Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL
- automata-based LTL model checking
- complexity of LTL model checking ⟵

Computation-Tree Logic

Equivalences and Abstraction

Complexity of LTL model checking

main steps of automata-based LTL model checking:

construction of an NBA $\mathcal{A}$ for $\neg\varphi$

persistence checking in the product $\mathcal{T} \otimes \mathcal{A}$

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The **LTL** model checking problem is **PSPACE**-complete

LTL model checking problem

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LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

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we show

- just for fun: **coNP**-hardness
- **PSPACE**-completeness

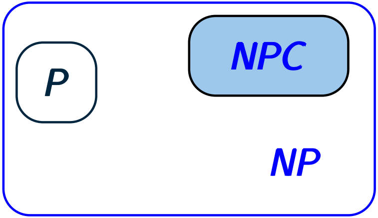
Recall: complexity classes

Complexity classes P, NP

P = class of decision problem solvable in deterministic polynomial time

NP = class of decision problem solvable in nondeterministic polynomial time

Complexity classes *P*, *NP*



NPC = class of ***NP***-complete problems

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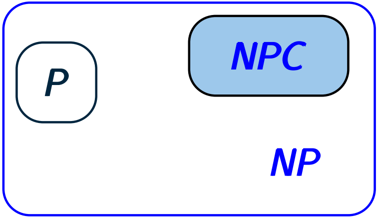

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(1) $L \in NP$

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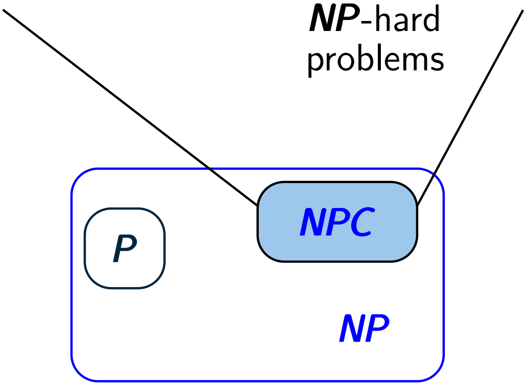

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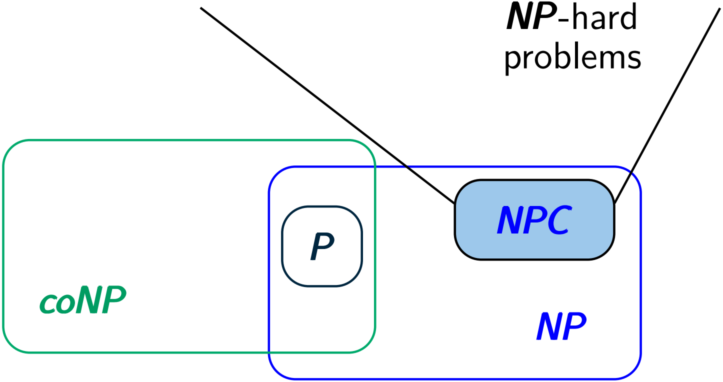
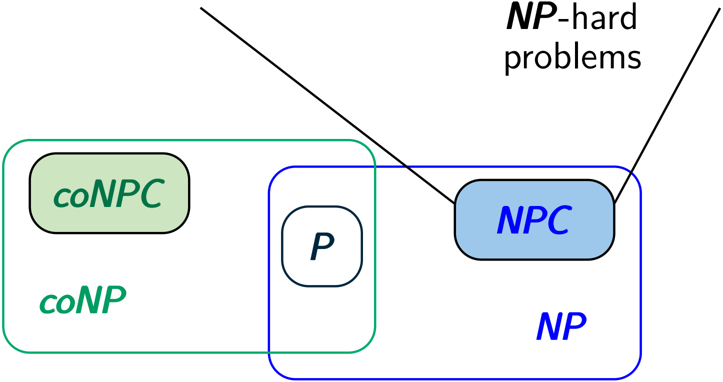


$$coNP \;=\; \{\,\overline{L} : L \in NP\,\}$$

$\uparrow$

complement of L

Complexity classes *P*, *NP*, *coNP*



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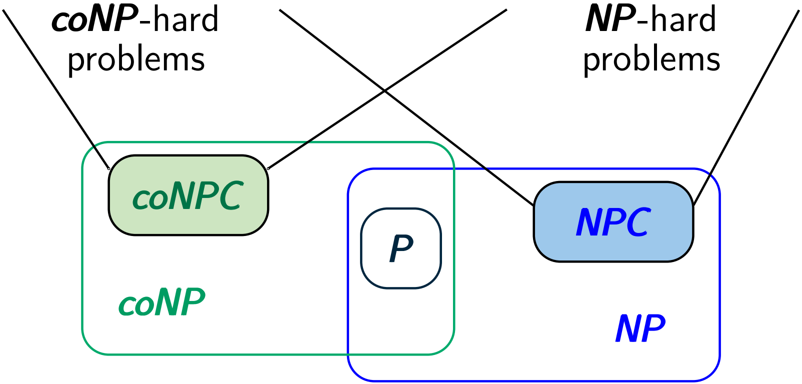

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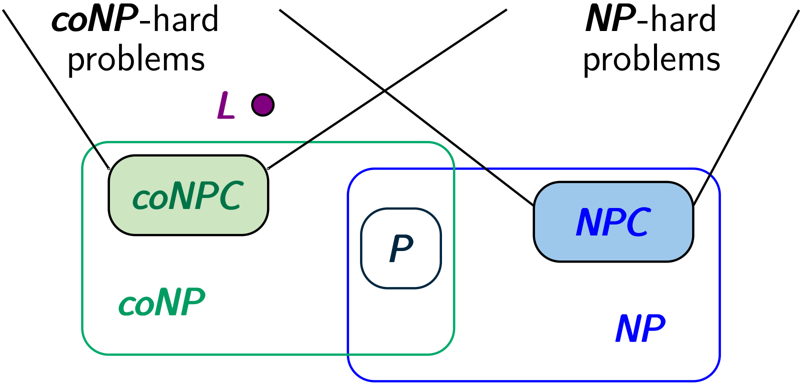

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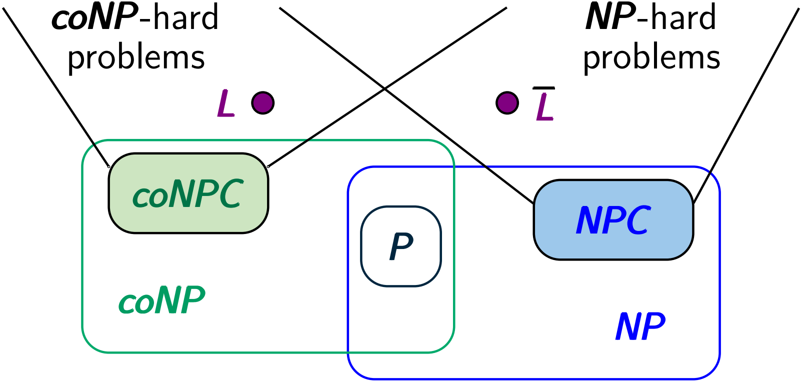

