Assignment #2 Sample Solutions

Question 1: Hilarious Numbers

Let us define that a positive integer greater than 1 is called *hilarious* if the sum of its unique prime factors is a prime itself. Therefore, all primes are hilarious. Also, for example, 12 is hilarious, because its unique prime factors are 2 and 3, and their sum 5 is a prime, too. On the other hand, 84 is not hilarious, because $84 = 2^2 \cdot 3 \cdot 7$ and $2 + 3 + 7 = 12$, which is not a prime.

a) Write a program in Java, C, C++, Python, or pseudocode that finds for a given number $n$ the first $n$ hilarious numbers. Attach a printout of your program to your assignment.

Here is a solution in plain, old C:

```c
#include <stdio.h>

int isHilarious(int n)
{
    int i, foundPrimeFactor, primeFactorSum = 0, currFactor = 1;
    while (n > 1)
    {
        currFactor++;
        foundPrimeFactor = 0;
        while (n % currFactor == 0)
        {
            n /= currFactor;
            foundPrimeFactor = 1;
        }
        if (foundPrimeFactor)
            primeFactorSum += currFactor;
    }
    for (i = 2; i <= primeFactorSum / 2; i++)
        if (primeFactorSum % i == 0)
            return 0;
    return 1;
}
```
int main()
{
    int i = 1, num;

    printf("How many hilarious numbers would you like? ");
    scanf_s("%d", &num);
    printf("Here you go:
");

    while (num)
    {
        i++;
        if (isHilarious(i))
        {
            printf("%d ", i);
            num--;
        }
    }
    printf("\n");
    return 0;
}

This program decides whether a number \( n \) is hilarious by trying to divide it by 2, then by 3, 4, 5, and so on. Whenever it finds that there is no remainder, it divides \( n \) by that number, puts the result back into \( n \), and keeps doing this for the same divisor until the division leaves a remainder. Then it adds the divisor to the sum of prime factors and resumes with finding greater and greater divisors of \( n \). Once \( n = 1 \), the sum is complete, and we just have to check whether it is prime. Basically, the algorithm performs prime factorization while keeping track of the sum of prime factors discovered so far.

b) What are the first 20 hilarious numbers as discovered by your program? If you used pseudocode, you need to find these numbers “by hand.”

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 23, 24

It turns out that many numbers are hilarious…

**Question 2: Prime Factor Examples**

Write down the prime factorization (in ascending order) of each of the following integers (Example: \( 720 = 2^4 \cdot 3^2 \cdot 5 \)).

a) 258 = 2 \cdot 3 \cdot 43
b) 100000 = 2^5 \cdot 5^5
c) 6250 = 2 \cdot 5^5
d) 104 = 8 \cdot 13
**Question 3: Euclidean Algorithm**

Use the Euclidean algorithm to determine the following greatest common divisors. Write down every step in your calculation.

a) \( \text{gcd}(3300, 550) \)

\[
3300 \mod 550 = 0
\]

\[\Rightarrow \text{gcd}(3300, 550) = 550\]

b) \( \text{gcd}(177, 300) \)

\[
300 \mod 177 = 123
\]
\[
177 \mod 123 = 54
\]
\[
123 \mod 54 = 15
\]
\[
54 \mod 15 = 9
\]
\[
15 \mod 9 = 6
\]
\[
9 \mod 6 = 3
\]
\[
6 \mod 3 = 0
\]

\[\Rightarrow \text{gcd}(177, 300) = 3\]

c) \( \text{gcd}(912, 625) \)

\[
912 \mod 625 = 287
\]
\[
625 \mod 287 = 51
\]
\[
287 \mod 51 = 32
\]
\[
51 \mod 32 = 19
\]
\[
32 \mod 19 = 13
\]
\[
19 \mod 13 = 6
\]
\[
13 \mod 6 = 1
\]
\[
6 \mod 1 = 0
\]

\[\Rightarrow \text{gcd}(912, 625) = 1\]
Question 4: Matrices

a) Find a matrix \( M \) such that

\[
\begin{bmatrix}
2 & 4 \\
1 & 3
\end{bmatrix} M + \begin{bmatrix}
-6 & 3 \\
0 & -2
\end{bmatrix} = \begin{bmatrix}
4 & 27 \\
6 & 15
\end{bmatrix}
\]

Hint: For each column of \( A \) you have to solve a system of linear equations. Write down all of these equations and every step of their solution.

