Question 1:

a) Write down the code of Program $P$ in the language $L$ for which $\#(P) = 1831$.

The prime factorization of $\#(P) + 1 = 1832$ is $2^3 \cdot 229$. As it turns out, $229 = p_{50}$, and thus our program consists of instruction 3, followed by 48 times instruction 0, and finally instruction 1.

Instruction 3:

$3 = \langle 2, 0 \rangle = \langle 2, \langle 0, 0 \rangle \rangle$, so $a = 2$ (label B), $b = 0$ ($V \leftarrow V$), and $c = 0$ (variable Y):

[B] $Y \leftarrow Y$

Instruction 0:

$0 = \langle 0, 0 \rangle = \langle 0, \langle 0, 0 \rangle \rangle$, so $a = 0$ (no label), $b = 0$ ($V \leftarrow V$), and $c = 0$ (variable Y):

$Y \leftarrow Y$

Instruction 1:

$1 = \langle 1, 0 \rangle = \langle 1, \langle 0, 0 \rangle \rangle$, so $a = 1$ (label A), $b = 0$ ($V \leftarrow V$), and $c = 0$ (variable Y):

[A] $Y \leftarrow Y$

So the whole program is

[B] $Y \leftarrow Y$

$Y \leftarrow Y$ (48x the same instruction)

... 

[A] $Y \leftarrow Y$
b) What is the number of the following program?

```
IF X ≠ 0 GOTO A
Y ← Y + 1
[A]  Y ← Y + 1
```

You do not have to compute the numerical values of expressions such as $3^{27}$ that would result in huge numbers.

(1) IF $X ≠ 0$ GOTO A

$a = 0, b = 3, c = 1$

$<0, <3, 1>> = <0, 23> = 46$

(2) $Y ← Y + 1$

$a = 0, b = 1, c = 0$

$<0, <1, 0>> = <0, 1> = 2$

(3) [A]  $Y ← Y + 1$

$a = 1, b = 1, c = 0$

$<1, <1, 0>> = <1, 1> = 5$

Program number: $[46, 2, 5] − 1 = 2^{46} \cdot 3^2 \cdot 5^5 - 1$

**Question 2:**

Do you remember how we used the pairing function and the Gödel numbering to associate each program in the language $L$ with a unique natural number? To be precise, we demanded that every program in $L$ is associated with unique number, and we also required that every natural number is associated with a valid program in $L$. Now it is your task to develop such one-to-one mappings for other things. If you think that a mapping cannot be defined, please give a reason.

a) Define such a mapping for $2 \times 2$ matrices $[a_{ij}]$ of natural numbers.

We can simply use a pairing of pairings, so for a $2 \times 2$ matrix $A = [a_{ij}]$ the mapping $f$ would be:

$f(A) = <<a_{11}, a_{12}>, <a_{21}, a_{22}}>>$
b) Define such a mapping for odd natural numbers.

\[ f(n) = \frac{n - 1}{2} \]

c) Define such a mapping for the set of natural numbers \( \{0, 1, \ldots, 1000\} \).

There is no such mapping, because for one-to-one correspondences between two sets, the sets have to have the same cardinality.