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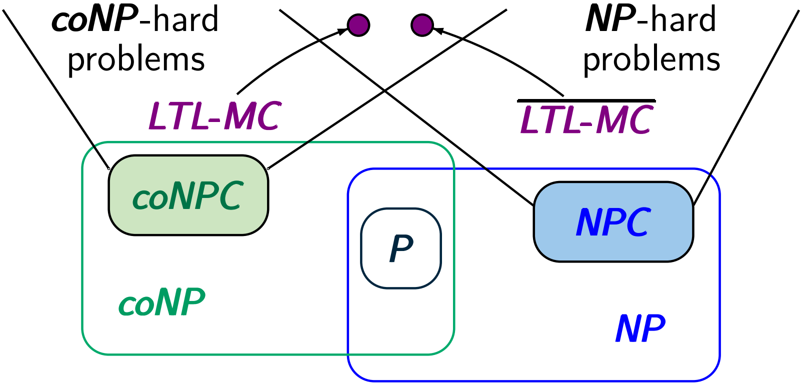

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***coNP*-hardness**

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proof by a polynomial reduction

$$HP \quad \le_{poly} \quad \overline{LTL\text{-}MC}$$

Hamilton path problem

complement of the **LTL** model checking problem

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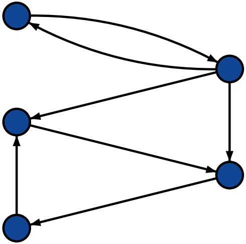

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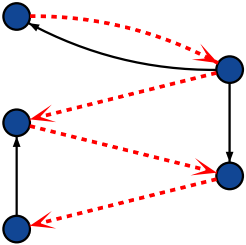

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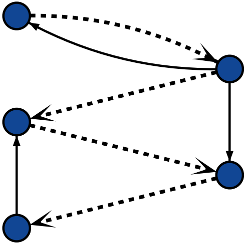

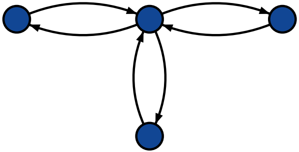

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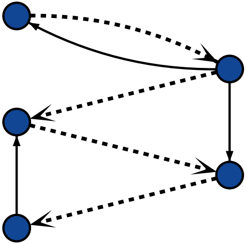

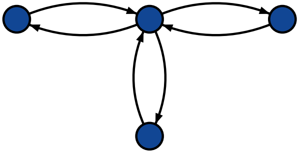

has no Hamilton path

HP is known to be ***NP***-complete

Polynomial reduction

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complement of the **LTL** model checking problem

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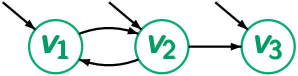
node-set V of G $\hat{=}$ states of $\mathcal{T}$



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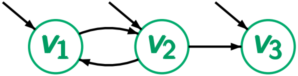

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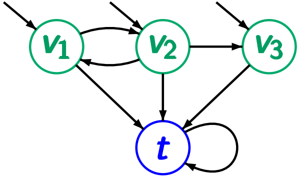
additional trap state t



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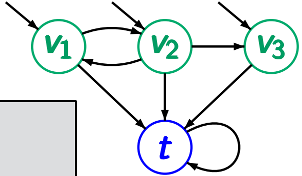

$\varphi = \,?$

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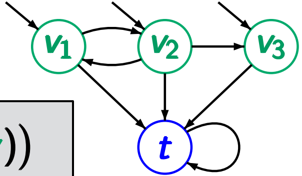

$$\varphi = \bigwedge_{v \in V} \big(\Diamond v \wedge \Box(v \rightarrow \bigcirc \Box \neg v) \big)$$

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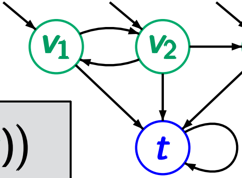

$$\varphi = \neg \bigwedge_{v \in V} \left(\Diamond v \wedge \Box(v \rightarrow \bigcirc \Box \neg v) \right)$$

Complexity of LTL model checking

We just saw:

The **LTL** model checking problem is ***coNP***-hard

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We now prove:

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DFS-based analysis of the computation tree
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DFS-based analysis of the computation tree of an ***NP***-algorithm

space requirements:

recursion depth $\widehat{=}$ height of computation tree

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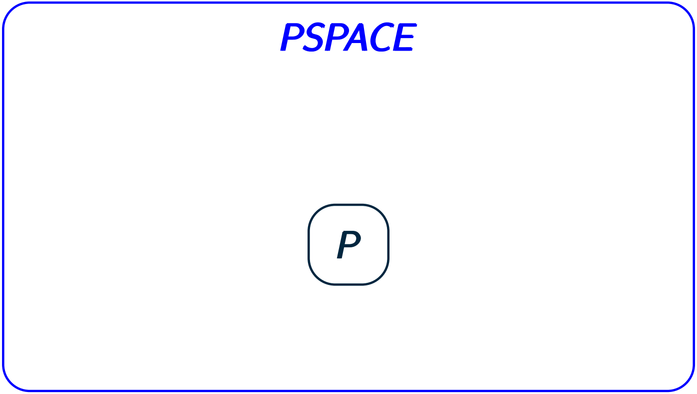
To prove $L \in$ *PSPACE* it suffices to provide a nondeterministic polynomially space-bounded algorithm for the complement $\overline{L}$ of L

Complexity classes *P*, *NP*, *coNP*, *PSPACE*



PSPACE = class of decision problems that are decidable in deterministic polynomial space

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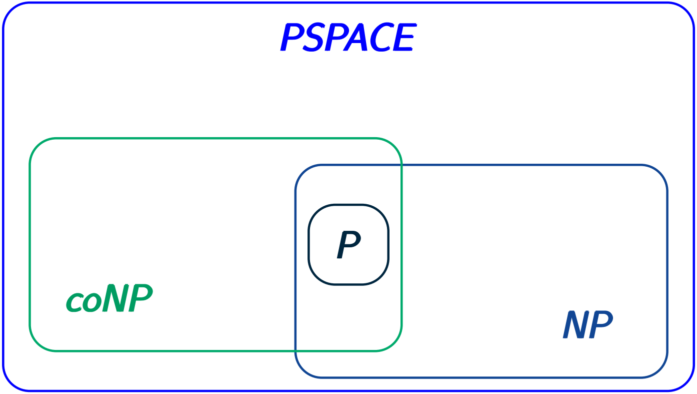

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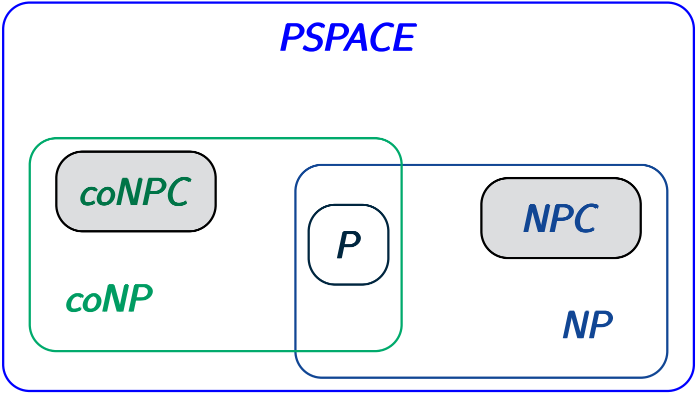

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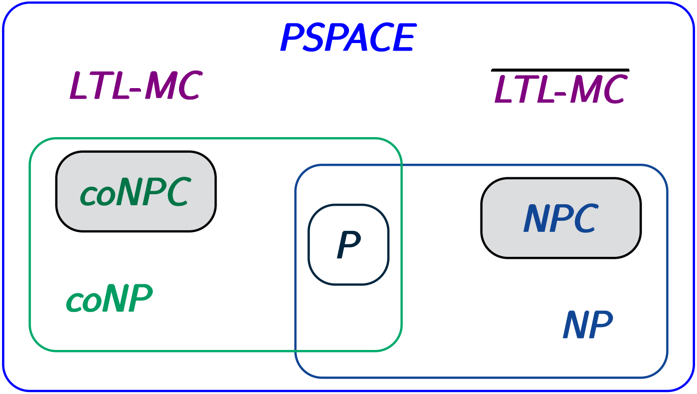

