Overview

Introduction

Modelling parallel systems

Linear Time Properties

- state-based and linear time view
- definition of linear time properties
- invariants and safety ⟵
- liveness and fairness

Regular Properties

Linear Temporal Logic

Computation-Tree Logic

Equivalences and Abstraction

Invariant

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Let E be an LT property over AP.

E is called an invariant if there exists a propositional formula Φ over AP such that

$$E = \left\{ A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^{\omega} : \forall i \geq 0.\, A_i \models \Phi \right\}$$

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Φ is called the invariant condition of E.

Safety properties

state that "nothing bad will happen"

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- deadlock freedom: e.g., for dining philosophers
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- beverage machine:
  no drink must be released if the user did not enter a coin before

  the total number of entered coins is never less than the total number of released drinks

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invariants:

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e.g., $\ldots \{●\}\{●\}$

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the total number of entered coins is never less than the total number of released drinks

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bad prefix, e.g., {*pay*} {*drink*} {*drink*}

Definition of safety properties

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$$\sigma = A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^\omega \setminus E$$

there exists a finite prefix $A_0 A_1 \ldots A_n$ of σ such that *none* of the words $A_0 A_1 \ldots A_n B_{n+1} B_{n+2} B_{n+3} \ldots$ belongs to E

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$E =$ set of all infinite words that do *not* have a bad prefix

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briefly: $BadPref$

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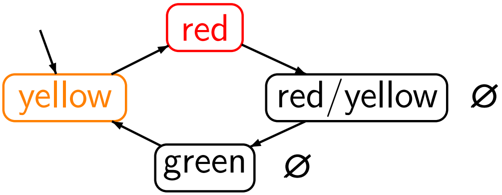
minimal bad prefixes: any word $A_0 \ldots A_i \ldots A_n \in \mathit{BadPref}$ s.t. no proper prefix $A_0 \ldots A_i$ is a bad prefix for E

Safety property for a traffic light



$AP = \{red, yellow\}$

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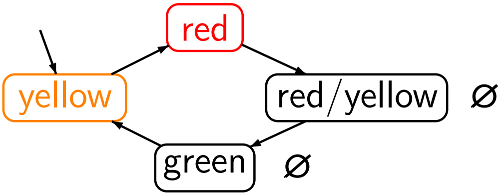

"every red phase is preceded by a yellow phase"

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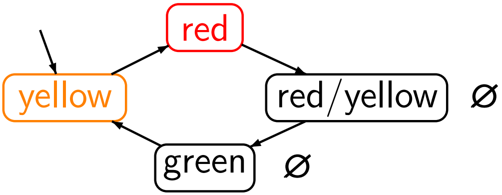

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hence: $\mathcal{T} \models E$

E = set of all infinite words $A_0\, A_1\, A_2 \ldots$ over 2^{AP} such that for all $i \in \mathbb{N}$:

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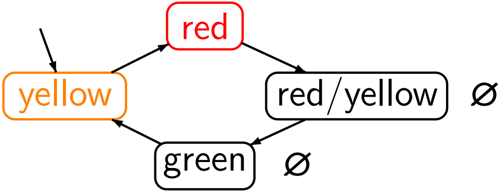

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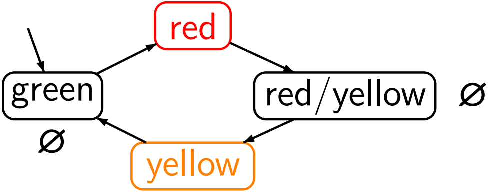

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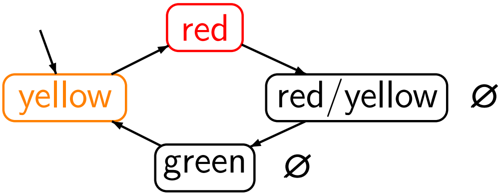

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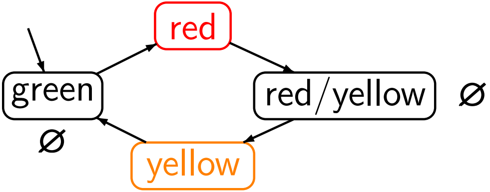

