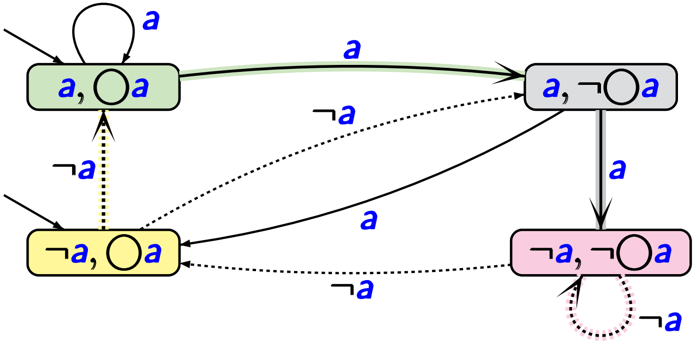

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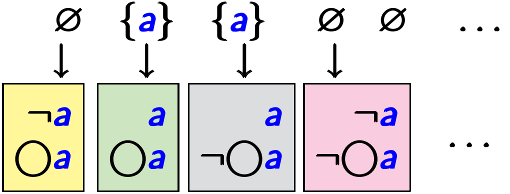
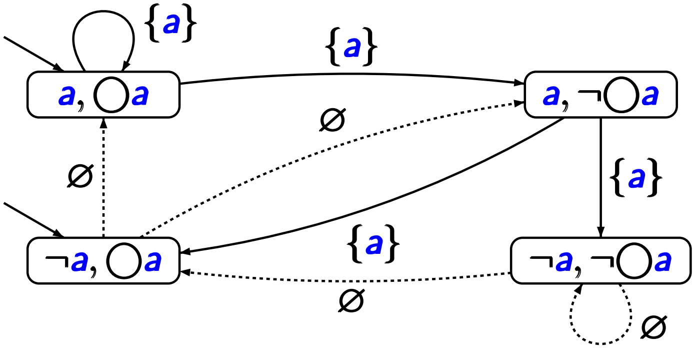


$\models \bigcirc a$

accepting run

Soundness of the GNBA for $\varphi = \bigcirc a$



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

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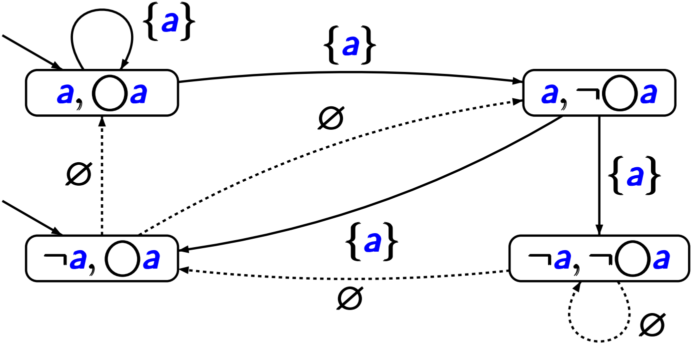

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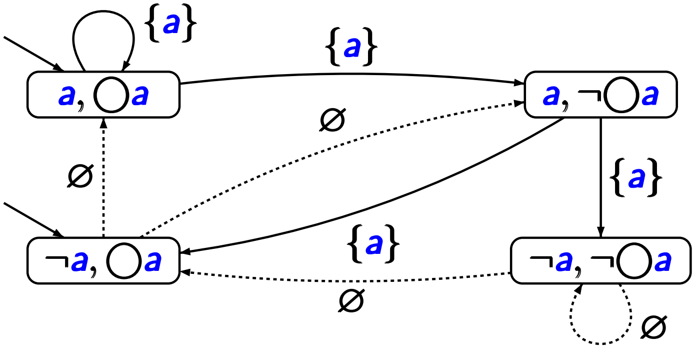

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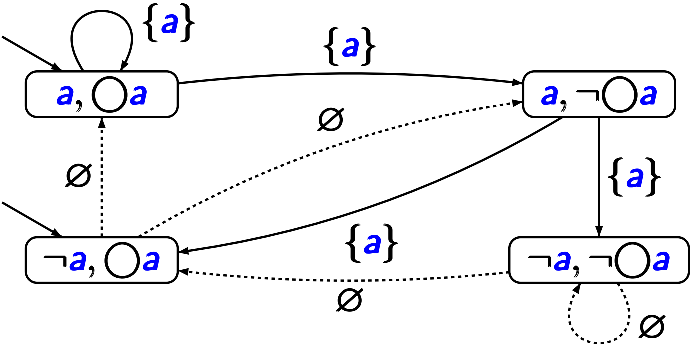

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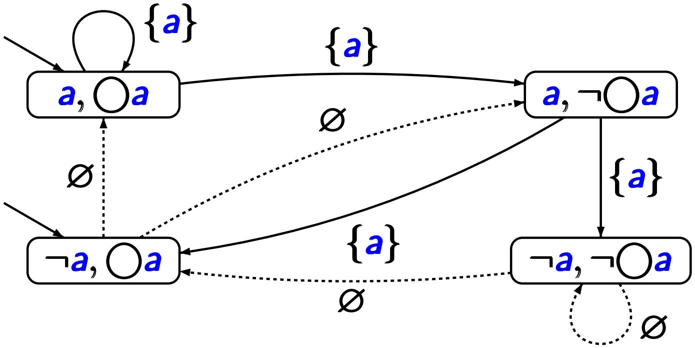

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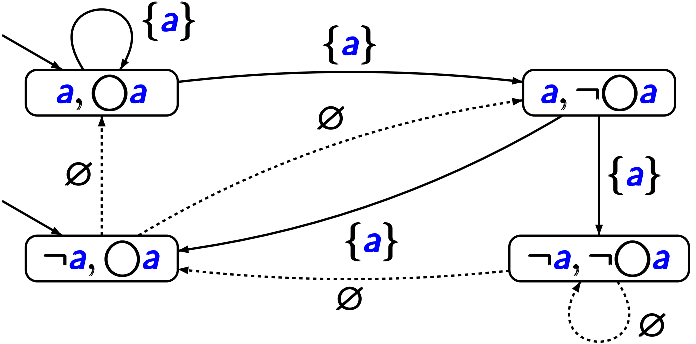

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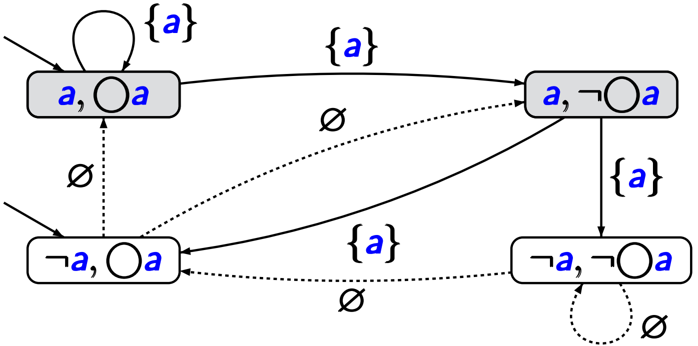