Let \( M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \). Then we get the following equations:

1. \( 2m_{11} + 4m_{21} + (-6) = 4 \Rightarrow 2m_{11} + 4m_{21} = 10 \)
2. \( m_{11} + 3m_{21} + 0 = 6 \Rightarrow m_{11} + 3m_{21} = 6 \)
3. \( 2m_{12} + 4m_{22} + 3 = 27 \Rightarrow 2m_{12} + 4m_{22} = 24 \)
4. \( m_{12} + 3m_{22} + (-2) = 15 \Rightarrow m_{12} + 3m_{22} = 17 \)

From (2) we get:

5. \( m_{11} = 6 - 3m_{21} \)

If we use this to substitute \( m_{11} \) in (1) we get:

\[
12 - 6m_{21} + 4m_{21} = 10 \Rightarrow -2m_{21} = -2 \Rightarrow m_{21} = 1
\]

From (5) it follows:

\[
m_{11} = 6 - 3 \cdot 1 = 3
\]

From (4) we get:

6. \( m_{12} = 17 - 3m_{22} \)

If we use this to substitute \( m_{12} \) in (3) we get:

\[
34 - 6m_{22} + 4m_{22} = 24 \Rightarrow -2m_{22} = -10 \Rightarrow m_{22} = 5
\]

From (6) it follows:

\[
m_{12} = 17 - 3 \cdot 5 = 2
\]
Therefore, the solution is $M = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$.

b) Show that for any zero-one matrix $A$ it is true that
- $A \land A = A$
- $A \lor A = A$

\[
A \land A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \land \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \land a_{11} & \ldots & a_{1n} \land a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \land a_{m1} & \ldots & a_{mn} \land a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = A
\]

\[
A \lor A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \lor \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \lor a_{11} & \ldots & a_{1n} \lor a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \lor a_{m1} & \ldots & a_{mn} \lor a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = A
\]

**Question 5: Rules of Inference**

Use rules of inference to show whether the following arguments are valid or not:

a) If a course is boring and taught at a late time, the students will fall asleep. The course is boring, and the students do not fall asleep. Therefore, the course is not taught at a late time.

B: The course is boring.

L: The course is taught at a late time.

F: The students fall asleep.

Hypotheses:

$(B \land L) \rightarrow F$

B

$\neg F$

Conclusion:

$\neg L$
Step 1: \((B \land L) \rightarrow F\)

Step 2: \(\neg F\)

Step 3: \(\neg(B \land L) \iff \neg B \lor \neg L\) \hspace{1cm} \text{(Modus Tollens of Steps 1 and 2)}

Step 4: \(B\)

Step 5: \(\neg L\) \hspace{1cm} \text{(Disjunctive Syllogism of Steps 3 and 4)}

The argument is valid.

b) All UMB students are Red Sox fans. Whenever the Red Sox win, all Red Sox fans celebrate. The Red sox won. Therefore, Kermit, who is a UMB student, celebrates.

\(U(x): x\) is a UMB student.

\(R(x): x\) is a Red Sox fan.

\(W: \) The Red Sox win.

\(C(x): x\) celebrates.

Hypotheses:

\(\forall x [U(x) \rightarrow R(x)]\)

\(W \rightarrow \forall x [R(x) \rightarrow C(x)]\)

\(W\)

\(U(\text{Kermit})\)

Conclusion:

\(C(\text{Kermit})\)

Step 1: \(\forall x [U(x) \rightarrow R(x)]\)

Step 2: \(U(y) \rightarrow R(y)\) for any \(y\) \hspace{1cm} \text{(Universal Instantiation of Step 1)}

Step 3: \(W \rightarrow \forall x [R(x) \rightarrow C(x)]\)
Step 4: W

Step 5: \( \forall x \ [R(x) \rightarrow C(x)] \) (Modus Ponens of Steps 3 and 4)

Step 6: \( R(y) \rightarrow C(y) \) for any \( y \) (Universal Instantiation of Step 5)

Step 7: \( U(y) \rightarrow C(y) \) for any \( y \) (Hypothetical Syllogism of Steps 2 and 6)

Step 8: \( U(\text{Kermit}) \)

Step 9: \( C(\text{Kermit}) \) (Modus Ponens of Steps 7 and 8)

The argument is valid.

c) If time travel were possible, someone from the future would have visited us. If someone from the future had visited us, we would have heard about it. We have not heard about it. Therefore, time travel is impossible.

T: Time travel is possible
F: Someone from the future has visited us
H: We heard about someone from the future visiting us.

Hypotheses:
\[ T \rightarrow F \]
\[ F \rightarrow H \]
\[ \neg H \]

Conclusion:
\[ \neg T \]

Step 1: \( T \rightarrow F \)
Step 2: \( F \rightarrow H \)
Step 3: \( T \rightarrow H \) (Hypothetical Syllogism of Steps 1 and 2)
Step 4: \( \neg H \)
Step 5: \( \neg T \) (Modus Tollens of Steps 3 and 4)
Question 6: Proofs

Prove or disprove the following statements:

a) \(2^n + 1\) is prime for all positive integers \(n\).

This statement is false. Proof by counterexample:

For \(n = 3\), \(2^n + 1 = 9\), which is not prime, since \(3 \cdot 3 = 9\).

b) Prove that for any integer \(x\), \(x^2 + x + 1\) is odd.

We know that if an integer \(n\) is even, then \(n = 2k\) for some integer \(k\). Similarly, if an integer \(n\) is odd, then \(n = 2k + 1\) for some integer \(k\).

Then the square of an even integer \(n\) is even:

\[
\begin{align*}
n &= 2k \\
n^2 &= 4k^2 \\
\end{align*}
\]

\(n^2 = 2 \cdot (2k^2)\), where \(2k^2\) is clearly an integer.

