Question 1:

(a)

#(I) = <a, <b, c>>,

where

b = 0 means instruction type $V \leftarrow V$
b = 1 means instruction type $V \leftarrow V + 1$
b = 2 means instruction type $V \leftarrow V - 1$
b = 3 means instruction type $V \leftarrow 0$ // our new instruction type!
b > 3 means instruction type IF $V \neq 0$ GOTO L, where b = #(L) + 3

(b)

	$Z \leftarrow X_{n+1} + 1$
	$S \leftarrow \Pi_{i=1}^{n} (p_{2i})^{X_i}$
	$K \leftarrow 1$
[C]	IF $K = Lt(Z) + 1 \vee K = 0$ GOTO F
	$U \leftarrow r((Z)_K)$
	$P \leftarrow p_{r(U)+1}$
	IF $l(U) = 0$ GOTO N
	IF $l(U) = 1$ GOTO A
	IF $\sim(P \mid S)$ GOTO N
	IF $l(U) = 2$ GOTO M
	IF l(U) = 3 GOTO Z
	$K \leftarrow \min_{i \leq Lt(Z)} [l((Z)_i) + \mathbf{3} = l(U)]$
	GOTO C
[Z]	**$S \leftarrow \lfloor S/P \rfloor$**
	IF P | S GOTO Z
	GOTO N
[M]	$S \leftarrow \lfloor S/P \rfloor$
	GOTO N
[A]	$S \leftarrow S \cdot P$
[N]	$K \leftarrow K+1$
	GOTO C
[F]	$Y \leftarrow (S)_1$

Question 2:

Let A, B be sets. Prove or disprove:

a) If $A \cup B$ is r.e., then A and B are both r.e.

The statement is false. Assume that $A = N$, $B = \overline{K}$. Then $A \cup B = N$ is r.e., but B is not r.e.

b) If $A \subseteq B$ and B is r.e., then A is r.e.

The statement is false. Assume that $A = \overline{K}$, $B = N$. Then $A \subseteq B$ and B is r.e., but A is not r.e.

Question 3:

a) boston

A = { a,b,n,o,s,t } |A| = 6 => n = 6 base 6

A = {s1,s2,s3,s4,s5,s6}

Integer correspondence 1, 2, 3, 4, 5 , 6

boston = 2 * 6^5 + 4 * 6^4 + 5 * 6^3 + 6 * 6^2 + 4 *6^1 + 3 *6^0 = 22059

b) B = { c,e,i,k,m,r,s,t } |B| = 8 => n = 8 base 8

B = { c ,e ,i ,k ,m ,r ,s ,t }

Integer correspondence 1, 2, 3, 4, 5 ,6, 7 ,8

	Q+(142688, 8) = 17835
i0	R+(142688,8) = 8
	Q+(17835, 8) = 2229
i1	R+(17835,8) = 3
	Q+(2229, 8) = 278
i2	R+(2229,8) = 5
	Q+(278, 8) = 34
i3	R+(278,8) = 6
	Q+(34, 8) = 4
i4	R+(34,8) = 2
	Q+(4, 8) = 0
i5	R+(4,8) = 4

Decimal 142688 in base 8 is 426538 which corresponds to string "kermit" in alphabet B.

c) Let A and B be alphabets with A $\subseteq$ B. Is it true for any natural number n that the string on the alphabet A associated with n is of equal or greater length than the string on the alphabet B associated with n? Prove or disprove.

Let us assume that the statement is false. Then, according to the definition of length, there must be values of |A|, |B|, and n such that:

$$\min_{x \le n}\left[\sum_{j=0}^{x} |A|^{j} > n\right] < \min_{x \le n}\left[\sum_{j=0}^{x} |B|^{j} > n\right]$$

Obviously, this can only be true if there is an x such that

$$\sum_{j=0}^{x} |A|^j > \sum_{j=0}^{x} |B|^j$$

But from this it follows that

$$|A| > |B|\ ,$$

which contradicts our initial assumption. Hence, the statement is true.

d) Let n be a natural number and w_A and w_B be the strings associated with n on the alphabets A and B, respectively. Is it true for any n that whenever the length of w_A is greater than or equal to the length of w_B, then |A| ≤ |B|? Prove or disprove.

The statement is false. Counterexample: Alphabets A = {s_1, s_2}, B = {s_1}. If n = 1 then $w_A = s_1$ and $w_B = s_1$.

e) Compute $UPCHANGE_{2,5}(123)$.

123 in base-2 notation is…

Q+(123, 2) = 61	R+(123, 2) = 1
Q+(61, 2) = 30	R+(61, 2) = 1
Q+(30, 2) = 14	R+(30, 2) = 2
Q+(14, 2) = 6	R+(14, 2) = 2
Q+(6, 2) = 2	R+(123, 2) = 2
Q+(2, 2) = 0	R+(123, 2) = 2

… $s_2s_2s_2s_2s_1s_1$.

In base-5 notation, $s_2s_2s_2s_2s_1s_1$ translates into the following integer:

2*5^5 + 2*5^4 + 2*5^3 + 2*5^2 + 5^1 + 5^0 = 6250 + 1250 + 250 + 50 + 5 + 1 = 7806

f) Compute $DOWNCHANGE_{3,7}(165)$.

165 in base-7 notation is…

Q+(165, 7) = 23	R+(165, 7) = 4
Q+(23, 7) = 3	R+(23, 7) = 2
Q+(3, 7) = 0	R+(3, 7) = 3

… $s_3s_2s_4$.

In base-3 notation, there is no s_4, so it is removed. Then s_3s_2 translates into the following integer:

3*3^1 + 2*3^0 = 9 + 2 = 11

g) Compute $LTEND_3(29)$.

Let us first convert 29 to base-3 notation:

Q+(29, 3) = 9	R+(29, 3) = 2
Q+(9, 3) = 2	R+(9, 3) = 3
Q+(2, 3) = 0	R+(2, 3) = 2

It is $s_2s_3s_2$. Therefore, $LTEND_3(29) = 2$.

h) C = {x,y,z}
The greatest possible string of length four in C is zzzz.
The decimal representation of zzzz = 3 * 3^3 + 3 *3^2 +3*3^1 + 3*3^0 = 120

Question 4:

In other words, for a given program with number u, there are infinitely many programs with different numbers that compute the same function as u. This is easy to see, as we can just add the unlabeled instruction Y ← Y before the first instruction of u and thereby create a program with a number different from u, which still computes the same function as u. We can do the same thing to this new program, and so on, and create an infinite number of such programs.

Mathematically speaking, we can compute an infinite sequence of program numbers v_n that compute the same function as u in the following way:

$v_0 = [0, (u + 1)_1, (u + 1)_2, \ldots, (u + 1)_{Lt(u + 1)}] - 1$
$v_{n+1} = [0, (v_n + 1)_1, (v_n + 1)_2, \ldots, (v_n + 1)_{Lt(v_{n + 1})}] - 1$