Practice Midterm Exam

Sample Solutions

Question 1:       out of       points

Question 2:       out of       points

Question 3:       out of       points

Question 4:       out of       points

Question 5:       out of       points

Total Score:

Grade:
Question 1: What do we know?

Let $p$, $q$, and $r$ be propositions; $p$ is known to be true, $q$ is known to be false, and $r$’s truth value is unknown at this time. Tell whether each of the following compound propositions is true, is false, or has an unknown truth value at this time by circling the appropriate word.

a) $p \lor r$  true  false  unknown

b) $p \land r$  true  false  unknown

c) $p \rightarrow r$  true  false  unknown

d) $q \rightarrow r$  true  false  unknown

e) $r \rightarrow p$  true  false  unknown

f) $r \rightarrow q$  true  false  unknown

g) $(p \land r) \leftrightarrow r$  true  false  unknown

h) $(p \lor r) \leftrightarrow r$  true  false  unknown

i) $(q \land r) \leftrightarrow r$  true  false  unknown

j) $(q \lor r) \leftrightarrow r$  true  false  unknown
Question 2: Some Simple Tasks…

Show your calculations for each task.

a) Use the prime factorization method to determine gcd(5000, 600).

\[ 5000 = 2^3 \cdot 5^4 \]
\[ 600 = 2^3 \cdot 3^1 \cdot 5^2 \]
Therefore, \( \text{gcd}(5000, 600) = 2^3 \cdot 3^0 \cdot 5^2 = 200 \)

b) Use the prime factorization method to determine lcm(30, 42).

\[ 30 = 2^1 \cdot 3^1 \cdot 5^1 \]
\[ 42 = 2^1 \cdot 3^1 \cdot 7^1 \]
Therefore, \( \text{lcm}(30, 42) = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 210 \)

c) Convert the integer \((1101001)_2\) from binary expansion to decimal expansion.

\[ (1101001)_2 = 2^6 + 2^5 + 2^3 + 2^0 = 64 + 32 + 8 + 1 = 105 \]

d) Convert the integer 5322 from decimal expansion to hexadecimal expansion.

\[ 5322/16 = 332 \text{ R } 10 \]
\[ 332/16 = 20 \text{ R } 12 \]
\[ 20/16 = 1 \text{ R } 4 \]
\[ 1/16 = 0 \text{ R } 1 \]
Result: \((14CA)_{16}\)

Question 3: Rules of Inference

Use rules of inference to show that the arguments below are valid, i.e., that their conclusion follows from their hypotheses. Apply the step-by-step method we used in class and list all those steps in your answer.

a) Hypotheses: If there is gas in the car, then I will go to the store. If I go to the store, then I will get a soda. There is gas in the car. Conclusion: I will get a soda.

\[ g: \text{“there is gas in the car”} \]
\[ s: \text{“I go to the store”} \]
\[ d: \text{“I will get a soda”} \]
Hypotheses:
\[ g \rightarrow s \]
\[ s \rightarrow d \]
\[ g \]

Conclusion:
\[ d \]

Step 1: \( g \) Hypothesis
Step 2: \( g \rightarrow s \) Hypothesis
Step 3: \( s \) R.I. using Steps 1 and 2
Step 4: \( s \rightarrow d \) Hypothesis
Step 5: \( d \) R.I. using Steps 3 and 4

b) Hypotheses: When Prof. P. gets angry, he fails his entire class. When the entire class fails, the Chancellor gets complaints. When the Chancellor gets complaints, he will either fire Prof. P., cut his salary, or do both. Prof. P. got angry, and he was not fired. Conclusion: Prof. P.’s salary was cut.

a: “Prof. P. gets angry”
f: “Prof. P. fails the entire class”
c: “Chancellor gets complaints”
p: “Chancellor fires Prof. P.”
s: “Chancellor cuts Prof. P.’s salary”

Hypotheses:
\[ a \rightarrow f \]
\[ f \rightarrow c \]
\[ c \rightarrow (p \lor s) \]
\[ a \]
\[ \neg p \]

Conclusion:
\[ s \]

Step 1: \( a \) Hypothesis
Step 2: \( a \rightarrow f \) Hypothesis
Step 3: \( f \) R.I. Steps 1 and 2
Step 4: \( f \rightarrow c \) Hypothesis
Step 5: \( c \) R.I. Steps 3 and 4
Step 6: \( c \rightarrow (p \lor s) \) Hypothesis
Step 7: \( p \lor s \) R.I. Steps 5 and 6
Step 8: \( \neg p \) Hypothesis
Step 9: \( s \) R.I. Steps 7 and 8
**Question 4: Induction**

Use the principle of mathematical induction to show that the following equation is true for all positive integers $n$:

$$1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n + 1)! - 1$$

**Basis step:** Show that the equation is true for $n = 1$:

$$1(1!) = 2! - 1$$

1 = 1. True.

**Inductive step:** Show that if the equation is true for $n$, it is also true for $n + 1$.

**Assumption:**

$$1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n + 1)! - 1$$

Add $(n + 1)((n + 1)!)$ on both sides:

$$1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) + (n + 1)((n + 1)!) = (n + 1)! - 1 + (n + 1)((n + 1)!)$$

$$= (n + 2)(n + 1)! - 1$$

$$= (n + 2)! - 1$$

$$= ((n + 1) + 1)! - 1$$

Therefore, if the assumption holds for $n$, it will also hold for $n + 1$.

**Conclusion:** The equation is true for all positive integers $n$.

**Question 5: Possibilities**

a) Paul wants to come to the U.S. to first get his Bachelor’s degree, then his Master’s degree, and finally his Ph.D. He considers seven possible universities for his studies, but he does not want to receive more than one degree from the same university. How many different academic career paths are possible for Paul, assuming that he does not switch universities during each program and actually passes all three programs?

In a career path, the order of universities matters, so there are $7 \cdot 6 \cdot 5 = 210$ choices.

b) Ten years later, Paul becomes a professor at UMass Boston. He receives seven applications from students who want to work in his new Time Travel Lab, but his research grants are limited, and so he will hire only four of them. How many choices in the recruitment of his research team does Paul have?

The order of recruitment does not change the research team, so there are $C(7, 4) = 35$ choices.
c) Paul’s most famous research article is about four-letter strings made up of the letters ‘a’, ‘b’, and ‘c’. He discovers the number of such strings in which an ‘a’ is never immediately followed by another ‘a’ or a ‘b’, and a ‘b’ or ‘c’ is never immediately followed by a ‘c’. Use a tree diagram to replicate Paul’s finding. How many different strings of this kind are there?

There are 13 different strings:

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<th>First letter</th>
<th>Second letter</th>
<th>Third letter</th>
<th>Fourth letter</th>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>c</td>
<td>a</td>
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