Representing Relations Using Digraphs

**Definition:** A directed graph, or digraph, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs). The vertex $a$ is called the initial vertex of the edge $(a, b)$, and the vertex $b$ is called the terminal vertex of this edge.

We can use arrows to display graphs.

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Example: Display the digraph with $V = \{a, b, c, d\}$, $E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$.

An edge of the form $(b, b)$ is called a loop.

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Representing Relations Using Digraphs

Obviously, we can represent any relation $R$ on a set $A$ by the digraph with $A$ as its vertices and all pairs $(a, b) \in R$ as its edges.

Vice versa, any digraph with vertices $V$ and edges $E$ can be represented by a relation on $V$ containing all the pairs in $E$.

This one-to-one correspondence between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

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Closures of Relations

What is the closure of a relation?

**Definition:** Let $R$ be a relation on a set $A$. $R$ may or may not have some property $P$, such as reflexivity, symmetry, or transitivity. If there is a relation $S$ that contains $R$ and has property $P$, and $S$ is a subset of every relation that contains $R$ and has property $P$, then $S$ is called the closure of $R$ with respect to $P$.

Note that the closure of a relation with respect to a property may not exist.

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Example I: Find the reflexive closure of relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$.

**Solution:** We know that any reflexive relation on $A$ must contain the elements $(1, 1), (2, 2),$ and $(3, 3)$. By adding $(2, 2)$ and $(3, 3)$ to $R$, we obtain the reflexive relation $S$, which is given by $S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)\}$.

$S$ is reflexive, contains $R$, and is contained within every reflexive relation that contains $R$.

Therefore, $S$ is the reflexive closure of $R$.

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Example II: Find the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers.

**Solution:** The symmetric closure of $R$ is given by $R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$.
Closures of Relations

Example III: Find the transitive closure of the relation \( R = \{(1, 3), (1, 4), (2, 1), (3, 2)\} \) on the set \( A = \{1, 2, 3, 4\} \).

Solution: R would be transitive, if for all pairs \( (a, b) \) and \( (b, c) \) in R there were also a pair \( (a, c) \) in R. If we add the missing pairs \( (1, 2), (2, 3), (2, 4), \) and \( (3, 1) \), will R be transitive? 
No, because the extended relation R contains \( (3, 1) \) and \( (1, 4) \), but does not contain \( (3, 4) \). By adding new elements to R, we also add new requirements for its transitivity. We need to look at paths in digraphs to solve this problem.

Closures of Relations

Definition: A path from a to b in the directed graph G is a sequence of one or more edges \( (x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n) \) in G, where \( x_0 = a \) and \( x_n = b \). In other words, a path is a sequence of edges where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path. This path is denoted by \( x_0, x_1, x_2, \ldots, x_n \) and has length \( n \). A path that begins and ends at the same vertex is called a circuit or cycle.

Example: Let us take a look at the following graph:

\[ \text{Is } c, a, b, d, b, a \text{ a path in this graph? Yes.} \]
\[ \text{Is } d, b, b, b, d, b, d \text{ a circuit in this graph? Yes.} \]
\[ \text{Is there any circuit including } c \text{ in this graph? No.} \]

Closures of Relations

Definition: Let R be a relation on a set A. The connectivity relation \( R^* \) consists of the pairs (a, b) such that there is a path between a and b in R. We know that \( R^n \) consists of the pairs (a, b) such that a and b are connected by a path of length \( n \). Therefore, \( R^* \) is the union of \( R^n \) across all positive integers \( n \):

\[
R^* = \bigcup_{n=1}^{\infty} R^n = R \cup R^2 \cup R^3 \cup \ldots
\]
Closures of Relations

**Theorem:** The transitive closure of a relation \( R \) equals the connectivity relation \( R^* \).

But how can we compute \( R^* \)?

**Lemma:** Let \( A \) be a set with \( n \) elements, and let \( R \) be a relation on \( A \). If there is a path in \( R \) from \( a \) to \( b \), then there is such a path with length not exceeding \( n \).

Moreover, if \( a \neq b \) and there is a path in \( R \) from \( a \) to \( b \), then there is such a path with length not exceeding \( n - 1 \).

This lemma is based on the observation that if a path from \( a \) to \( b \) visits any vertex more than once, it must include at least one **circuit**.

These circuits can be **eliminated** from the path, and the reduced path will still connect \( a \) and \( b \).

**Theorem:** For a relation \( R \) on a set \( A \) with \( n \) elements, the transitive closure \( R^* \) is given by:

\[
R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n
\]

For matrices representing relations we have:

\[
M_R = M_R \vee M_R^2 \vee M_R^3 \vee \ldots \vee M_R^n
\]

Let us finally solve **Example III** by finding the transitive closure of the relation \( R = \{(1, 3), (1, 4), (2, 1), (3, 2)\} \) on the set \( A = \{1, 2, 3, 4\} \).

\( R \) can be represented by the following matrix \( M_R \):

\[
M_R = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

**Solution:** The transitive closure of the relation \( R = \{(1, 3), (1, 4), (2, 1), (3, 2)\} \) on the set \( A = \{1, 2, 3, 4\} \) is given by the relation

\[
\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}
\]

**Equivalence Relations**

**Equivalence relations** are used to relate objects that are similar in some way.

**Definition:** A relation on a set \( A \) is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation \( R \) are called **equivalent**.
Equivalence Relations

Since $R$ is **symmetric**, $a$ is equivalent to $b$ whenever $b$ is equivalent to $a$.