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$\Longrightarrow \quad \{a\} = B_1 \cap AP = A_1$

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$\neg a, \neg b, \neg(a \mathsf{U} b)$

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$a, \neg b, \neg(a \mathsf{U} b)$

$\neg a, b, a \mathsf{U} b$

locally inconsistent: $\{a, b, \neg(a \mathsf{U} b)\}$

$\{\neg a, b, \neg(a \mathsf{U} b)\}$

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$\neg a, b, a \, \mathsf{U} \, b$

initial states: B with $\varphi = a \, \mathsf{U} \, b \in B$

Example: GNBA for $\varphi = a \mathsf{U} b$

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$\neg a, \neg b, \neg(a \mathsf{U} b)$

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initial states: B with $\varphi = a \, \mathsf{U} \, b \in B$

acceptance condition: just one set of accept states

F = set of all B with $\varphi \notin B$ or $b \in B$

$\longrightarrow$ $a, b, a \, \mathsf{U} \, b$

$\neg a, \neg b, \neg(a \, \mathsf{U} \, b)$

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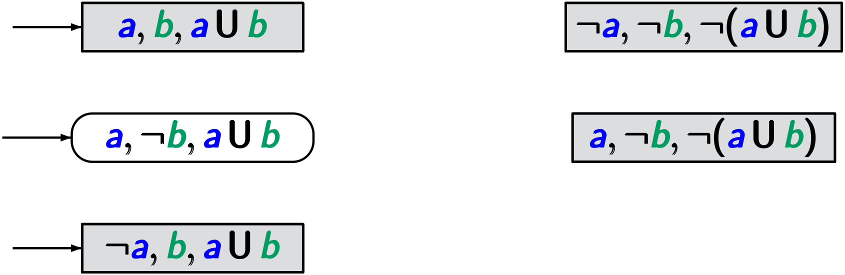

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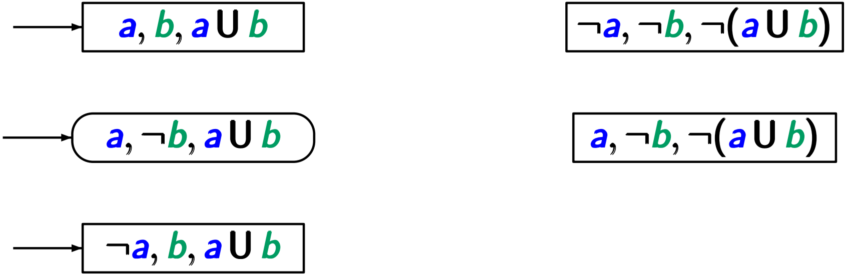

transition relation: $B' \in \delta(B, B \cap AP)$ iff

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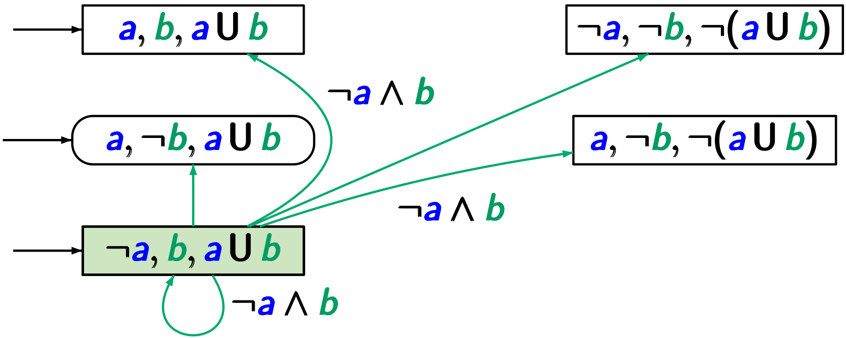

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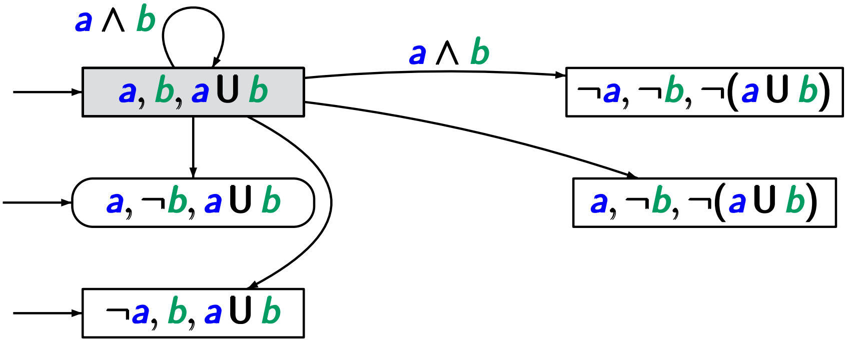

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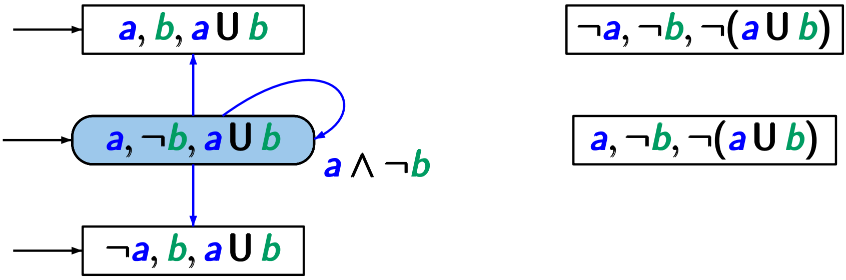

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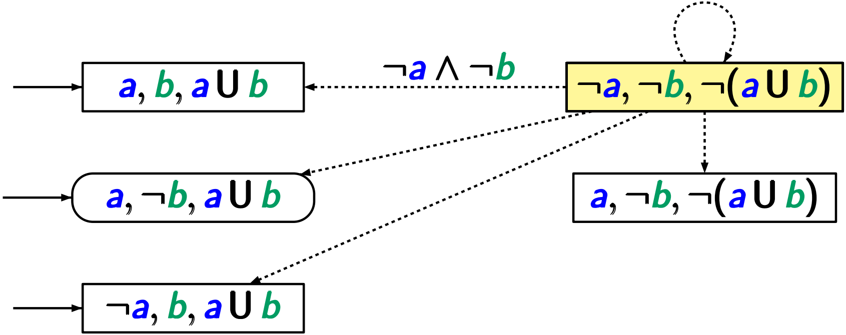