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PSPACE-completeness

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decision problem L is *PSPACE*-complete iff

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$L \in \mathit{PSPACE}$ $\iff$ $\overline{L} \in \mathit{NPSPACE}$

PSPACE-completeness of *LTL-MC*

LTL-MC LTL model checking problem

"does $\pi \models \varphi$ hold for all paths π of $\mathcal{T}$ **?**"

$\overline{\textit{LTL-MC}}$ = complement of ***LTL-MC***

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show: ***∃LTL-MC*** $\in$ ***NPSPACE***

∃LTL-MC is ***PSPACE***-hard

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show: ***∃LTL-MC*** $\in$ ***NPSPACE*** $\Longrightarrow$ $\boxed{\textit{LTL-MC} \in \textit{PSPACE}}$

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show: ***∃LTL-MC*** $\in$ ***NPSPACE***

∃LTL-MC is ***PSPACE***-hard $\Longrightarrow$ ***LTL-MC*** is ***PSPACE***-hard

Existential LTL model checking problem

given:	$\mathcal{T}$ be a finite transition system
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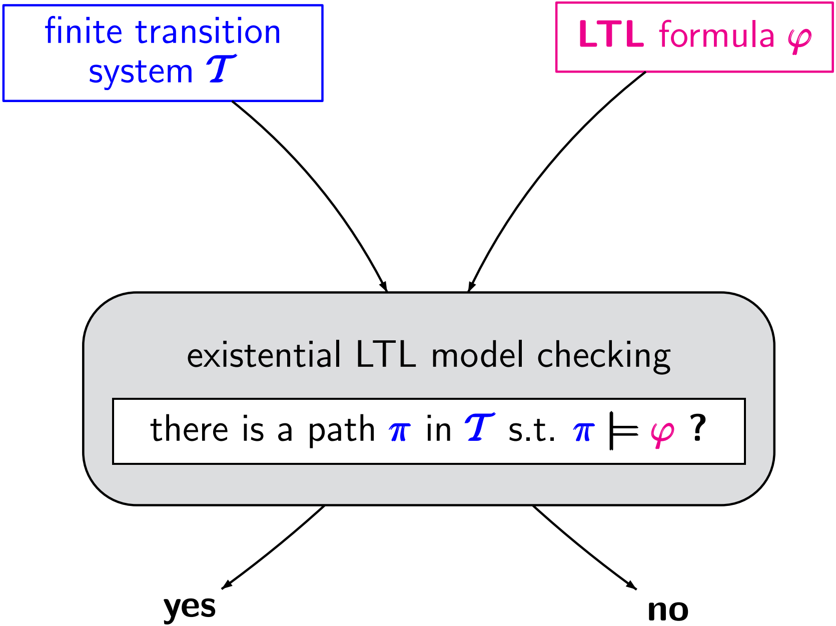
idea: use the **GNBA** $\mathcal{G}$ for φ (constructed by our LTL-2-GNBA algorithm)

Existential LTL model checking



Existential LTL model checking

finite transition system $\mathcal{T}$

LTL formula φ

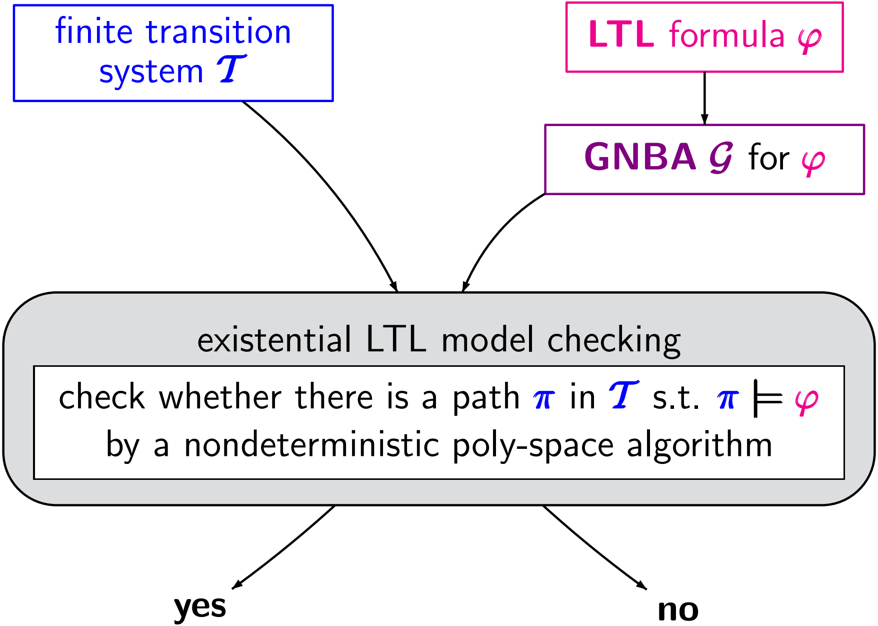
existential LTL model checking

check whether there is a path π in $\mathcal{T}$ s.t. $\pi \models \varphi$
by a nondeterministic poly-space algorithm

yes

no

Existential LTL model checking



Existential LTL model checking

finite transition system $\mathcal{T}$

LTL formula φ

GNBA $\mathcal{G}$ for φ

nondeterministic poly-space algorithm

guess an ultimatively periodic path

$$\pi = u_0 \ldots u_{n-1} (u_n \ldots u_{n+m})^\omega \text{ in } \mathcal{T} \otimes \mathcal{G}$$

check whether $\pi \models \bigwedge_{F \in \mathcal{F}} \Box \Diamond F$

yes

no

Recall: elementary formula-sets

closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. **U**

For $\varphi = a\,\mathsf{U}(\neg a \wedge b)$, the elementary sets are:

$\{\ a,\ \ b, \neg(\neg a \wedge b), \varphi\}$ $\quad$ $\{\ a,\ \ b, \neg(\neg a \wedge b), \neg\varphi\}$

$\{\ a, \neg b, \neg(\neg a \wedge b), \varphi\}$ $\quad$ $\{\ a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

$\{\neg a,\ \ b,\ \ \neg a \wedge b\ , \varphi\}$ $\quad$ $\{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Recall: GNBA for LTL-formula φ

$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Criterion for existential LTL properties

There exists a path π in $\mathcal{T}$ with $\pi \models \varphi$ iff

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NPSPACE-algorithm for $\exists$*LTL-MC*

The existential LTL model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ
question: is there a path $\pi \in \boldsymbol{Paths(\mathcal{T})}$ with $\pi \models \varphi$?

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$\boxed{\text{GNBA for } \varphi \text{ obtained by our LTL-2-GNBA algorithm}}$

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  GNBA for φ obtained by our LTL-2-GNBA algorithm

- check whether the guessed path meets the acceptance condition of $\mathcal{G}$

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guess two natural numbers $n, m \leq k$ s.t. $m \geq 1$
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- $\left\{ \psi_2 : \psi_1 \mathsf{U} \psi_2 \in \bigcup\limits_{n<i\leq n+m} B_i \right\} \subseteq \bigcup\limits_{n<i\leq n+m} B_i$

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If so then return "**yes**". Otherwise return "**no**".

