Coding Programs by Numbers

Finally, for a program \( \varphi \) that consists of the instructions \( I_1, I_2, \ldots, I_k \), we set
\[
\#(\varphi) = [\#(I_1), \#(I_2), \ldots, \#(I_k)] - 1.
\]
This way we associated every possible program in \( \mathcal{L} \) with a unique number.

Gödel numbers are usually very large, even for small programs. Let us look at the following example:
\[
[A] \quad X \leftarrow X+1
\]
\[
\text{IF } X=0 \text{ GOTO A}
\]
\[
\#(I_1) = \langle 1, \langle 1, 1 \rangle \rangle = \langle 1, 5 \rangle = 21
\]
\[
\#(I_2) = \langle 0, \langle 3, 1 \rangle \rangle = \langle 0, 23 \rangle = 46
\]
So the number of our small program is
\[
2^{21} \cdot 3^{46} - 1.
\]

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Note that the number of the unlabeled instruction \( Y \leftarrow Y \) is \( \langle 0, \langle 0, 0 \rangle \rangle = \langle 0, 0 \rangle = 0 \).

Thus, the number of a program will be unchanged if an unlabeled instruction \( Y \leftarrow Y \) is appended to it.

Although this ambiguity is harmless, we avoid it by adding a sentence to our definition of programs of \( \mathcal{L} \):

The final instruction in a program is not permitted to be the unlabeled statement \( Y \leftarrow Y \).

Then each number determines a unique program. As an example, let us determine the program whose number is 199:
\[
199 + 1 = 200 = 2^3 \cdot 3^3 \cdot 5^2 = \langle 3, 0, 2 \rangle.
\]
So if \( \#(\varphi) = 199 \), \( \varphi \) consists of 3 instructions, the second of which is the unlabeled statement \( Y \leftarrow Y \).
\[
3 = \langle 2, 0 \rangle = \langle 2, \langle 0, 0 \rangle \rangle
\]
\[
2 = \langle 0, 1 \rangle = \langle 0, \langle 1, 0 \rangle \rangle
\]
Thus, the program is:
\[
[B] \quad Y \leftarrow Y
\]
\[
Y \leftarrow Y
\]
\[
Y \leftarrow Y+1
\]

The Halting Problem

Let us define the predicate \( \text{HALT}(x, y) \).

For a given number \( y \), let \( \varphi \) be the program such that \( \#(\varphi) = y \).

Then \( \text{HALT}(x, y) \) is true if \( \psi_{\varphi}(1)(x) \) is defined and false if \( \psi_{\varphi}(1)(x) \) is undefined.

In other words:
\[
\text{HALT}(x, y) \iff \text{program number } y \text{ eventually halts on input } x.
\]