Primitive Recursive Predicates

Theorem 6.1: Let $\mathcal{C}$ be a PRC class.
If f(t, x_1, …, x_n) belongs to $\mathcal{C}$, then so do the functions

$$g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

$$h(y, x_1, \ldots, x_n) = \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

To prove this theorem, we will show that g and h can be obtained from f by primitive recursion.

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Proof I:

$$g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

can be obtained by recursion in the following way:

g(0, x_1, …, x_n) = f(0, x_1, …, x_n)
g(t + 1, x_1, …, x_n) = g(t, x_1, …, x_n) + f(t + 1, x_1, …, x_n)

Since + is primitive recursive, g belongs to $\mathcal{C}$.

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Proof II:

$$h(y, x_1, \ldots, x_n) = \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

can be obtained by recursion in the following way:

h(0, x_1, …, x_n) = f(0, x_1, …, x_n)
h(t + 1, x_1, …, x_n) = h(t, x_1, …, x_n) · f(t + 1, x_1, …, x_n)

Since · is primitive recursive, h belongs to $\mathcal{C}$.

Primitive Recursive Predicates

In some cases we might want to begin the iteration at 1 instead of 0:

$$g(y, x_1, ..., x_n) = \sum_{t=1}^{y} f(t, x_1, ..., x_n)$$

$$h(y, x_1, ..., x_n) = \prod_{t=1}^{y} f(t, x_1, ..., x_n)$$

Then we just need to replace the initial recursion equations:
g(0, x_1, …, x_n) = 0
h(0, x_1, …, x_n) = 1

Primitive Recursive Predicates

As a “by-product” of the preceding idea we obtained the following corollary:

Corollary 6.2: If f(t, x_1, …, x_n) belongs to the PRC class $\mathcal{C}$, then so do the following two functions:

$$g(y, x_1, ..., x_n) = \sum_{t=1}^{y} f(t, x_1, ..., x_n)$$

$$h(y, x_1, ..., x_n) = \prod_{t=1}^{y} f(t, x_1, ..., x_n)$$

Primitive Recursive Predicates

Theorem 6.3: If the predicate P(t, x_1, …, x_n) belongs to some PRC class $\mathcal{C}$, then so do the following two predicates:
$(\forall t)_{\le y}$ P(t, x_1, …, x_n) and $(\exists t)_{\le y}$ P(t, x_1, …, x_n)

Proof:

$$(\forall t)_{\le y} P(t, x_1, ..., x_n) \Leftrightarrow \left[\prod_{t=0}^{y} P(t, x_1, ..., x_n)\right] = 1$$

$$(\exists t)_{\le y} P(t, x_1, ..., x_n) \Leftrightarrow \left[\sum_{t=0}^{y} P(t, x_1, ..., x_n)\right] \ne 0$$

Primitive Recursive Predicates

Sometimes we may want to use the quantifiers $(\forall t)_{<y}\, P(t, x_1, \ldots, x_n)$ and $(\exists t)_{<y}\, P(t, x_1, \ldots, x_n)$.

Then Theorem 6.3 is still valid, which is obvious from the following two relations:

$(\forall t)_{<y}\, P(t, x_1, \ldots, x_n) \Leftrightarrow (\forall t)_{\leq y}\, [t = y \vee P(t, x_1, \ldots, x_n)]$

$(\exists t)_{<y}\, P(t, x_1, \ldots, x_n) \Leftrightarrow (\exists t)_{\leq y}\, [t \neq y \,\&\, P(t, x_1, \ldots, x_n)]$

Primitive Recursive Predicates

Example 12: y | x

y | x means "y is a divisor of x."

For example,

4 | 15 is **false.**
3 | 9 is **true.**

This predicate is primitive recursive because
$y \mid x \Leftrightarrow (\exists t)_{\leq x}\, (y \cdot t = x)$.

Primitive Recursive Predicates

Example 13: Prime(x)

Prime(x) is the predicate "x is a prime."

It is primitive recursive since
$\text{Prime}(x) \Leftrightarrow x > 1 \,\&\, (\forall t)_{\leq x}\, [t = 1 \vee t = x \vee \sim (t \mid x)]$.

Minimalization

Let the predicate $P(t, x_1, \ldots, x_n)$ belong to some PRC class $\mathcal{C}$.

Then by Theorem 6.1 the function

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$

also belongs to $\mathcal{C}$.