We saw that:

The existential LTL model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ
question: is there a path π in $\mathcal{T}$ with $\pi \models \varphi$ **?**

belongs to ***NPSPACE***

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It remains to prove the *PSPACE*-hardness

PSPACE-hardness of $\exists$*LTL-MC*

we show that for all problems $K \in PSPACE$:

$$K \leq_{poly} \exists LTL\text{-}MC$$

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Let

- $\mathcal{M}$ be a deterministic Turing machine (DTM) that decides K,
- P a polynomial

such that $\mathcal{M}$ started with an input word w visits at most $P(|w|)$ tape cells we show that for all problems $K \in$ *PSPACE*:

$$K \leq_{poly} \exists LTL\text{-}MC$$

Given DTM $\mathcal{M}$ that decides K with polynomial space bound $P(n)$, provide a polynomial reduction:

input word w for $\mathcal{M}$	$\xrightarrow{\text{poly time}}$	TS $\mathcal{T}$ LTL-formula φ

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Given DTM $\mathcal{M}$ that decides K with polynomial space bound $P(n)$, provide a polynomial reduction:

input word w for $\mathcal{M}$	$\xrightarrow{\text{poly time}}$	TS $\mathcal{T}$ LTL-formula φ
$\mathcal{M}$ accepts w, i.e., $w \in K$	iff	there is path π of $\mathcal{T}$ with $\pi \models \varphi$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

DTM $\mathcal{M}$ visits at the most the tape cells $1, 2, \ldots, P(n)$ for inputs of length n (where P is a polynomial)

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$\$$	A_1	$\ldots$	A_{n-1}	A_n	$\sqcup$	$\sqcup$	$\ldots$	$\sqcup$	$\ldots$
1	2		n	$n+1$					

initial tape configuration for input $w = A_1\, A_2 \ldots A_n$

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w.l.o.g. $P(n) > n$

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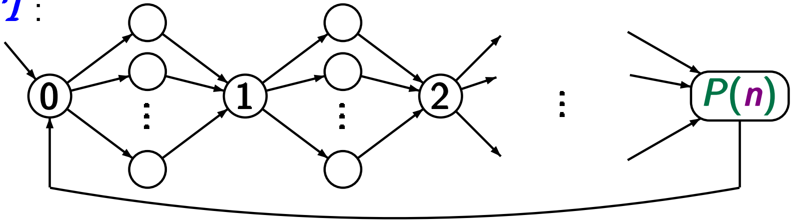
TS $\mathcal{T}$:



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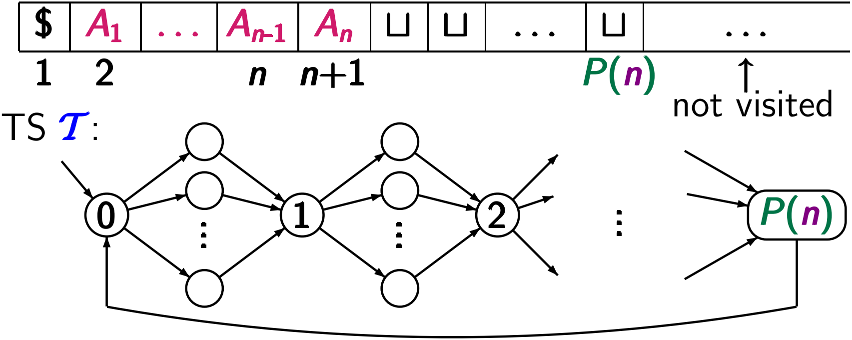

states of $\mathcal{T}$: $0, 1, \dots, P(n)$,

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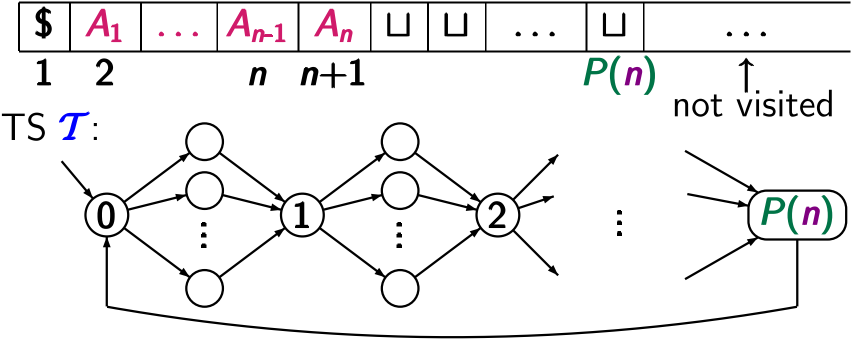

states of $\mathcal{T}$: $0, 1, \ldots, P(n)$, $\langle q, A, i \rangle$, $\langle *, A, i \rangle$

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$	A_1	...	A_{n-1}	A_n	⊔	⊔	...	⊔	...
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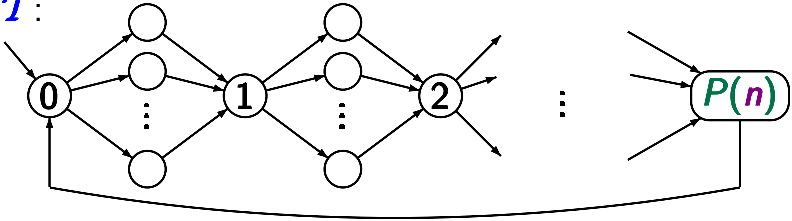

states of $\mathcal{T}$: $0, 1, \ldots, P(n)$, $\langle q, A, i \rangle$, $\langle *, A, i \rangle$
where q is a state of $\mathcal{M}$, A a tape symbol, $1 \leq i \leq P(n)$

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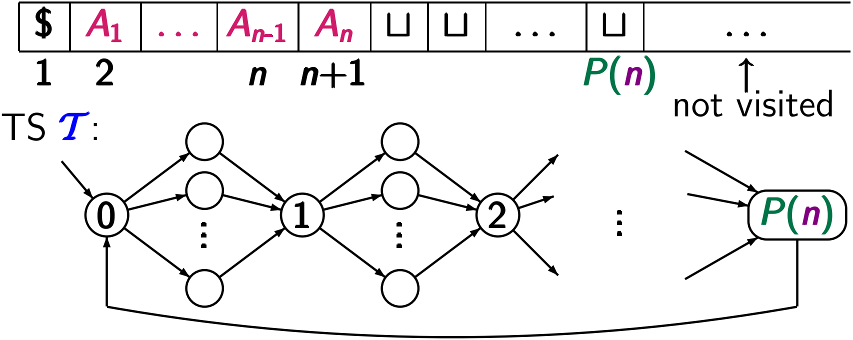
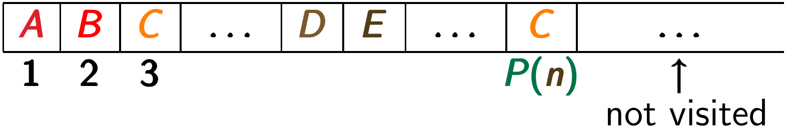


idea: TS $\mathcal{T}$ encodes each configuration of $\mathcal{M}$ by a path fragment from state $\mathbf{0}$ to state $P(n)$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



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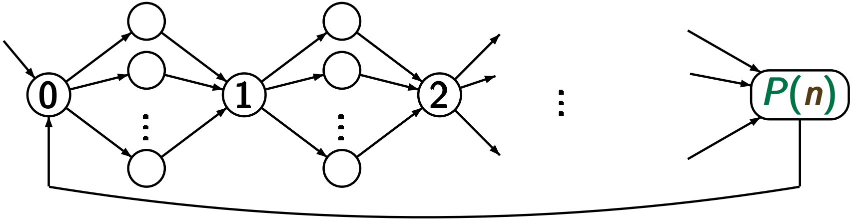