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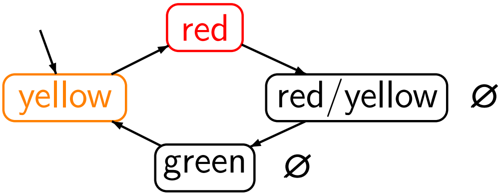

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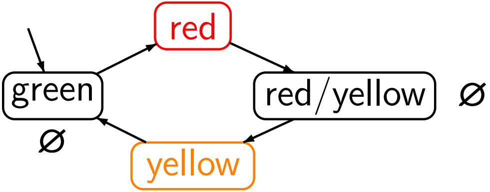

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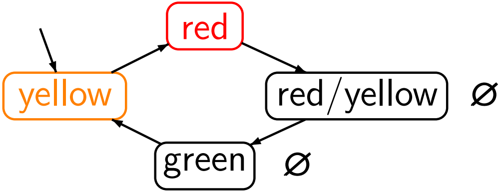

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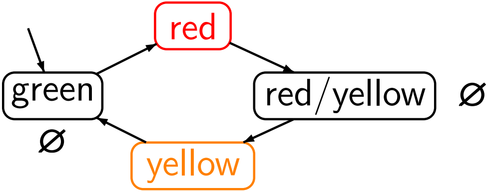

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bad prefix, e.g., $\varnothing\,\{red\}\,\varnothing\,\{yellow\}$

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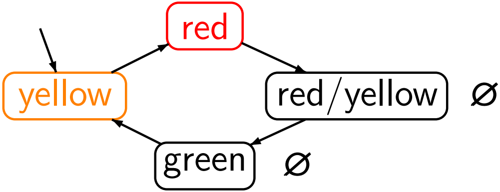

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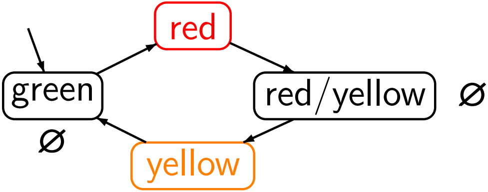

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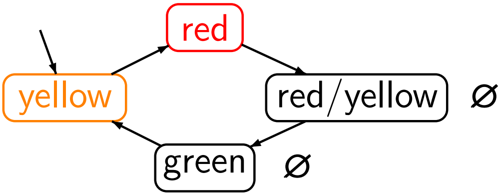

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is a safety property over $AP = \{red, yellow\}$ with

$$BadPref = \text{set of all finite words } A_0 A_1 \ldots A_n \text{ over } 2^{AP} \text{ s.t. for some } i \in \{0, \ldots, n\}\text{:}$$
$$red \in A_i \wedge (i{=}0 \vee yellow \notin A_{i-1})$$

Satisfaction of safety properties

IS2.5-11A

Satisfaction of safety properties

Let $E \subseteq (2^{AP})^\omega$ be a safety property, $\mathcal{T}$ a TS over AP.

$$\boxed{\mathcal{T} \models E \quad \text{iff} \quad \mathit{Traces}(\mathcal{T}) \subseteq E}$$

$\mathit{Traces}(\mathcal{T}) \quad = \quad$ set of traces of $\mathcal{T}$

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$= \{\ \mathit{trace}(\widehat{\pi}) : \widehat{\pi}$ is an initial, finite path fragment of $\mathcal{T} \}$

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$\mathit{BadPref}$ = set of all bad prefixes of E

$\mathit{MinBadPref}$ = set of all minimal bad prefixes of E

$\mathit{Traces}(\mathcal{T})$ = set of traces of $\mathcal{T}$

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Correct or wrong?

Every invariant is a safety property.

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finite words $A_0\, A_1 \ldots A_{n-1}\, A_n$ such that

$$A_i \models \Phi \text{ for } i = 0, 1, \ldots, n{-}1, \text{ and } A_n \not\models \Phi$$

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$$\text{“For all words} \in \underbrace{(2^{AP})^\omega \setminus (2^{AP})^\omega}_{=\varnothing} \ldots\text{”}$$

Prefix closure

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For a given infinite word $\sigma = A_0\, A_1\, A_2\, \ldots$, let

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For $E \subseteq (2^{AP})^\omega$, let $pref(E) \overset{\text{def}}{=} \bigcup\limits_{\sigma \in E} pref(\sigma)$

