

Recursively Enumerable Sets

Definition: We write:

$$W_n = \{x \in \mathbb{N} \mid \Phi(x, n) \downarrow\}.$$

Theorem 4.6 (Enumeration Theorem):

A set B is r.e. if and only if there is an n for which $B = W_n$.

This is an immediate consequence of the definition of $\Phi(x, n)$.

The theorem gets its name from the fact that the sequence W_0, W_1, W_2, \dots is an enumeration of all r.e. sets.

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We further define:

$$K = \{n \in \mathbb{N} \mid n \in W_n\}.$$

Then

$$n \in W_n$$

$$\Leftrightarrow \Phi(n, n) \downarrow$$

$$\Leftrightarrow \text{HALT}(n, n).$$

K is the set of all numbers n such that program number n eventually halts on input n .

Theorem 4.7:

K is r.e. but not recursive.

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Proof:

Since $K = \{n \in \mathbb{N} \mid \Phi(n, n) \downarrow\}$, and by the universality theorem (Theorem 3.1), $\Phi(n, n)$ is partially computable, K is obviously r.e.

If K were recursive, then $\neg K$ would be r.e.

If that were the case, then by the enumeration theorem there would have to be some number i so that $\neg K = W_i$.

But then:

$$i \in K$$

$$\Leftrightarrow i \in W_i$$

$$\Leftrightarrow i \in \neg K.$$

Contradiction!

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There are alternative ways of describing r.e. sets:

Theorem 4.8:

Let B be an r.e. set. Then there is a primitive recursive predicate $R(x, t)$ such that $B = \{x \in \mathbb{N} \mid (\exists t) R(x, t)\}$.

Proof:

Let $B = W_n$. Then

$$B = \{x \in \mathbb{N} \mid (\exists t) \text{STP}^{(1)}(x, n, t)\},$$

and $\text{STP}^{(1)}$ is primitive recursive by Theorem 3.2.

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Theorem 4.9:

Let S be a nonempty r.e. set. Then there is a primitive recursive function $f(u)$ such that $S = \{f(n) \mid n \in \mathbb{N}\} = \{f(0), f(1), f(2), \dots\}$. In other words, S is the range of f .

Theorem 4.10:

Let $f(x)$ be a partially computable function, and let $S = \{f(x) \mid f(x) \downarrow\}$ (so S is the range of f).

Then S is r.e.

If we combine Theorems 4.9 and 4.10, we get:

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Theorem 4.11:

Consider a set $S \neq \emptyset$. The following statements are all equivalent:

1. S is r.e.;
2. S is the range of a primitive recursive function;
3. S is the range of a recursive function;
4. S is the range of a partial recursive function.

This theorem motivates the term **recursively enumerable**.

A nonempty r.e. set is enumerated by a recursive function.

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The Parameter Theorem

The **parameter theorem** is also called **iteration theorem** and **s-m-n theorem**.

It is important to the theory of computation as it relates the functions $\Phi^{(n)}(x_1, \dots, x_n, y)$ for different values of n .

Theorem 5.1 (Parameter Theorem):

For each $n, m > 0$ there is a primitive recursive function $S_m^n(u_1, \dots, u_n, y)$ such that

$$\Phi^{(m+n)}(x_1, \dots, x_m, u_1, \dots, u_n, y) =$$

$$\Phi^{(m)}(x_1, \dots, x_m, S_m^n(u_1, \dots, u_n, y)).$$