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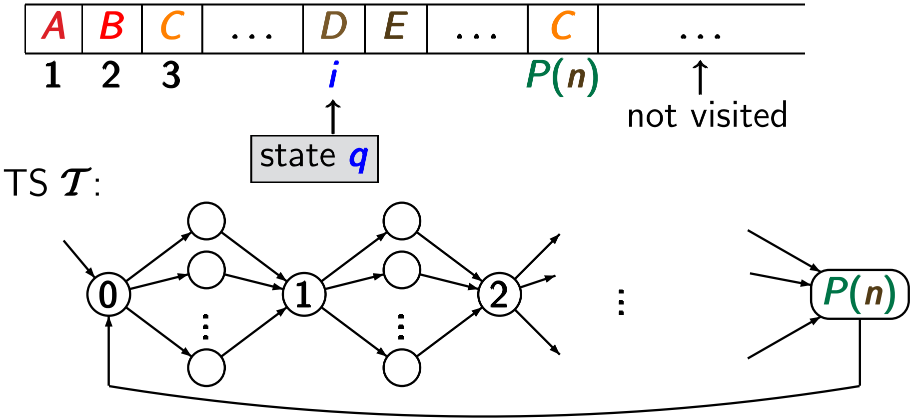

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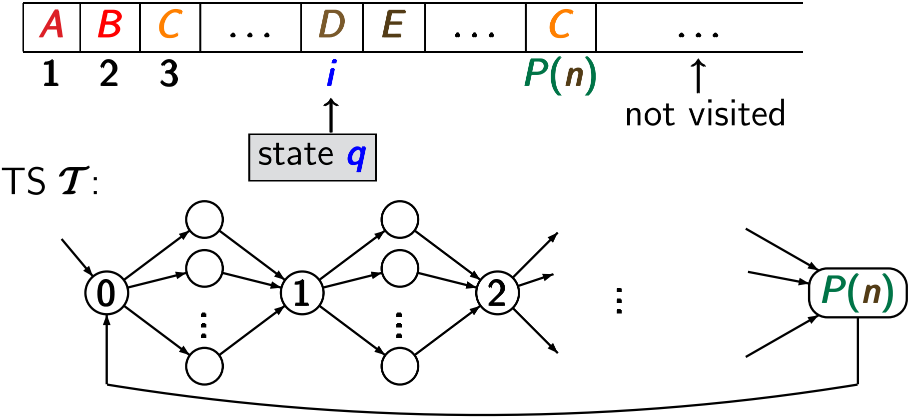
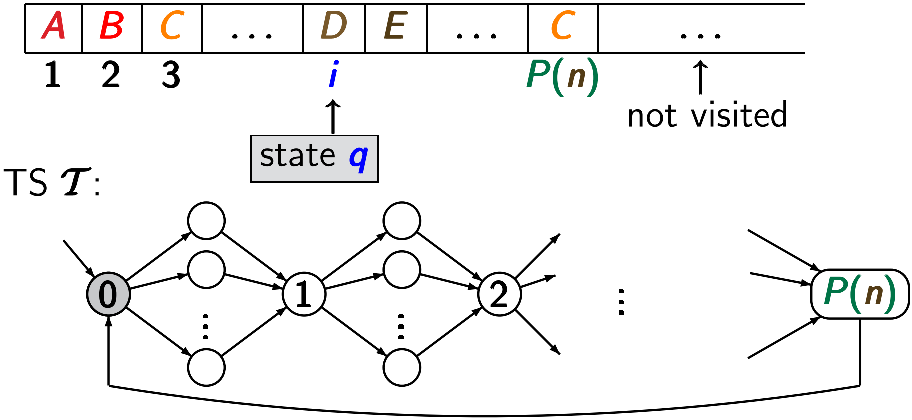


$$\boxed{\text{suppose } \delta(q, D) = (p, B, +1)}$$

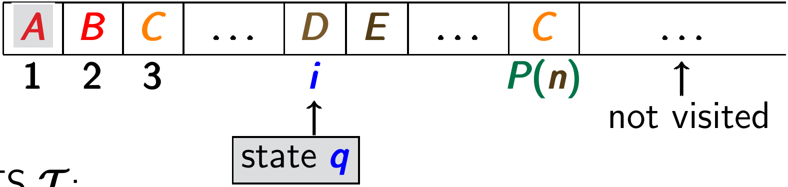
Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



0

path fragment for the configuration $ABC\ldots\ q\ D\ldots C$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



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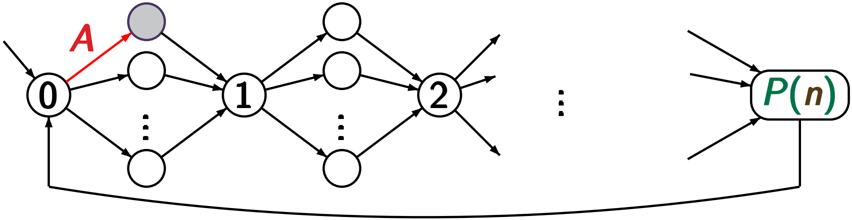

$0 \ \langle *, A, 1 \rangle$

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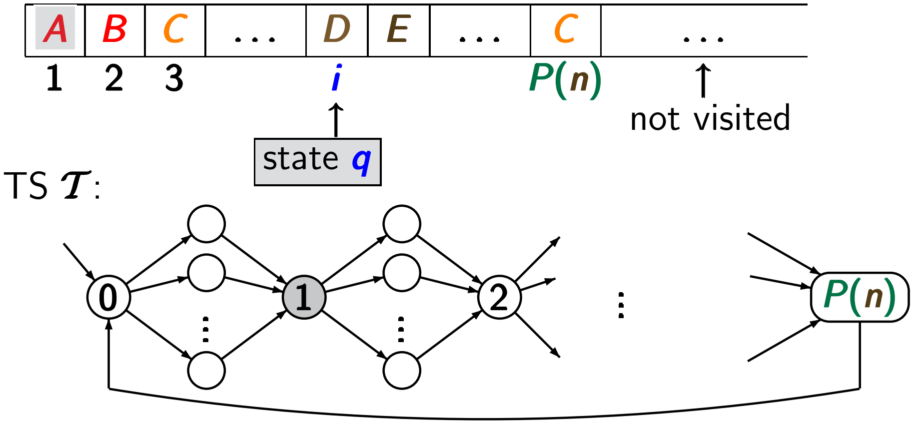

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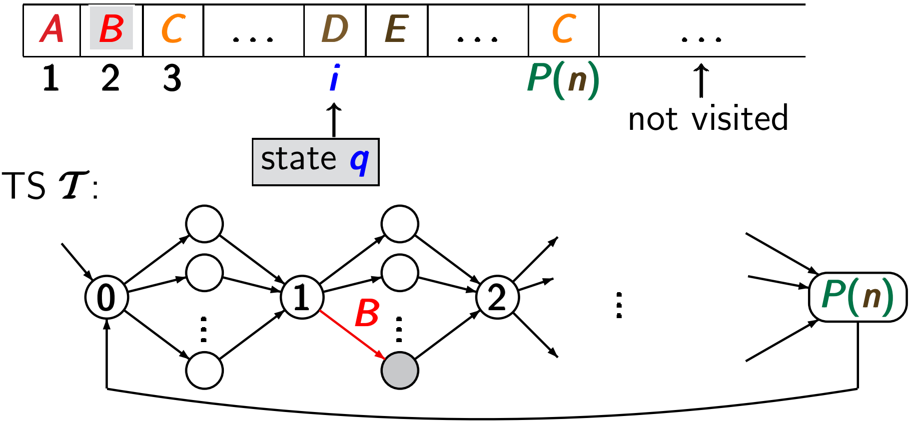