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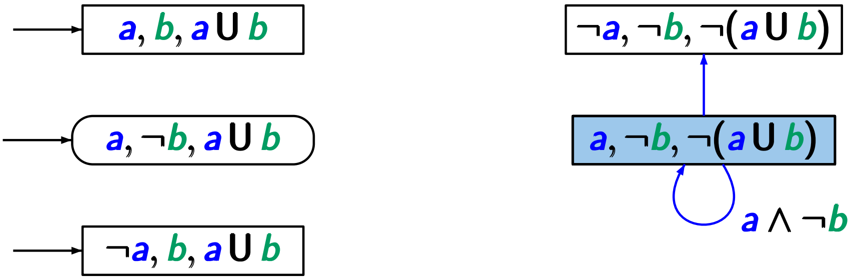

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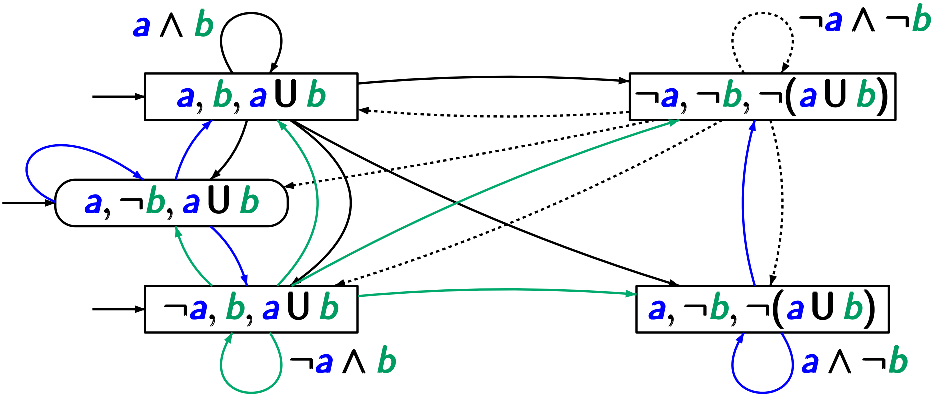
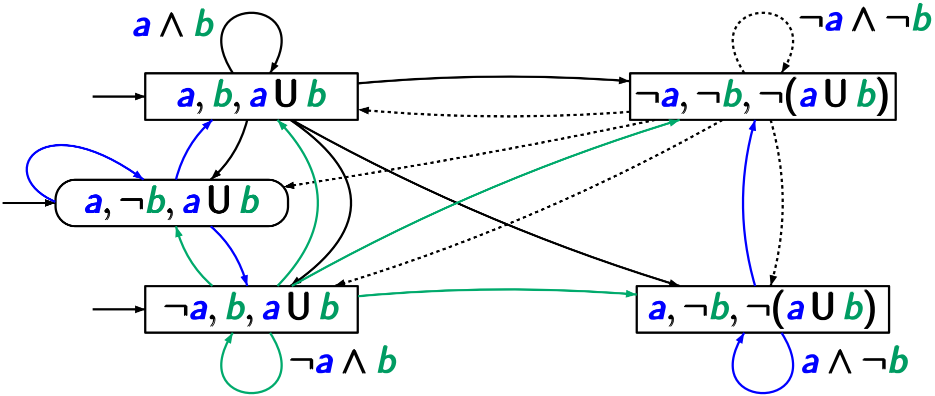
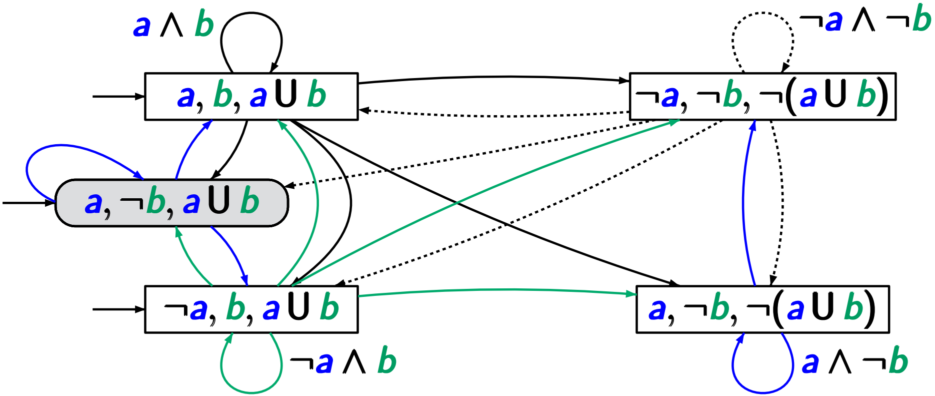


Example: (G)NBA for $\varphi = a \mathsf{U} b$



$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \mathsf{U} b$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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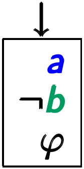

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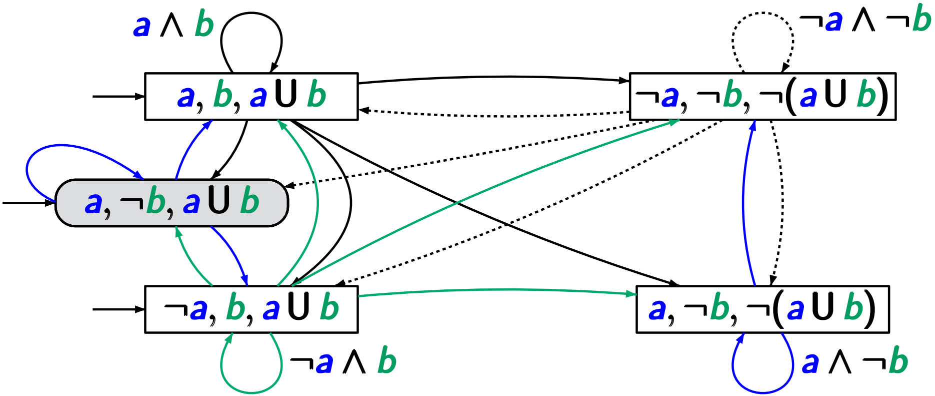

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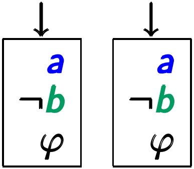