Since $R$ is **reflexive**, every element is equivalent to itself.

Since $R$ is **transitive**, if $a$ and $b$ are equivalent and $b$ and $c$ are equivalent, then $a$ and $c$ are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Classes

**Definition:** Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the **equivalence class** of $a$.

The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$.

When only one relation is under consideration, we will delete the subscript $R$ and write $[a]$ for this equivalence class.

If $b \in [a]_R$, $b$ is called a **representative** of this equivalence class.

Equivalence Classes

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

(i) $aRb$
(ii) $[a] = [b]$
(iii) $[a] \cap [b] \neq \emptyset$

**Definition:** A **partition** of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union. In other words, the collection of subsets $A_i$, $i \in I$, forms a partition of $S$ if and only if

(i) $A_i \neq \emptyset$ for $i \in I$
(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$
(iii) $\bigcup_{i \in I} A_i = S$

Equivalence Classes

**Example:** Suppose that $R$ is the relation on the set of strings that consist of English letters such that $aRb$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string $x$. Is $R$ an equivalence relation?

**Solution:**

- $R$ is reflexive, because $l(a) = l(a)$ and therefore $aRa$ for any string $a$.
- $R$ is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if $aRb$ then $bRa$.
- $R$ is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so $aRb$ and $bRc$ implies $aRc$.

$R$ is an equivalence relation.

Equivalence Classes

**Example:** In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by $[\text{mouse}]$?

**Solution:** $[\text{mouse}]$ is the set of all English words containing five letters.

For example, ‘horse’ would be a representative of this equivalence class.

Equivalence Classes

**Examples:** Let $S$ be the set $\{u, m, b, r, o, c, k, s\}$. Do the following collections of sets partition $S$?

- $\{\{m, o, c, k\}, \{r, u, b, s\}\}$ yes.
- $\{\{c, o, m, b\}, \{u, s\}, \{r\}\}$ no (k is missing).
- $\{\{b, r, o, c, k\}, \{m, u, s, t\}\}$ no (t is not in S).
- $\{\{u, m, b, r, o, c, k, s\}\}$ yes.
- $\{\{b, o, r, k\}, \{r, u, m\}, \{c, s\}\}$ No (r is in two sets)
- $\{\{u, m, b\}, \{r, o, c, k, s\}, \emptyset\}$ no (\emptyset not allowed).
Equivalence Classes

**Theorem:** Let $R$ be an equivalence relation on a set $S$. Then the **equivalence classes** of $R$ form a **partition** of $S$. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$, $i \in I$, as its equivalence classes.

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**Example:** Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

Let $R$ be the equivalence relation $\{(a, b) \mid a$ and $b$ live in the same city$\}$ on the set $P = \{\text{Frank, Suzanne, George, Stephanie, Max, Jennifer}\}$.

Then $R = \{(\text{Frank, Frank}), (\text{Frank, Suzanne}), (\text{Suzanne, Frank}), (\text{Suzanne, Suzanne}), (\text{Suzanne, George}), (\text{George, Frank}), (\text{George, Suzanne}), (\text{George, George}), (\text{Stephanie, Stephanie}), (\text{Stephanie, Max}), (\text{Max, Stephanie}), (\text{Max, Max}), (\text{Jennifer, Jennifer})\}$.

Then the **equivalence classes** of $R$ are:

$\{\{\text{Frank, Suzanne, George}\}, \{\text{Stephanie, Max}\}, \{\text{Jennifer}\}\}$.

This is a **partition** of $P$.

The equivalence classes of any equivalence relation $R$ defined on a set $S$ constitute a partition of $S$, because every element in $S$ is assigned to **exactly one** of the equivalence classes.

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**Another example:** Let $R$ be the relation $\{(a, b) \mid a \equiv b \pmod{3}\}$ on the set of integers.

Is $R$ an equivalence relation?

Yes, $R$ is reflexive, symmetric, and transitive.

What are the equivalence classes of $R$?

$\{\{\ldots, -6, -3, 0, 3, 6, \ldots\}, \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \{\ldots, -4, -1, 2, 5, 8, \ldots\}\}$

Again, these three classes form a partition of the set of integers.

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**Partial Orderings**

Sometimes, relations do not specify the equality of elements in a set, but define an **order** on them.

**Definition:** A relation $R$ on a set $S$ is called a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive.

A set $S$ together with a partial ordering $R$ is called a **partially ordered set**, or **poset**, and is denoted by $(S, R)$.

**Example:** Consider the “greater than or equal” relation $\geq$ (defined by $\{(a, b) \mid a \geq b\}$).

Is $\geq$ a **partial ordering** on the set of integers?

- $\geq$ is **reflexive**, because $a \geq a$ for every integer $a$.
- $\geq$ is **antisymmetric**, because if $a \neq b$, then $a \geq b \land b \geq a$ is false.
- $\geq$ is **transitive**, because if $a \geq b$ and $b \geq c$, then $a \geq c$.

Consequently, $(\mathbb{Z}, \geq)$ is a partially ordered set.
Partial Orderings

Another example: Is the "inclusion relation" \( \subseteq \) a partial ordering on the power set of a set \( S \)?

- \( \subseteq \) is reflexive, because \( A \subseteq A \) for every set \( A \).
- \( \subseteq \) is antisymmetric, because if \( A \neq B \), then \( A \subseteq B \land B \subseteq A \) is false.
- \( \subseteq \) is transitive, because if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \).

Consequently, \( (P(S), \subseteq) \) is a partially ordered set.