Given an LT property E, the prefix closure of E is:

$$cl(E) \overset{\text{def}}{=} \{\sigma \in (2^{AP})^\omega : pref(\sigma) \subseteq pref(E)\}$$

Prefix closure and safety

For any infinite word $\sigma \in \left(2^{AP}\right)^{\omega}$, let

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For any LT property $E \subseteq \left(2^{AP}\right)^{\omega}$, let

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Theorem:

E is a safety property iff $cl(E) = E$

Safety and finite trace inclusion

remind: LT properties and trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2)$$

iff for all LT properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

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iff for all LT properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

safety properties and finite trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longrightarrow$": obvious, as for safety property E:

$$\mathcal{T} \models E \quad \text{iff} \quad Traces_{fin}(\mathcal{T}) \cap BadPref = \varnothing$$

$$\mathit{Traces}_{fin}(\mathcal{T}_1) \subseteq \mathit{Traces}_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longrightarrow$": obvious, as for safety property E:

$$\mathcal{T} \models E \quad \text{iff} \quad \mathit{Traces}_{fin}(\mathcal{T}) \cap \mathit{BadPref} = \varnothing$$

Hence:

If $\mathcal{T}_2 \models E$ and $\mathit{Traces}_{fin}(\mathcal{T}_1) \subseteq \mathit{Traces}_{fin}(\mathcal{T}_2)$ then:

Safety and finite trace inclusion

$$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longrightarrow$": obvious, as for safety property E:

$$\mathcal{T} \models E \quad \text{iff} \quad \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \cap \mathit{BadPref} = \varnothing$$

Hence:

If $\mathcal{T}_2 \models E$ and $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2)$ then:

$$\begin{aligned} & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \cap \mathit{BadPref} \\ \subseteq \ & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2) \cap \mathit{BadPref} = \varnothing \end{aligned}$$

Safety and finite trace inclusion

$$\mathit{Traces}_{fin}(\mathcal{T}_1) \subseteq \mathit{Traces}_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longrightarrow$": obvious, as for safety property E:

$$\mathcal{T} \models E \quad \text{iff} \quad \mathit{Traces}_{fin}(\mathcal{T}) \cap \mathit{BadPref} = \varnothing$$

Hence:

If $\mathcal{T}_2 \models E$ and $\mathit{Traces}_{fin}(\mathcal{T}_1) \subseteq \mathit{Traces}_{fin}(\mathcal{T}_2)$ then:

$$\boxed{\begin{aligned} &\mathit{Traces}_{fin}(\mathcal{T}_1) \cap \mathit{BadPref} \\ \subseteq\ &\mathit{Traces}_{fin}(\mathcal{T}_2) \cap \mathit{BadPref} = \varnothing \end{aligned}}$$

and therefore $\mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2))$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

for each transition system $\mathcal{T}$:

$$pref(Traces(\mathcal{T})) = Traces_{fin}(\mathcal{T})$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property

as $cl(E) = E$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property

$\uparrow$

as $cl(E) = E$

set of bad prefixes: $(2^{AP})^+ \setminus Traces_{fin}(\mathcal{T}_2)$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$\mathit{Traces_{fin}}(\mathcal{T}_1) \subseteq \mathit{Traces_{fin}}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = \mathit{cl}(\mathit{Traces}(\mathcal{T}_2)) = \big\{\sigma : \mathit{pref}(\sigma) \subseteq \mathit{Traces_{fin}}(\mathcal{T}_2)\big\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $\mathit{Traces}(\mathcal{T}_1) \subseteq E$.

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $Traces(\mathcal{T}_1) \subseteq E$.

Hence: $Traces_{fin}(\mathcal{T}_1) = pref(Traces(\mathcal{T}_1))$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $Traces(\mathcal{T}_1) \subseteq E$.

Hence: $Traces_{fin}(\mathcal{T}_1) = pref(Traces(\mathcal{T}_1))$

$$\subseteq pref(E)$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $Traces(\mathcal{T}_1) \subseteq E$.

Hence: $Traces_{fin}(\mathcal{T}_1) = pref(Traces(\mathcal{T}_1))$

$$\subseteq pref(E) = pref(cl(Traces(\mathcal{T}_2)))$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $Traces(\mathcal{T}_1) \subseteq E$.