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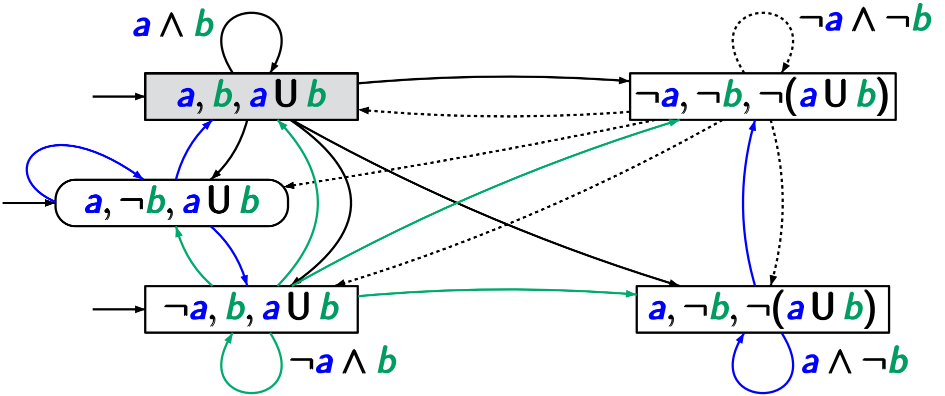

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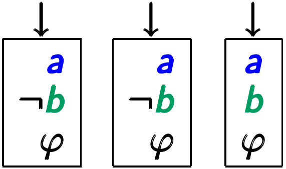

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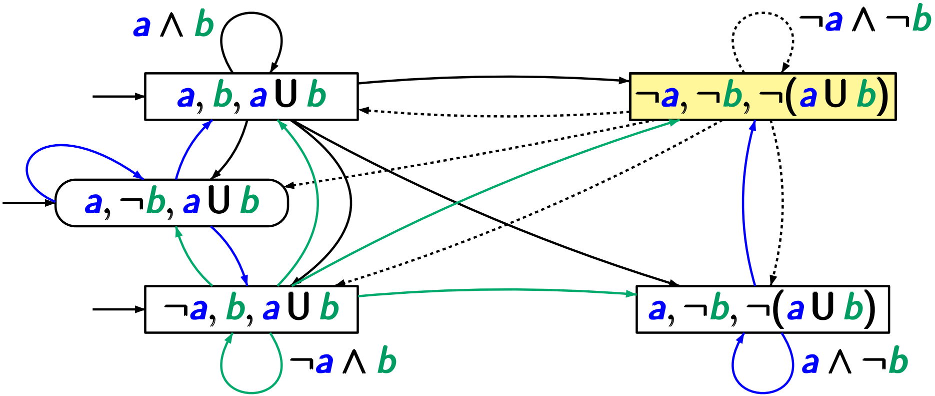

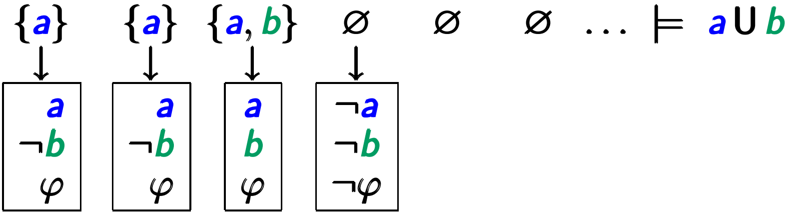

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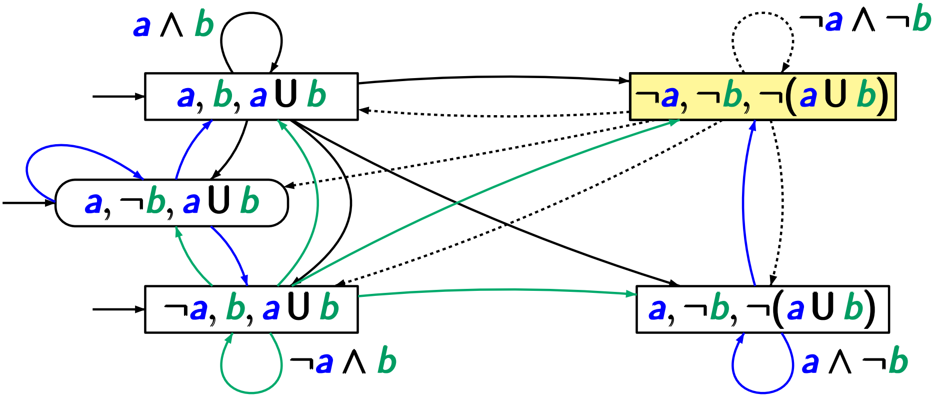