(Remember the primitive recursive "negation" function α we defined earlier.)

Minimalization

Let us take a closer look at the function

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$

Let us assume that there is a value t_0 that is the smallest value of $t \leq y$ for which $P(t, x_1, \ldots, x_n)$ is true:

$P(t, x_1, \ldots, x_n) = 0$ for $t < t_0$

$P(t_0, x_1, \ldots, x_n) = 1$. Then

$$\prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n)) = \begin{cases} 1 & \text{if} \quad u < t_0 \\ 0 & \text{if} \quad u \geq t_0 \end{cases}$$

Hence, $g(y, x_1, \ldots, x_n) = \sum_{u<t_0} 1 = t_0$

so that $g(y, x_1, \ldots, x_n)$ is the least value of t for which $P(t, x_1, \ldots, x_n)$ is true.

Minimalization

Then we define:

$$\min_{t \leq y} P(t, x_1, \ldots, x_n) = \begin{cases} g(y, x_1, \ldots, x_n) & \text{if} \quad (\exists t)_{\leq y}\, P(t, x_1, \ldots, x_n) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\min_{t \leq y} P(t, x_1, \ldots, x_n)$ is the smallest value of $t \leq y$ for which $P(t, x_1, \ldots, x_n)$ is true, if such $t \leq y$ exists, otherwise it is 0.

Using Theorems 5.4 and 6.3, we have:

Theorem 7.1: If $P(t, x_1, \ldots, x_n)$ belongs to some PRC Class $\mathcal{C}$ and $f(y, x_1, \ldots, x_n) = \min_{t \leq y} P(t, x_1, \ldots, x_n)$, then f also belongs to $\mathcal{C}$.

We will call the operation "$\min_{t \leq y}$" **bounded minimalization**.

Minimalization

Example 14: $\lfloor x/y \rfloor$

$\lfloor\ \rfloor$ is the **floor function**.
For example, $\lfloor 8/3 \rfloor = 2$.
So $\lfloor x/y \rfloor$ is the **integer part** of the quotient x/y.

$\lfloor x/y \rfloor$ is primitive recursive as shown by the equation

$\lfloor x/y \rfloor = \min_{t \le y} [(t + 1) \cdot y > x]$

Note that this equation gives us $\lfloor x/0 \rfloor = 0$.

Minimalization

Example 15: R(x, y)

R(x, y) is the **remainder** when x is divided by y.
We can also write R(x, y) = x mod y (**"modulo"**).

Obviously, it is true that
$x/y = \lfloor x/y \rfloor + R(x, y)/y$

Therefore, we can write:

$R(x, y) = x \dot{-} (y \cdot \lfloor x/y \rfloor)$

This shows that R(x, y) is primitive recursive.

Note that R(x, 0) = x.

Minimalization

Example 16: p_n

Here, for n > 0, p_n is the **n-th prime number** (in order of size).

In order to make p_n a **total function**, we set $p_0 = 0$.
Then we have:

$p_0 = 0$

$p_1 = 2$

$p_2 = 3$

$p_3 = 5$

$p_4 = 7$

Minimalization

Now consider the following recursion equations:
$p_0 = 0$
$p_{n+1} = \min_{t \le p_n!+1}[\text{Prime}(t) \ \& \ t > p_n]$.

To see that these equations are correct, we must verify the following inequality:
$p_{n+1} \le p_n! + 1$.

Minimalization

$p_{n+1} \le p_n! + 1$.

Note that for $0 < i \le n$ we have:
$(p_n! + 1)/p_i = p_n!/p_i + 1/p_i = K + 1/p_i$,
where K is an integer.

Why is $p_n!$ always divisible by p_i ?

Example: n = 4, i = 2

Then $p_n! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7$, and $p_i = 3$,
so $p_n!/p_i = 1\cdot2\cdot4\cdot5\cdot6\cdot7 = K$.

In other words, p_i is always one of the factors in $p_n!$.

According to the equation $(p_n! + 1)/p_i = K + 1/p_i$,
$p_n! + 1$ is not divisible by any of the primes $p_1, \ldots, p_n$.

Minimalization

So either $p_n! + 1$ is itself a prime or it is divisible by a prime > p_n.

In either case there is a prime q such that
$p_n < q \le p_n! + 1$, which gives us the inequality that we wanted to verify:

$p_{n+1} \le p_n! + 1$.