Hence:
$$\begin{aligned} Traces_{fin}(\mathcal{T}_1) &= pref(Traces(\mathcal{T}_1)) \\ &\subseteq pref(E) = pref(cl(Traces(\mathcal{T}_2))) \\ &= Traces_{fin}(\mathcal{T}_2) \end{aligned}$$

Safety and finite trace equivalence

Safety and finite trace equivalence

safety properties and finite trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Safety and finite trace equivalence

safety properties and finite trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

safety properties and finite trace equivalence:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) = Traces_{fin}(\mathcal{T}_2)$$

iff $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the same safety properties

Summary: trace relations and properties

trace inclusion

$Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$ iff

for all LT properties E: $\mathcal{T}' \models E \Longrightarrow \mathcal{T} \models E$

finite trace inclusion

$Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$ iff

for all safety properties E: $\mathcal{T}' \models E \Longrightarrow \mathcal{T} \models E$

trace equivalence

$\mathit{Traces}(\mathcal{T}) = \mathit{Traces}(\mathcal{T}')$ iff

$\mathcal{T}$ and $\mathcal{T}'$ satisfy the same LT properties

finite trace equivalence

$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) = \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$ iff

$\mathcal{T}$ and $\mathcal{T}'$ satisfy the same safety properties

correct or wrong?

If $\mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}')$
then $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$.

If $\mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}')$
then $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$.

correct, since

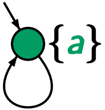
$$\begin{aligned} \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) &= \text{set of all finite nonempty prefixes of words in } \mathit{Traces}(\mathcal{T}) \\ &= \mathit{pref}(\mathit{Traces}(\mathcal{T})) \end{aligned}$$

correct or wrong?

If $\mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}')$
then $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$.

correct, since

$$\begin{aligned} \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) &= \text{set of all finite nonempty prefixes} \\ & \text{of words in } \mathit{Traces}(\mathcal{T}) \\ &= \mathit{pref}(\mathit{Traces}(\mathcal{T})) \end{aligned}$$



$$\begin{aligned} \mathit{Traces}(\mathcal{T}) &= \{ \{a\}^{\omega} \} \\ \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) &= \{ \{a\}^{n} : n \geq 1 \} \end{aligned}$$

is trace equivalence the same as
finite trace equivalence **?**

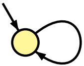
Finite trace relations versus trace relations

is trace equivalence the same as
finite trace equivalence **?**

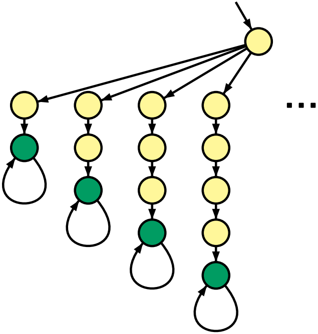
answer: **no**

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$



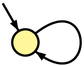
$\bigcirc \ \widehat{=}\ \varnothing \qquad \bullet \ \widehat{=}\ \{b\}$

set of propositions

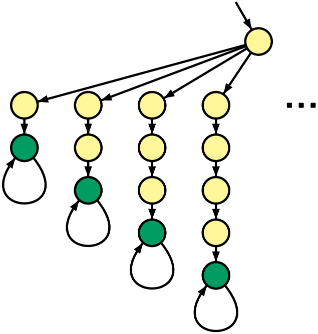
$AP = \{b\}$

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$



$$Traces(\mathcal{T}) \quad = \{\varnothing^\omega\}$$

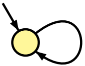
 $\widehat{=} \varnothing$  $\widehat{=} \{b\}$

set of propositions

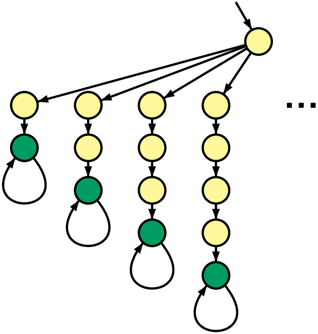
$$AP = \{b\}$$

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$

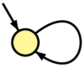


$$Traces(\mathcal{T}) = \{\varnothing^\omega\}$$
$$Traces_{fin}(\mathcal{T}) = \{\varnothing^n : n \geq 0\}$$