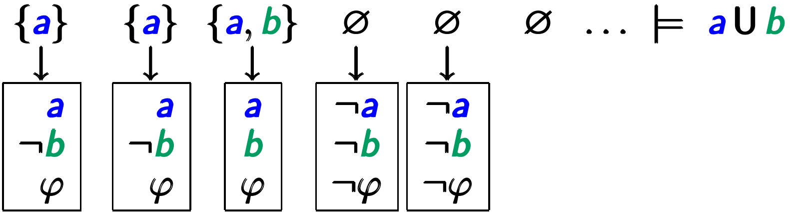

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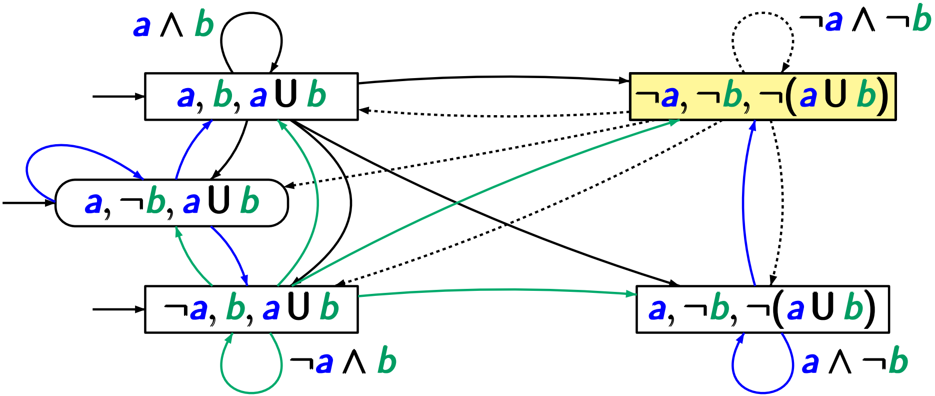
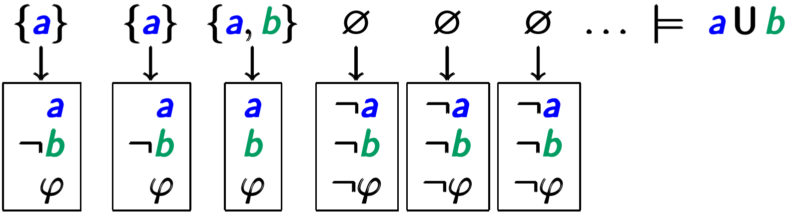
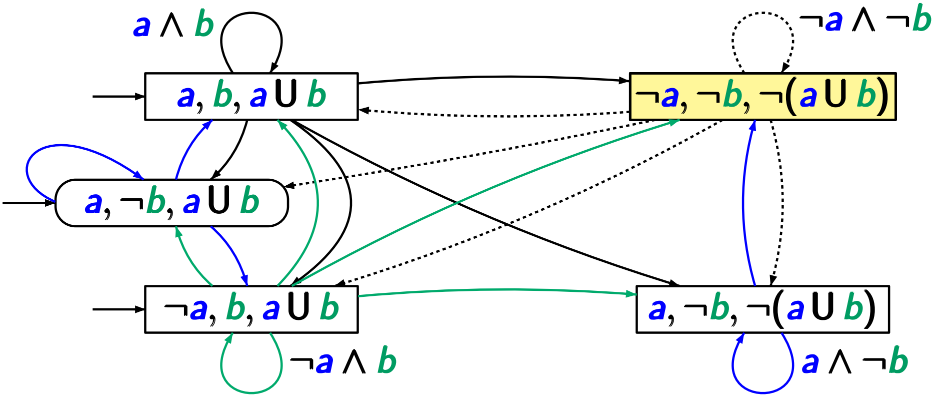
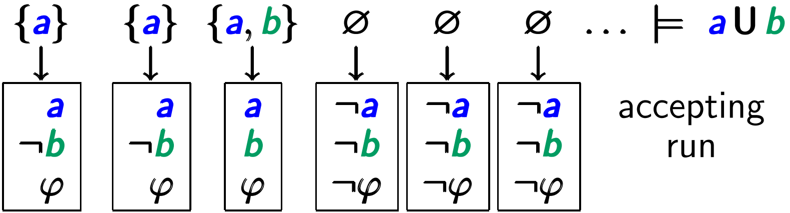




Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$





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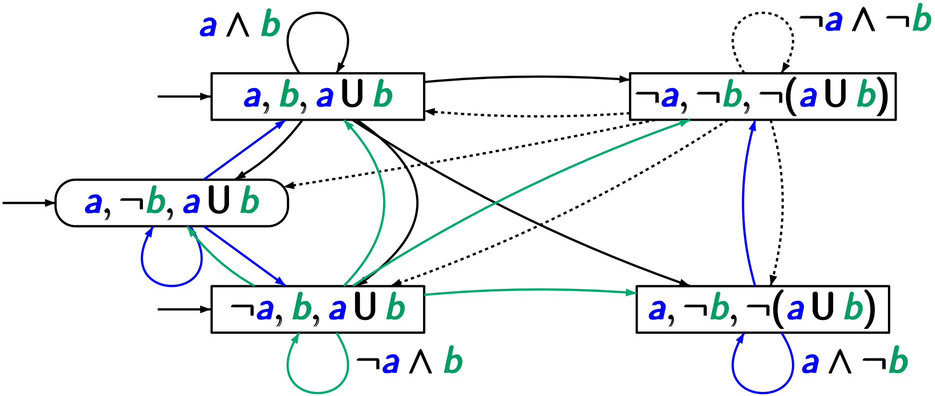

$$\{a\}\,\{a\}\,\{a\}\,\{a\}\,\ldots \not\models \varphi$$

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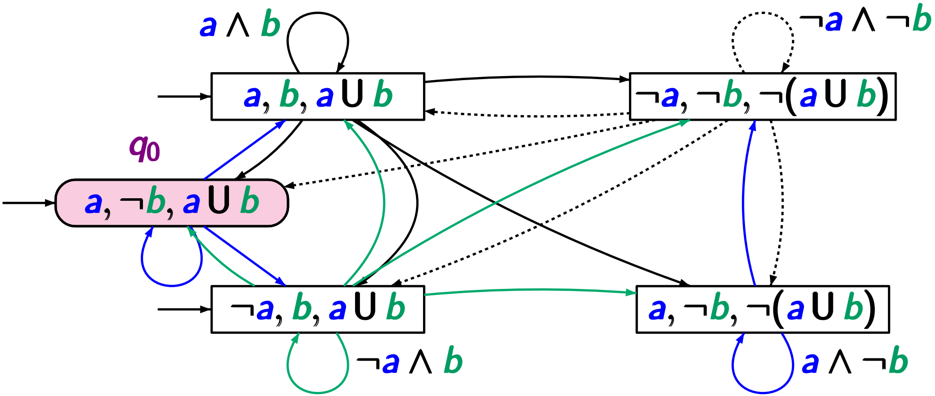

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only **1** infinite run: $q_0\, q_0\, q_0\, \ldots$

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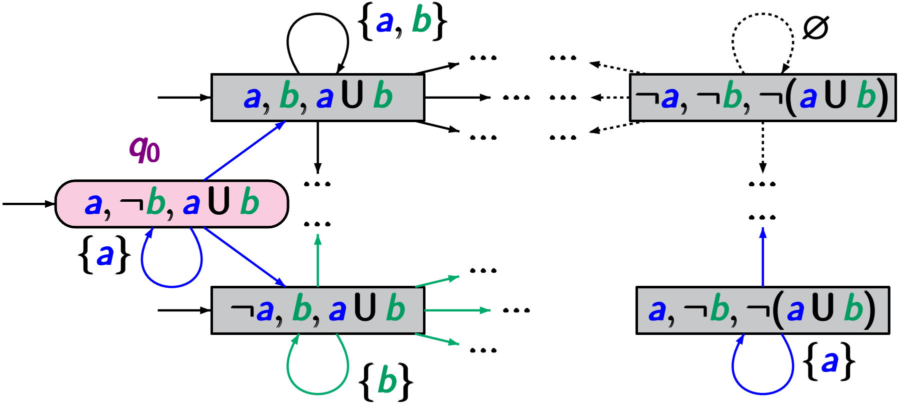