$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle$

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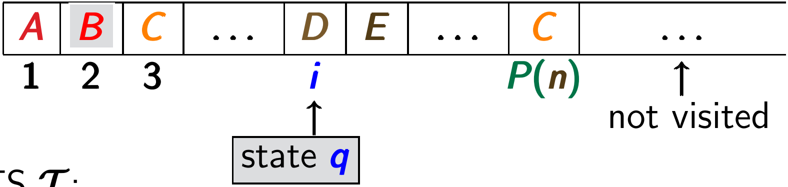

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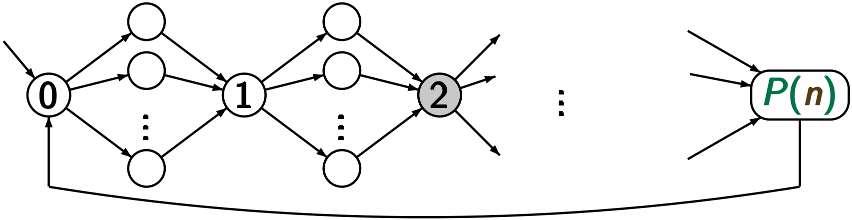

$$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2$$

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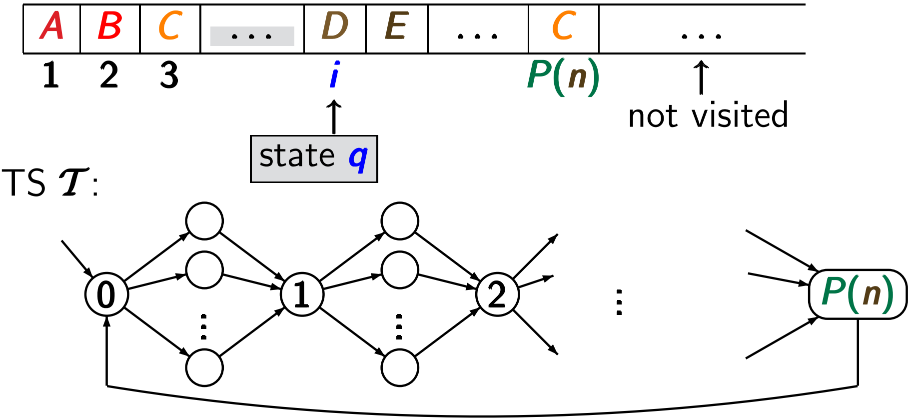
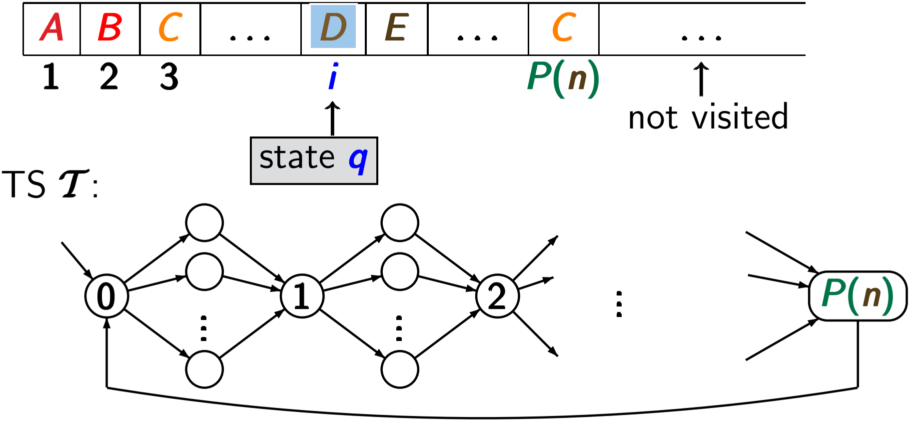


$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2\ \ldots\ (i{-}1)$

path fragment for the configuration $ABC\ldots\ q\ D\ldots C$



$$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2\ \ldots\ (i{-}1)\langle q, D, i\rangle$$

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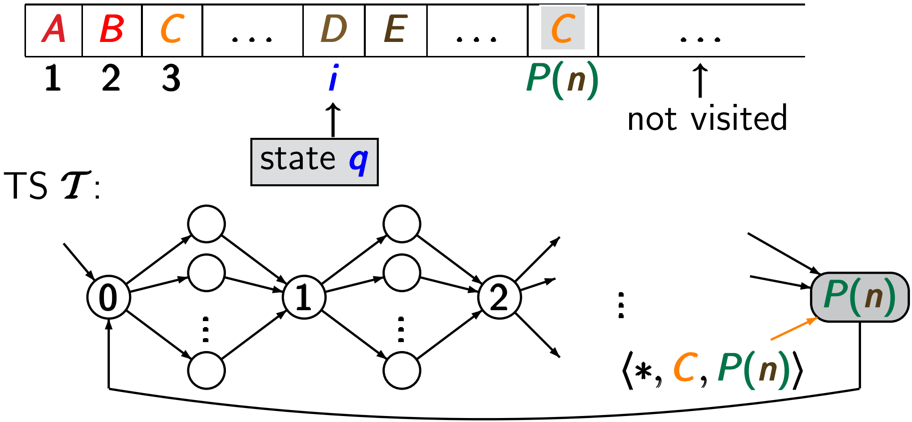

$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2\ \ldots\ (i{-}1)\langle q, D, i\rangle\ i\ \ \ldots\ \ \ P(n)$

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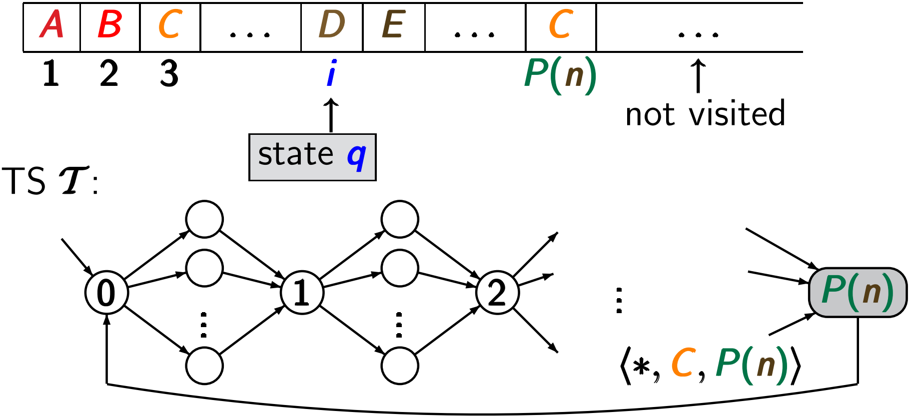


$0 \ \langle *, A, 1\rangle \ 1 \ \langle *, B, 2\rangle \ 2 \ \ldots \ (i{-}1)\langle q, D, i\rangle \ i \ \ \ldots \qquad P(n)$