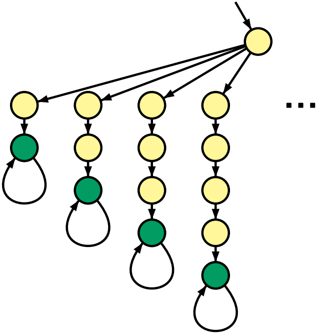
○ $\widehat{=}\ \varnothing$ ● $\widehat{=}\ \{b\}$

set of propositions
$AP = \{b\}$

$\mathcal{T}$



$\mathcal{T}'$



$$
\begin{aligned}
Traces(\mathcal{T}) &= \{\varnothing^\omega\} \\
Traces_{fin}(\mathcal{T}) &= \{\varnothing^n : n \geq 0\} \\
Traces(\mathcal{T}') &= \{\varnothing^n \{b\}^\omega : n \geq 2\}
\end{aligned}
$$

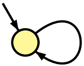
 $\hat{=}\ \varnothing$  $\hat{=}\ \{b\}$

set of propositions

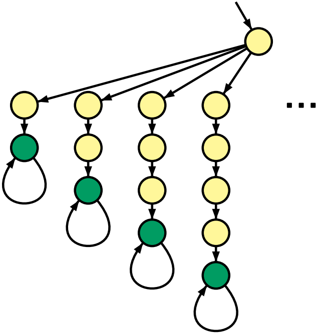
$AP = \{b\}$

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$



$$Traces(\mathcal{T}) = \{\varnothing^\omega\}$$

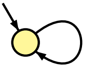
$$Traces_{fin}(\mathcal{T}) = \{\varnothing^n : n \geq 0\}$$

$$Traces(\mathcal{T}') = \{\varnothing^n \{b\}^\omega : n \geq 2\}$$

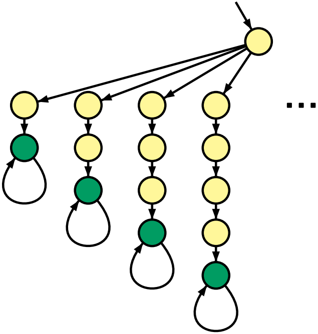
$$Traces_{fin}(\mathcal{T}') = \{\varnothing^n : n \geq 0\} \ \cup \ \{\varnothing^n \{b\}^m : n \geq 2 \wedge m \geq 1\}$$

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$

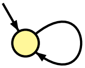


$$
\begin{aligned}
\mathit{Traces}(\mathcal{T}) &= \{\varnothing^\omega\} \\
\mathit{Traces}_{fin}(\mathcal{T}) &= \{\varnothing^n : n \geq 0\} \\
\mathit{Traces}(\mathcal{T}') &= \{\varnothing^n \{b\}^\omega : n \geq 2\} \\
\mathit{Traces}_{fin}(\mathcal{T}') &= \{\varnothing^n : n \geq 0\} \ \cup \\
& \quad \{\varnothing^n \{b\}^m : n \geq 2 \wedge m \geq 1\}
\end{aligned}
$$

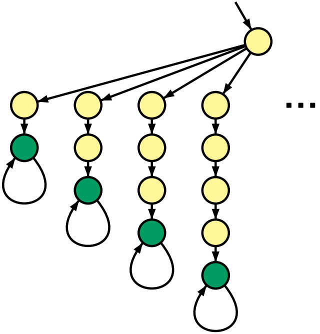
$\mathit{Traces}(\mathcal{T}) \not\subseteq \mathit{Traces}(\mathcal{T}')$, but
$\mathit{Traces}_{fin}(\mathcal{T}) \subseteq \mathit{Traces}_{fin}(\mathcal{T}')$

Finite trace relations versus trace relations

$\mathcal{T}$



$\mathcal{T}'$



$$
\begin{aligned}
Traces(\mathcal{T}) &= \{\varnothing^\omega\} \\
Traces_{fin}(\mathcal{T}) &= \{\varnothing^n : n \geq 0\} \\
Traces(\mathcal{T}') &= \{\varnothing^n \{b\}^\omega : n \geq 2\} \\
Traces_{fin}(\mathcal{T}') &= \{\varnothing^n : n \geq 0\} \ \cup \\
&\quad\ \{\varnothing^n \{b\}^m : n \geq 2 \wedge m \geq 1\}
\end{aligned}
$$

$Traces(\mathcal{T}) \not\subseteq Traces(\mathcal{T}')$, but

$Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

LT property

$E \mathrel{\widehat{=}}$ "eventually b"

$\mathcal{T} \not\models E, \quad \mathcal{T}' \models E$

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over $\boldsymbol{AP}$ such that

(1) $\mathcal{T}$ has no terminal states,

(2) $\mathcal{T}'$ is finite.