And the square of an odd integer is odd:

\[
\begin{align*}
n &= 2k + 1 \\
n^2 &= 4k^2 + 4k + 1 \\
\end{align*}
\]

\(n^2 = 2 \cdot (2k^2 + 2k) + 1\), where \(2k^2 + 2k\) is clearly an integer.

Whenever we add an even integer \(m = 2k\) and an odd integer \(n = 2q + 1\), we get an odd integer:

\[
\begin{align*}
m + n &= 2k + 2q + 1 = 2 \cdot (k + q) + 1, \text{ where } k + q \text{ is an integer.}
\end{align*}
\]

(It is OK if you did not provide the proofs above.)

Then we can distinguish two cases:

(1) \(x\) is even. Then \(x^2\) is even and \(x + 1\) is odd, so \(x^2 + x + 1\) is odd.

(2) \(x\) is odd. Then \(x^2\) is odd and \(x + 1\) is even, so \(x^2 + x + 1\) is odd.
There are no other cases, and therefore the statement is true for any integer $x$.

c) For every integer $n$, $n(n + 1)$ is even.

We first show that the product of an even number $m = 2k$ and an odd number $n = 2q + 1$ is even:

$$m \cdot n = 2k \cdot (2q + 1) = 4kq + 2k = 2(2kq + k),$$

where $2kq + k$ is an integer.

Now if $n$ is even, then $n + 1$ is odd, so $n(n + 1)$ is odd.

And if $n$ is odd, then $n + 1$ is even, so $n(n + 1)$ is odd.

There are no other possibilities, so we have proven the statement to be true for all integers $n$.

d) The product of three odd integers is odd.

For any odd integers $a$, $b$, $c$ we can write $a = 2p + 1$, $b = 2q + 1$, and $c = 2r + 1$ with integers $p$, $q$, and $r$. Then we have:

$$abc = (2p + 1)(2q + 1)(2r + 1) = 8pqr + 4pq + 4pr + 4qr + 2p + 2q + 2r + 1$$

$$abc = 2(4pqr + 2pq + 2pr + 2qr + p + q + r) + 1$$

Since the expression in parentheses is clearly an integer, the product $abc$ must be odd.

**Question 7: Formalization of Logical Expressions**

Here I am using “=” and “≠” to simplify things, and that is OK, but if we wanted to be formally correct, we should define and use an identity predicate, for example, $\text{ID}(x, y)$.

a) Franz and Erika are the only professors who play soccer.

$$\text{Professor(Franz)} \land \text{Professor(Erika)} \land \text{PlaysSoccer(Franz)} \land \text{PlaysSoccer(Erika)} \land \forall x \left[ \text{Professor}(x) \land \text{PlaysSoccer}(x) \rightarrow x = \text{Franz} \lor x = \text{Erika} \right]$$

b) All spouses of UMB students know each other.

$$\forall x,y,u,v \left[ \text{UMB_Student}(x) \land \text{UMB_Student}(y) \land \text{Spouses}(x, u) \land \text{Spouses}(y, v) \rightarrow \text{Knows}(u, v) \right]$$
c) Petra and Bert never take classes that are interesting.
\[ \forall x \ [ \text{Class}(x) \land \text{Interesting}(x) \rightarrow \neg \text{Takes}(\text{Petra}, x) \land \neg \text{Takes}(\text{Bert}, x)] \]

d) Any UMB student will fail if he/she does not complete Assignment #1.
\[ \forall x \ [ \text{UMB Student}(x) \land \neg \text{Completes}(x, \text{Assignment1}) \rightarrow \text{Fails}(x)] \]

e) Andreas beats up all professors that fail him and do not fail his sister Susanne.
\[ \forall x \ [ \text{Professor}(x) \land \text{Fails}(x, \text{Andreas}) \land \neg \text{Fails}(x, \text{Susanne}) \rightarrow \text{BeatsUp}(\text{Andreas}, x)] \]

f) There is at most one computer scientist who can dance.
\[ \neg \exists x, y \ [ x \neq y \land \text{CS}(x) \land \text{CS}(y) \land \text{CanDance}(x) \land \text{CanDance}(y)] \]

g) There is one UMB student who is taller than 7 feet and taller than all other UMB students.
\[ \exists x \ [ (\text{UMB Student}(x) \land \text{TallerThan}(x, 7 \text{ feet}) \land \forall y \ [ x \neq y \land \text{UMB Student}(y) \rightarrow \text{TallerThan}(x, y))] \]

h) All UMB students who work full-time prefer evening classes.
\[ \forall x \ [ \text{UMB Student}(x) \land \text{WorksFullTime}(x) \rightarrow \text{Prefers}(x, \text{Evening Classes})] \]

i) There are no scientists who are neither crazy nor dangerous.
\[ \forall x \ [ \text{Scientist}(x) \rightarrow \text{Crazy}(x) \lor \text{Dangerous}(x)] \]

j) George and Claudia always attend the same courses.
\[ \forall x \ [ \text{Course}(x) \rightarrow (\text{Takes}(\text{George}, x) \leftrightarrow \text{Takes}(\text{Claudia}, x))] \]