# Why does a negative times a negative make a positive?

Carl D. Offner

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## 1 Introduction

A long time ago—probably when you were in Junior High School or its equvalent; maybe when you were about 14 or 15 years old—you learned about negative numbers in your mathematics class. And your teacher explained to you that

- If you multiply a positive number by a negative number, the product is negative.
- If you multiply two negative numbers, the product is positive.

I'm also sure—though you may not really remember this—that the teacher didn't just assert this, or tell you that there was no reason for this other than "that's the way we do it". In fact, I'm sure that the teacher actually gave you a way to understand why this really has to be the case.

You say you don't remember this? Really?

No problem. I'm going to explain it again here, and since you are so much older and more mature and sophisticated now, you will, I am confident, understand this very easily. And it's a very powerful bit of reasoning, and well worth knowing.

## 2 Why the product of a positive and a negative number is negative

It's really important to understand that the reason this statement is true is **not** "because the teacher said so". That is not the reason!

There is a very good reason why this statement **has** to be true. The reason is based on two very basic properties of numbers:

- If you add a number to its negative, you get 0. That is, if $a$ is any number, then

  $$a + (-a) = 0 \tag{1}$$

  Of course $a$ itself could be either positive or negative, and $-a$ will then have the opposite sign. That is, if $a = 5$, then $-a = -5$, and if $a = -5$, then $-a = 5$. But in any case, the sum of $a$ and $-a$ is 0. And we really want this identity (1) to be true. If it wasn't, nothing about negative numbers would make any sense at all. So we take this as a necessary truth.

- We also need the **distributive property**: No matter what numbers $a$, $b$, and $c$ are,

  $$a(b+c) = ab + ac \tag{2}$$

  Everyone knows this is true, and it certainly works for positive numbers as you undoubtedly learned early in elementary school. (You may not have learned that it was called the "distributive property", but that doesn't really matter.)

  The point, however, is that we really want this property to continue to be true if some or all of the three numbers are negative. And that also seems like a very useful thing to insist on: we really **don't** want to have to write things like this

  $$a(b+c) = \begin{cases} ab+ac & \text{if } a, b, \text{ and } c \text{ are all positive} \\ \text{something else} & \text{if } a \text{ and } c \text{ are positive but } b \text{ is negative} \\ \text{another ``something else''} & \text{if } a \text{ and } b \text{ are positive but } c \text{ is negative} \\ \text{... and so on ...} & \end{cases}$$

  This sort of thing would be awful, and no one would be able to remember it. And it really would make little sense. So we will agree that the distributive property—the identity (2)—holds for all numbers positive or negative.

Now based on this we can prove our result very directly: Suppose that $a$ and $b$ are positive, so $-b$ is negative. Then we must have

$$a\big(b+(-b)\big) = \begin{cases} a(0) = 0 & \text{because we know that } b+(-b)=0 \\ ab + a(-b) & \text{by the distributive property} \end{cases}$$

And so we see that it must be the case that

$$ab + a(-b) = 0$$

which can only be true if

$$a(-b) = -(ab)$$

That is, $a(-b)$, which is the product of a positive and a negative number, must equal $-(ab)$, which is the negative of a positive number, and so is negative.

And that's what we needed to show.

## 3 Why the product of two negative numbers is positive

This is almost the same reasoning. Suppose $a$ and $b$ are positive (so $-a$ and $-b$ are negative). Then we have

$$(-a)\big(b+(-b)\big) = \begin{cases} (-a)(0) = 0 & \text{because we know that } b+(-b)=0 \\ (-a)b + (-a)(-b) & \text{by the distributive property} \end{cases}$$

That is we see that

$$(-a)b + (-a)(-b) = 0$$

Now $a$ and $b$ are both positive, and we just saw above that $(-a)b = -(ab)$. That is

$$-(ab) + (-a)(-b) = 0$$

and so

$$(-a)(-b) = -\big(-(ab)\big)$$

Now the right-hand side is just $ab$. (For instance, $-(-3) = 3$, right?) And so we see that

$$(-a)(-b) = ab$$

That is, the product of two negative numbers is positive, and that is what we wanted to show.