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states,
i.e., all paths of $\mathcal{T}$ are infinite

(2) $\mathcal{T}'$ is finite.

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states,
i.e., all paths of $\mathcal{T}$ are infinite

(2) $\mathcal{T}'$ is finite.

Then:

$$\begin{aligned} & \mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}') \\ \text{iff}\quad & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}') \end{aligned}$$

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states,
i.e., all paths of $\mathcal{T}$ are infinite

(2) $\mathcal{T}'$ is finite.

Then:

$$Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$$
$$\text{iff} \quad Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$$

"$\Longrightarrow$": holds for all transition systems,
no matter whether (1) and (2) hold

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states,
i.e., all paths of $\mathcal{T}$ are infinite

(2) $\mathcal{T}'$ is finite.

Then:
$$Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$$
$$\text{iff} \quad Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$$

"$\Longrightarrow$": holds for all transition systems

"$\Longleftarrow$": suppose that (1) and (2) hold and that

(3) $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

Show that $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states

(2) $\mathcal{T}'$ is finite

(3) $\mathit{Traces}_{fin}(\mathcal{T}) \subseteq \mathit{Traces}_{fin}(\mathcal{T}')$

Then $\mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}')$

Proof:

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states

(2) $\mathcal{T}'$ is finite

(3) $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

Then $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

Proof: Pick some path $\pi = s_0\, s_1\, s_2 \ldots$ in $\mathcal{T}$ and show that there exists a path

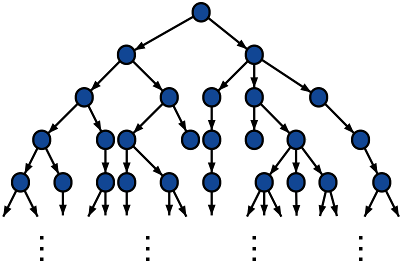
$$\pi' = t_0\, t_1\, t_2 \ldots \text{ in } \mathcal{T}'$$

such that $trace(\pi) = trace(\pi')$

Tracesfin versus traces

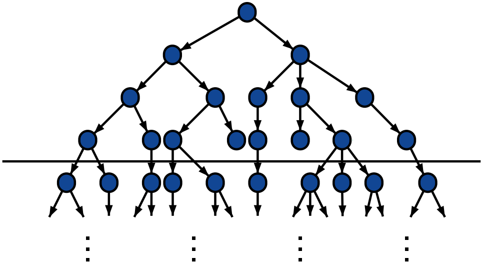
finite TS $\mathcal{T}'$

paths from state t_0
(unfolded into a tree)



finite TS $\mathcal{T}'$

paths from state t_0
(unfolded into a tree)



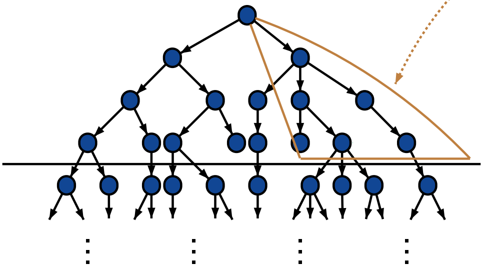
finite until
depth $\leq n$

Tracesfin versus traces

finite TS $\mathcal{T}'$

paths from state t_0
(unfolded into a tree)

contains all path fragments with trace $A_0\,A_1 \ldots A_n$



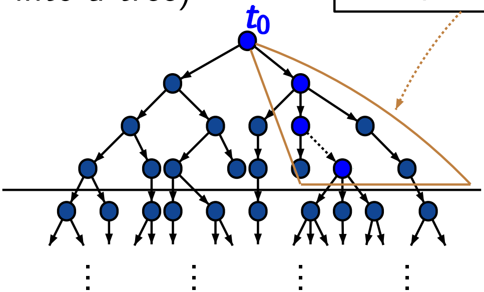
finite until depth $\leq n$

Tracesfin versus traces

finite TS $\mathcal{T}'$

paths from state t_0
(unfolded into a tree)

contains all path fragments with trace $A_0\,A_1 \ldots A_n$
in particular: $t_0\,t_1 \ldots t_n$