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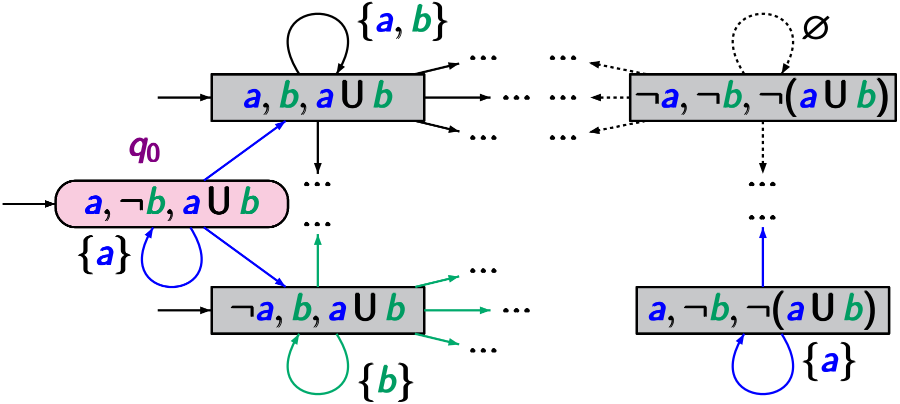

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GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

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$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

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Soundness

.... of the construction LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

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"$\subseteq$" show: each infinite word $A_0 A_1 A_2 \ldots \in (2^{AP})^\omega$

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Accepting runs for the elements of *Words*(φ)

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ elementary formula-sets $B \subseteq cl(\varphi)$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are states in $\mathcal{G}$, i.e., elementary subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

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$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$$\downarrow$$

$$\boxed{\begin{array}{r} a \\ \neg b \\ \neg \psi \\ \varphi \end{array}}$$

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$\{a\}$	$\{a\}$
$\downarrow$	$\downarrow$
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ

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$$\downarrow \quad \downarrow \quad \downarrow$$

a	a	a
$\neg b$	$\neg b$	b
$\neg \psi$	$\neg \psi$	$\neg \psi$
φ	φ	φ

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$				
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ	a, b, $\neg\psi$, φ	$\neg a$, b, ψ, φ				

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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
a	a	a	$\neg a$	$\neg a$			
$\neg b$	$\neg b$	b	b	$\neg b$			
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ	$\neg \psi$			
φ	φ	φ	φ	$\neg \varphi$			

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ	a, b, $\neg\psi$, φ	$\neg a$, b, ψ, φ	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\ldots$	

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

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Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$\psi \notin B$	iff	$\neg\psi \in B$
$\psi_1 \wedge \psi_2 \in B$	iff	$\psi_1 \in B$ and $\psi_2 \in B$
$\mathit{true} \in cl(\varphi)$	implies	$\mathit{true} \in B$

(ii) B is locally consistent with respect to until $\mathbf{U}$, i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \mathbf{U} \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \mathbf{U} \psi_2 \in B$

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\,A_1\,A_2 \ldots \in (2^{AP})^\omega$

with $A_0\,A_1\,A_2 \ldots \models \varphi$

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Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

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$\Longrightarrow$ there is an accepting run $B_0\, B_1\, B_2 \ldots$ for σ

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$\Longrightarrow$ $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t. $\varphi \in B_0$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

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as $B_0 \in Q_0$

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and (*) holds

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and $(*)$ holds

$\Longrightarrow$ $\sigma = A_0\,A_1\,A_2\ldots \models \varphi$

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

$\Longrightarrow$ there is an accepting run $B_0\, B_1\, B_2 \ldots$ for σ

$\Longrightarrow$ $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t. $\varphi \in B_0$ and $(*)$ holds

($\varphi \in B_0$ as $B_0 \in Q_0$)

$\Longrightarrow$ $\sigma = A_0\, A_1\, A_2 \ldots \models \varphi$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \dots \models \psi$$

Proof by structural induction on ψ

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0. \ B_j \in F \qquad (*)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Proof by structural induction on ψ

base of induction:

$\psi = true$

$\psi = a \in AP$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F \qquad (*)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Proof by structural induction on ψ

base of induction:

$\psi = \mathit{true}$

$\psi = a \in AP$

induction step:

$\psi = \neg\psi'$

$\psi = \psi_1 \wedge \psi_2$

$\psi = \bigcirc\psi'$

$\psi = \psi_1 \mathsf{U} \psi_2$

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$.

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$

note: ***true*** is contained in all elementary formula-sets

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

note: ***true*** is contained in all elementary formula-sets
true holds for all paths/traces

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

Let $\psi = a \in AP$.

Base of induction

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Let $\psi = a \in AP$. Then:

$a \in B_0$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

Let $\psi = a \in AP$. Then:

$$a \in B_0 \iff a \in A_0$$

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

$$A_0 = B_0 \cap AP$$

then for all formulas $\psi \in cl(\varphi)$:

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

$$A_0 A_1 A_2 \ldots \models \textit{true}$$

Let $\psi = a \in AP$. Then:

$$a \in B_0 \iff a \in A_0 \iff A_0 A_1 A_2 \ldots \models a$$

Induction step: negation

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

iff $A_0\, A_1\, A_2 \ldots \models \psi$ (semantics of $\neg$)

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\psi \notin B \quad \text{iff} \quad \neg\psi \in B$$
$$\psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B$$
$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until $\mathbf{U}$, i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

$$\text{if } \psi_1 \mathbf{U} \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B$$
$$\text{if } \psi_2 \in B \text{ then } \psi_1 \mathbf{U} \psi_2 \in B$$

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
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$$\textit{true} \in cl(\varphi) \quad \text{implies} \quad \textit{true} \in B$$

(ii) B is locally consistent with respect to until $\mathbf{U}$,
i.e., if $\psi_1 \,\mathbf{U}\, \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \,\mathbf{U}\, \psi_2 \in B$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

iff $A_0\, A_1\, A_2 \ldots \models \psi$ (semantics of $\wedge$)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$\bigcirc\psi \in B$ iff $\psi \in B'$

$\psi_1 \mathsf{U} \psi_2 \in B$ iff $(\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{B_1 \in \delta(B_0, A_0)}$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

iff $\psi' \in B_1$ (definition of δ)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

$$B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$\psi \in B_0$

iff $\psi' \in B_1$ (definition of δ)

iff $A_1\, A_2\, A_3 \ldots \models \psi'$ (induction hypothesis)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \; \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

$$B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$\psi \in B_0$

iff $\psi' \in B_1$ (definition of δ)

iff $A_1\, A_2\, A_3 \ldots \models \psi'$ (induction hypothesis)

iff $A_0\, A_1\, A_2\, A_3 \ldots \models \psi$ (semantics of $\bigcirc$)

Induction step: until

Recall: elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\psi \notin B \quad \text{iff} \quad \neg\psi \in B$$
$$\psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B$$
$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until $\mathbf{U}$,
i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \mathbf{U} \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \mathbf{U} \psi_2 \in B$

Recall: GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$\bigcirc\psi \in B$ iff $\psi \in B'$