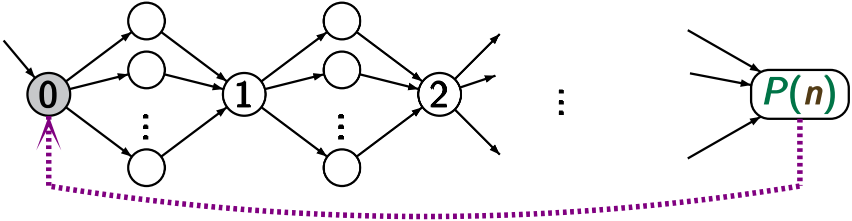
$$\boxed{\text{suppose } \delta(q, D) = (p, B, +1)}$$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

A	B	C	...	B	E	...	C	...
1	2	3		i	$i+1$		$P(n)$	not visited

state p

TS $\mathcal{T}$:



$0 \ \langle *, A, 1\rangle \ 1 \ \langle *, B, 2\rangle \ 2 \ \ldots \ \langle q, D, i\rangle \ i \ \langle *, E, i{+}1\rangle \ \ldots \ P(n)$

0

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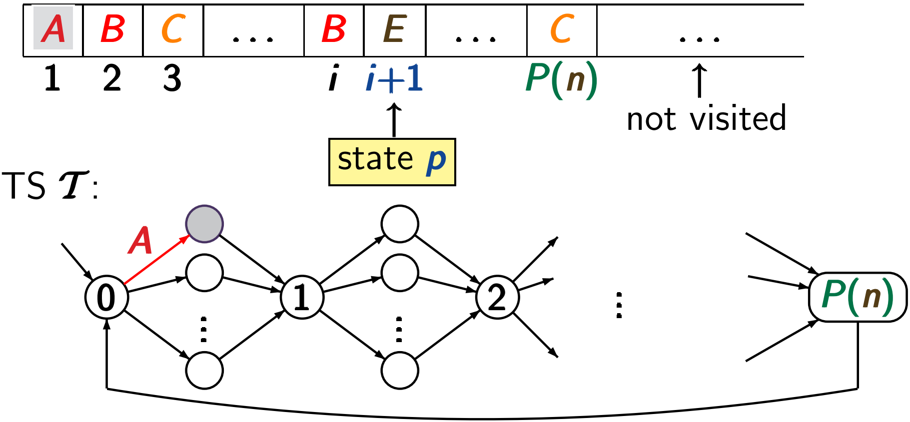
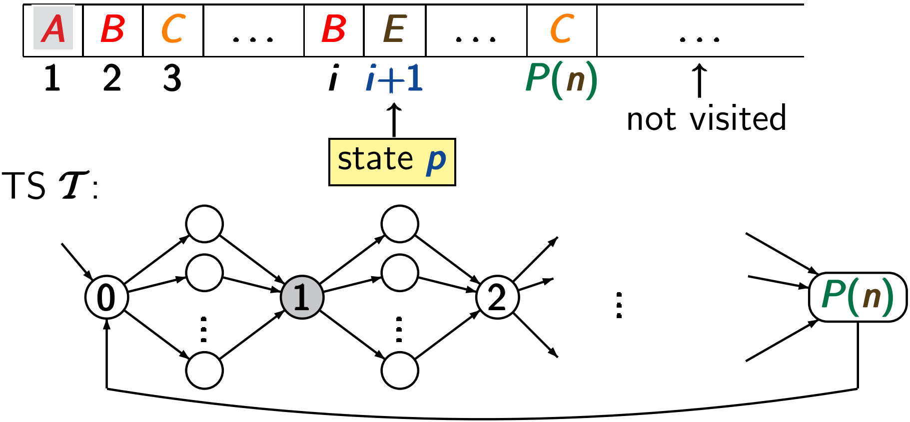


$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2\ \ldots\ \langle q, D, i\rangle\ i\ \langle *, E, i{+}1\rangle\ \ldots\ P(n)$

$0\ \langle *, A, 1\rangle$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



$0 \ \langle *, A, 1 \rangle \ 1 \ \langle *, B, 2 \rangle \ 2 \ \ldots \ \langle q, D, i \rangle \ i \ \langle *, E, i{+}1 \rangle \ \ldots \ P(n)$

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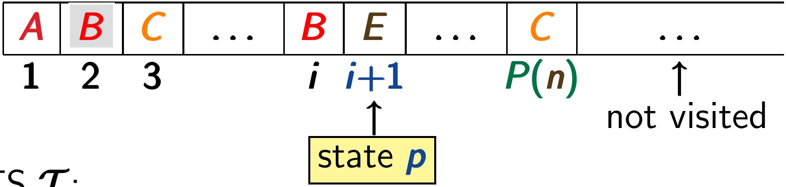

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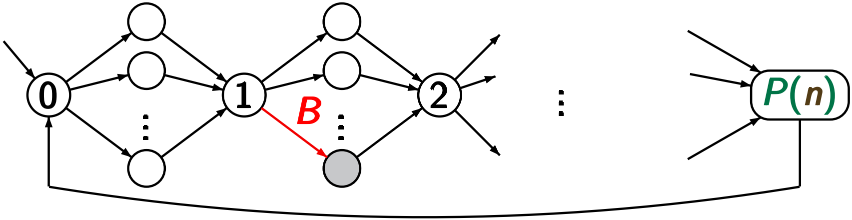

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$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle$

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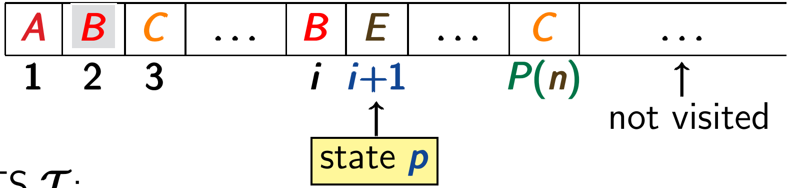

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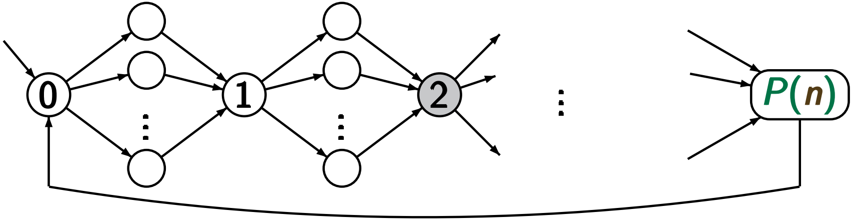

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$0\ \langle *, A, 1\rangle\ 1\ \langle *, B, 2\rangle\ 2$

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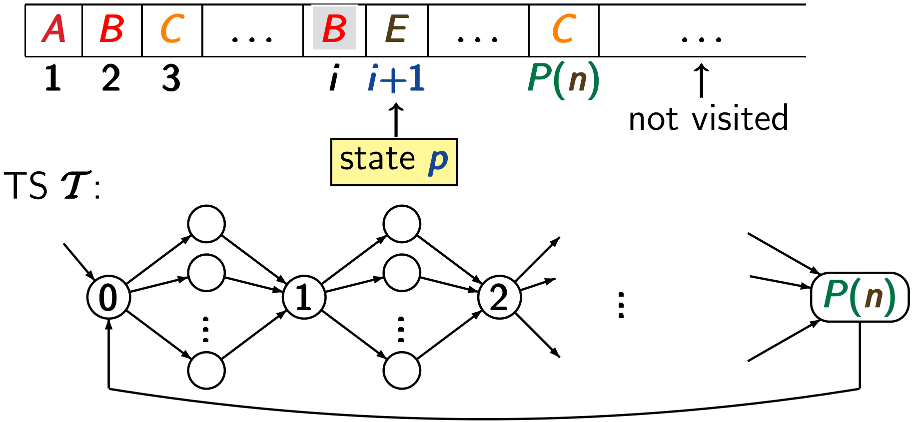