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finite TS $\mathcal{T}'$

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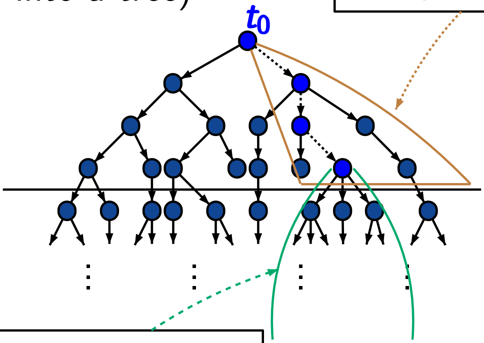

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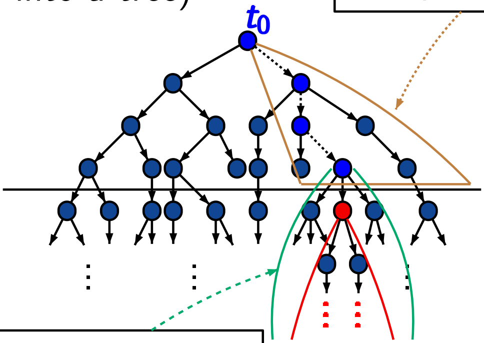

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there exists $t_{n+1} \in Post(t_n)$
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infinitely many m

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states

(2) $\mathcal{T}'$ is finite $\longleftarrow$ image-finiteness is sufficient

(3) $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

Then $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

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Trace equivalence vs. finite trace equivalence

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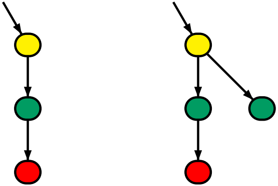

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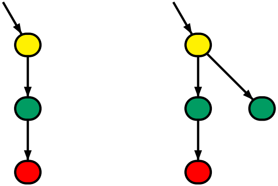

finite trace equivalent,

but *not* trace equivalent

Whenever $\mathit{Traces}(\mathcal{T}) = \mathit{Traces}(\mathcal{T}')$ then
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The reverse implication holds under additional assumptions, e.g.,

- if $\mathcal{T}$ and $\mathcal{T}'$ are finite and have no terminal states
- or, if $\mathcal{T}$ and $\mathcal{T}'$ are AP-deterministic

Overview

Liveness

“liveness: something good will happen.”

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"event a will occur eventually"

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e.g., termination for sequential programs

"event ***a*** will occur infinitely many times"

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"whenever event ***b*** occurs then event ***a***
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e.g., every waiting process enters eventually
its critical section

which property type?

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Liveness

many different formal definitions of liveness
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here: one just example for a formal definition
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Definition of liveness properties

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recall: $pref(E) =$ set of all finite, nonempty prefixes of words in E

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Examples:

- each process will eventually enter its critical section
- each process will enter its critical section infinitely often
- whenever a process has requested its critical section then it will eventually enter its critical section

Examples for liveness properties

An LT property E over AP is called a liveness property if $pref(E) = (2^{AP})^+$

Examples for $AP = \{crit_i : i = 1, \ldots, n\}$:

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$$\forall i \in \{1, \ldots, n\}\ \forall j \geq 0.\ wait_i \in A_j \longrightarrow \exists k > j.\ crit_i \in A_k$$

Recall: safety properties, prefix closure

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> $$\left\{\sigma' \in E : A_0 A_1 \ldots A_n \in \mathit{pref}(\sigma')\right\} = \varnothing$$

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remind:

$$\mathit{pref}(\sigma) = \text{set of all finite, nonempty prefixes of } \sigma$$

$$\mathit{pref}(E) = \bigcup_{\sigma \in E} \mathit{pref}(\sigma)$$

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iff $cl(E) = E$

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Being safe and live

Which LT properties are both
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If E is a safety property too, then $cl(E) = E$.

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Hence $E = cl(E) = (2^{AP})^\omega$.