$\psi_1 \mathsf{U} \psi_2 \in B$ iff $(\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

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Induction step: until

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathsf{U} \psi_2$:

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$$A_j\, A_{j+1}\, A_{j+2} \ldots \models \psi_2$$
$$A_{j-1}\, A_j\, A_{j-1} \ldots \models \psi_1$$
$$A_{j-2}\, A_{j-1}\, A_j \ldots \models \psi_1$$
$$\vdots$$
$$A_0\, A_1\, A_2\, A_3 \ldots \models \psi_1$$

Induction step: until (part "$\Longleftarrow$")

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A_j\, A_{j+1}\, A_{j+2} \ldots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j \\
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A_{j-2}\, A_{j-1}\, A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} \\
\vdots & & & \vdots \\
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B_j is elementary

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A_{j-2}\, A_{j-1}\, A_j \ldots \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & & \\
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$$\Rightarrow \ \psi \in B_1$$

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$$\Rightarrow \; \psi \in B_1 \;\wedge\; \psi_2 \notin B_1$$

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Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

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$$\begin{array}{ll} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 \end{array}$$

Induction step: until (part "$\Longrightarrow$")

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Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

"$\Longrightarrow$" Suppose $\psi \in B_0$. There exists $j \geq 0$ with $\psi_2 \in B_j$, since otherwise $\forall j \geq 0.\, \psi_2 \notin B_j$ and therefore:

$$\left.\begin{array}{l} \quad\ \psi \in B_0 \ \wedge\ \psi_2 \notin B_0 \\ \Rightarrow \psi \in B_1 \ \wedge\ \psi_2 \notin B_1 \\ \Rightarrow \psi \in B_2 \ \wedge\ \psi_2 \notin B_2 \\ \qquad\qquad \vdots \end{array}\right\} \Longrightarrow \forall j \geq 0.\ B_j \notin F_\psi \text{ where}$$

$$F_\psi = \{B : \psi \notin B \text{ or } \psi_2 \in B\}$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\left.\begin{array}{rl} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 \ \wedge \ \psi_2 \notin B_2 \\ & \vdots \end{array}\right\}$$

$\Longrightarrow \forall j \geq 0. \ B_j \notin F_\psi$ where

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Contradiction!

Induction step: until (part "$\Longrightarrow$")

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Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$$\begin{array}{ll} \neg\psi_2 & \in B_{j-1} \\ \neg\psi_2 & \in B_{j-2} \\ \quad\vdots & \\ \neg\psi_2 & \in B_1 \\ \neg\psi_2 & \in B_0 \end{array}$$

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Let $\boxed{\psi \in B_0}$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

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$\neg\psi_2 \in B_{j-1}$

$\neg\psi_2 \in B_{j-2}$

$\vdots$

$\neg\psi_2 \in B_1$

$\neg\psi_2, \ \psi \in B_0 \quad \longleftarrow$ by assumption

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Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

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$$\begin{array}{ll}
\neg\psi_2 & \in B_{j-1} \\
\neg\psi_2 & \in B_{j-2} \\
\vdots & \\
\neg\psi_2 & \in B_1 \\
\neg\psi_2, \psi_1, \psi & \in B_0
\end{array}$$

$\longleftarrow$ local consistency w.r.t. $\mathbf{U}$

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$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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Complexity: LTL $\rightsquigarrow$ NBA

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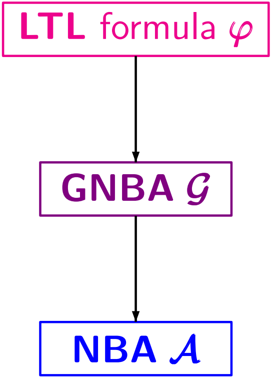

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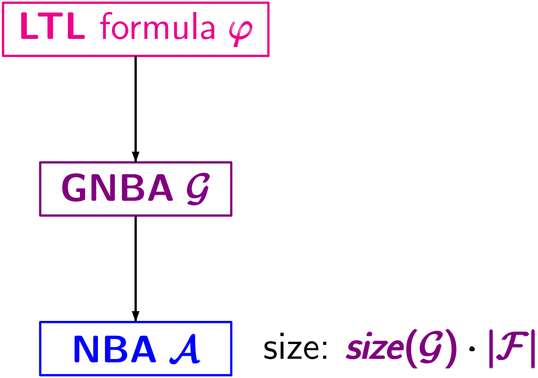

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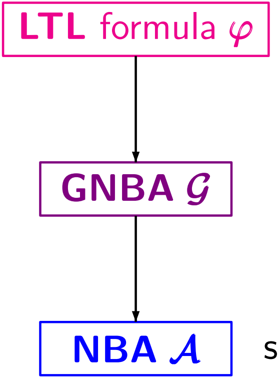

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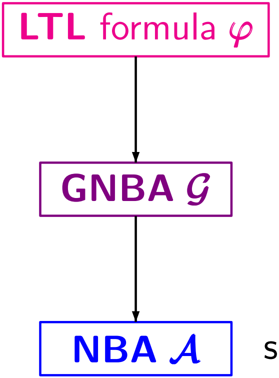

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LTL formula φ

↓

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↓

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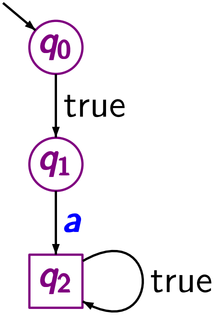

constructed GNBA has **4** states and **8** edges

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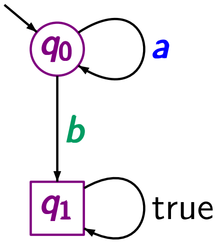
NBA for a **U** b



constructed (G)NBA has
5 states and **20** edges

Size of NBA for LTL formulas

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... but there exists LTL formulas φ_n such that

- $|\varphi_n| = \mathcal{O}(poly(n))$
- each NBA for φ_n has at least 2^n states

LT-properties that have no "small" NBA

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consider the following family of LT-properties $(E_n)_{n \geq 1}$:

$$E_n = \left\{ \begin{array}{l} \text{set of all infinite words over } 2^{AP} \text{ of the form} \\ \quad A_1 A_2 A_3 \ldots A_n A_1 A_2 A_3 \ldots A_n B_1 B_2 B_3 B_4 \ldots \end{array} \right.$$

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$= xx$ for some $x \in (2^{AP})^*$ of length n

$\in (2^{AP})^\omega$ arbitrary

LTL formula φ_n with $Words(\varphi_n) = E_n$

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LT-property E_n for n=1

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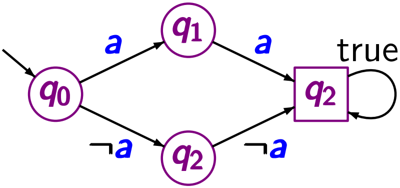