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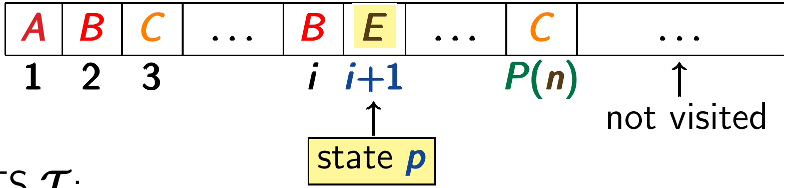

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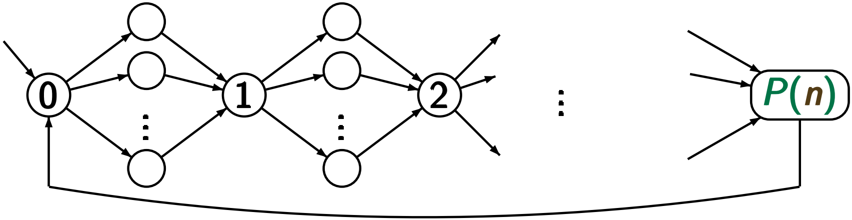

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$$0 \ \langle *, A, 1\rangle \ 1 \ \langle *, B, 2\rangle \ 2 \ \ldots \ \langle *, B, i\rangle \ i \ \langle p, E, i{+}1\rangle$$

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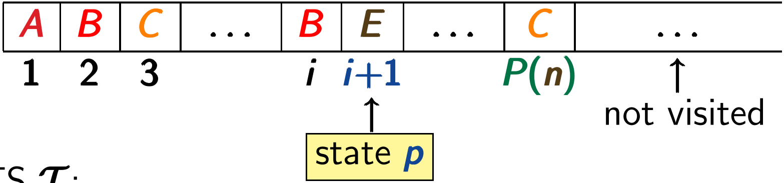

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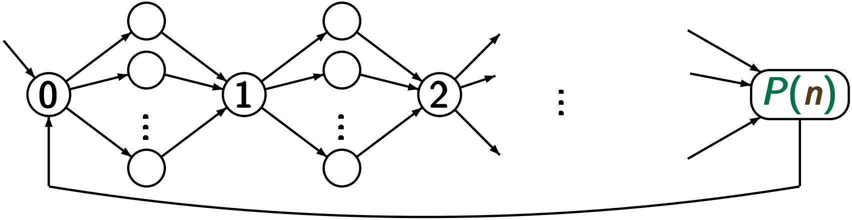

$$0 \; \langle *, A, 1\rangle \; 1 \; \langle *, B, 2\rangle \; 2 \; \ldots \; \langle q, D, i\rangle \; i \; \langle *, E, i{+}1\rangle \; \ldots \; P(n)$$

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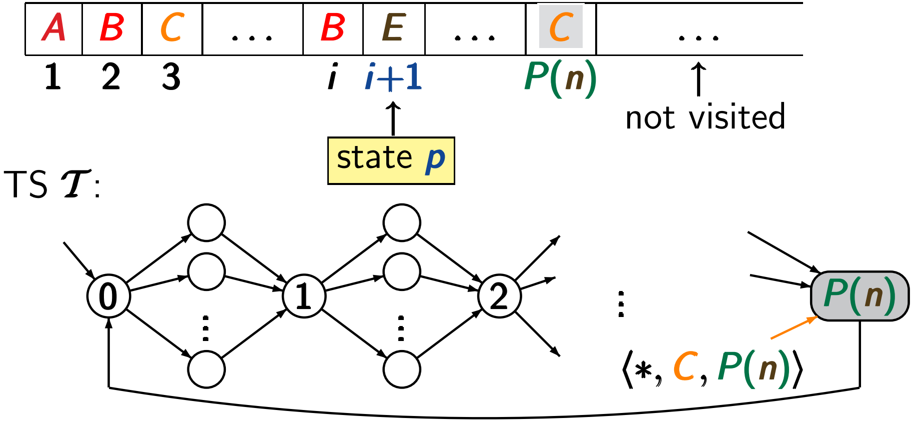

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Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

Let $\mathcal{M}$ be a DTM with polynomial space bound $P(n)$

- state space Q
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- set of accept states F
- tape alphabet Γ
- input alphabet $\Sigma \subseteq \Gamma$
- blank symbol $\sqcup$

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$$\boxed{\text{input word } w \text{ for } \mathcal{M}} \xrightarrow{\text{poly time}} \boxed{\text{TS } \mathcal{T} \text{ LTL-formula } \varphi}$$

$\mathcal{M}$ accepts w, i.e., $w \in K$ iff there is path π of $\mathcal{T}$ with $\pi \models \varphi$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, F)$ be a DTM with polynomial space bound $P(n)$, and $w \in \Sigma^*$, $|w|=n$.

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$$S = \{0, 1, \ldots, P(n)\} \cup \{\langle q, A, i\rangle, \langle *, A, i\rangle : q \in Q, \\ A \in \Gamma, 1 \leq i \leq P(n)\}$$

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transitions: $\left.\begin{array}{l} i{-}1 \longrightarrow \langle q, A, i\rangle \\ \langle q, A, i\rangle \longrightarrow i \end{array}\right\}$ for $1 \leq i \leq P(n)$ and $q \in Q \cup \{*\}$

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$$\text{LTL formula } \varphi \stackrel{\text{def}}{=} \varphi^{w}_{start} \wedge \varphi_{\delta} \wedge \varphi_{conf} \wedge \varphi_{accept}$$

Complexity of LTL model checking problem

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We saw that:

The existential **LTL** model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ

question: is there a path π in $\mathcal{T}$ with $\pi \models \varphi$?

is ***PSPACE***-complete.

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As ***PSPACE*** $=$ ***coPSPACE*** we get:

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Summary: LTL model checking problem

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additionally we proved ***coNP***-hardness
using an LTL-encoding of the Hamilton-path problem

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There is **no** LTL formula φ over $AP = \{a\}$ s.t.

$$Words(\varphi) = \text{set of words } A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^\omega \text{ s.t.}$$
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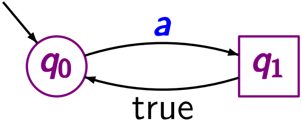

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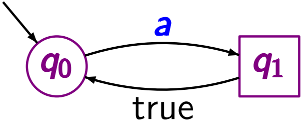

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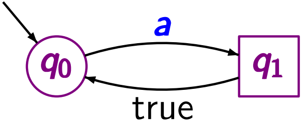

LTL formula $\varphi \;=\; a \wedge \Box(a \rightarrow \bigcirc\bigcirc a)$?

$\sigma \;=\; \{a\}\{a\}\{a\}\varnothing\{a\}^\omega \;\not\models\; \varphi$, but $\sigma \in \mathcal{L}_\omega(\mathcal{A})$

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question: is φ satisfiable **?**

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examples:

$\Diamond\Box a \wedge \Box\Diamond\neg a$ unsatisfiable

$a \,\mathsf{U}\, b \wedge \Box\neg b$ unsatisfiable

$\Diamond\Box a \wedge a \,\mathsf{U}(\Box b)$ satisfiable

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if yes, return "**yes**", otherwise "**no**" *given*: **LTL** formula φ over $\boldsymbol{AP}$

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$\uparrow$

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