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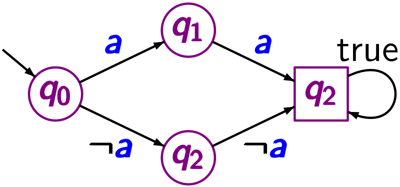

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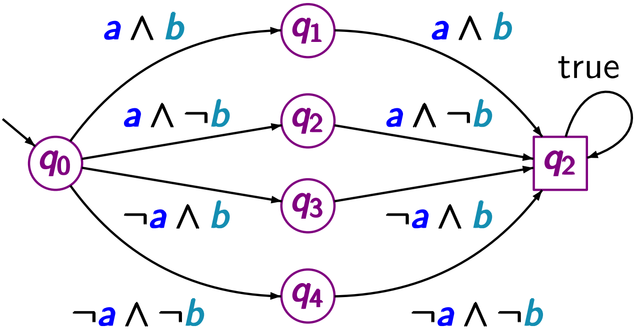

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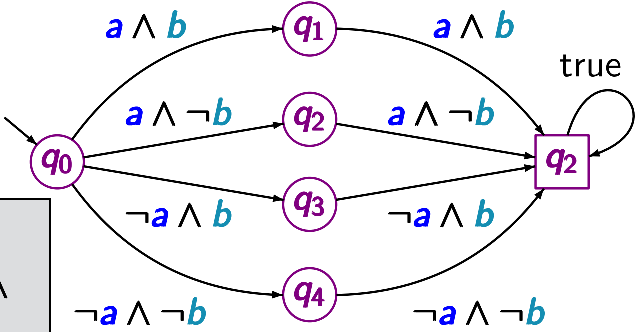

LTL-formula:

$(a \leftrightarrow \bigcirc a) \wedge$
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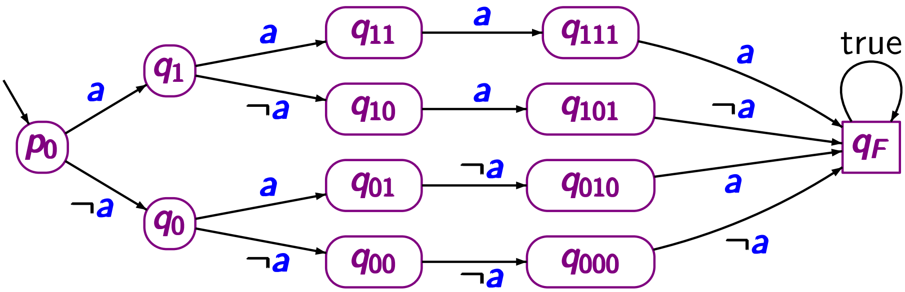
LT property E_n for n=2 and $AP = \{a\}$



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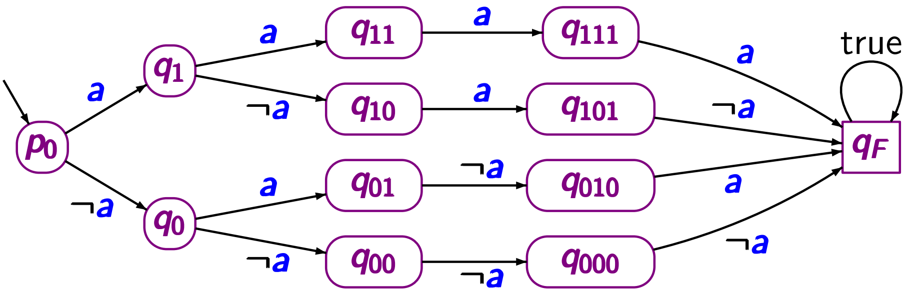

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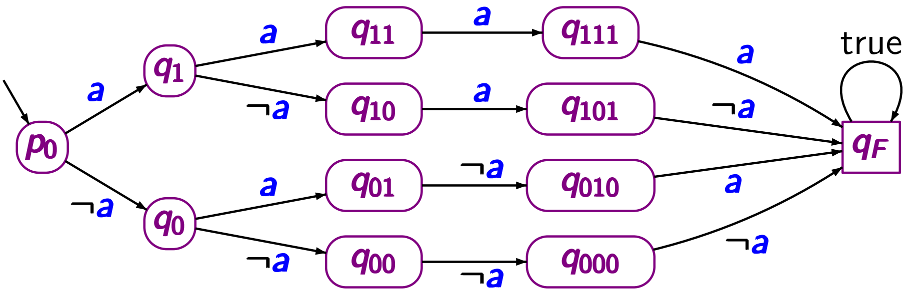

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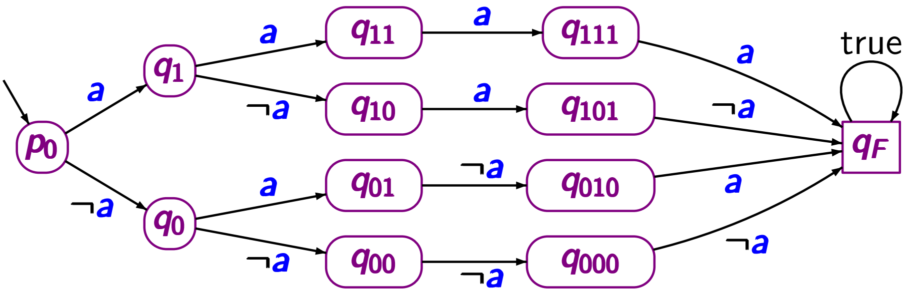

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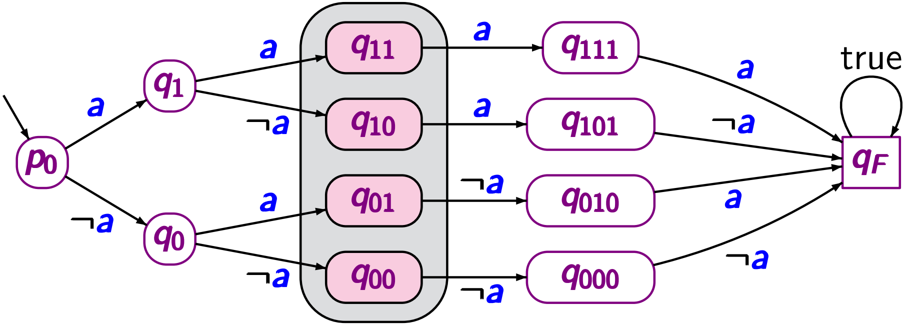

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