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Chapter 1

Θ-Stable Levi Subalgebras

Introduction

We begin by defining certain objects that will play a fundamental role in the thesis. We will use the notion of θ-stable parabolic subalgebras, Levi subalgebras, and the Kostant-Sekiguchi correspondence. The representation theory of $\mathfrak{sl}_2$ will be used extensively. In this chapter we shall prove among other things that minimal θ-stable Levi subalgebras containing a nilpotent element e are conjugate. We will also prove several denseness theorems which will give a pre-homogeneous structure (in the sense of Sato [**S-K**]) to certain subspaces of a θ-stable parabolic subalgebra.

1. Minimal θ-stable Levi subalgebras

Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Finally, let θ be the

corresponding Cartan involution of $\mathfrak{g}$ and σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

Definition. By a (σ,θ)-*stable Levi subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ we shall mean a Levi subalgebra of a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ in Vogan's sense **[Vo]**. In other words if l is a (σ,θ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ then there exists a θ-stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = l \oplus u$ such that $\theta(l) = l$ and $\sigma(l) = l$. Every such l is of the form $l = \mathfrak{g}_{\mathbb{C}}^z$ for some $z \in i\mathfrak{k}$.

A triple (x,e,f) in $\mathfrak{g}_{\mathbb{C}}$ is called a standard triple if $[x,e] = 2e$, $[x,f] = -2f$ and $[e,f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$, e and $f \in \mathfrak{p}_{\mathbb{C}}$ then (x,e,f) is a normal triple. It is a result of Kostant and Rallis **[K-R]** that any nilpotent e of $\mathfrak{p}_{\mathbb{C}}$ can be embedded in a standard normal triple (x,e,f). Moreover e is $K_{\mathbb{C}}$-conjugate to a nilpotent e' inside of a normal triple (x',e',f') with $\sigma(e') = f'$ **[Se]**. The triple (x',e',f') will be called a $Kostant - Sekiguchi$ or KS-triple .

Every nilpotent E' in $\mathfrak{g}$ is G-conjugate to a triple (H,E,F) in $\mathfrak{g}$ with the property that $\theta(H) = -H$ and $\theta(E) = -F$ **[Se]**. Such a triple will be called a KS-triple also.

Define a map c from the set of KS-triples of $\mathfrak{g}$ to the set of normal triples of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$x = c(H) = i(E - F)$$

$$e = c(E) = \tfrac{1}{2}(H + i(E + F))$$

$$f = c(F) = \tfrac{1}{2}(H - i(E+F))$$

The triple (x, e, f) is called the Cayley transform of (H, E, F). It is easy to verify that the triple (x, e, f) is a KS-triple and that $x \in i\mathfrak{k}$. The Kostant-Sekiguchi correspondence [**Se**] gives a one to one map between the set of G-conjugacy classes of nilpotents in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-conjugacy classes of nilpotents in $\mathfrak{p}_{\mathbb{C}}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a KS-triple to the one through the nilpositive element of its Cayley transform. Recently, Michèle Vergne [**Ve**] has proved that there is in fact a diffeomorphism between the G-conjugacy class and the $K_{\mathbb{C}}$-conjugacy class asssociated by the Kostant-Sekiguchi correspondence.

The KS-triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ corresponds to a real KS-triple (H, E, F) in $\mathfrak{g}$ under the Kostant-Sekiguchi map and the reductive centralizer $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{k}^{(H,E,F)} \oplus i\mathfrak{k}^{(H,E,F)}$ is θ-stable and σ-stable.

Let $\mathfrak{s} = \mathfrak{k}_{\mathbb{C}}^{e}$ be the centralizer of e in $\mathfrak{k}_{\mathbb{C}}$. Call any subalgebra of $\mathfrak{k}_{\mathbb{C}}$ consisting of semisimple elements toral. Then we have

Lemma 1.1.1. *Let $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{s}$, and $\mathfrak{t}$ consists exactly of the semisimple elements in $\mathfrak{s}^{\mathfrak{t}}$.*

Proof. This is essentially part of the proof of theorem 8.1.1 of [**C-Mc**]. We must show that $\mathfrak{s}^{\mathfrak{t}}$ is a self-normalizing nilpotent subalgebra of $\mathfrak{s}$. Let $Z \in \mathfrak{s}^{\mathfrak{t}}$. The semisimple and nilpotent parts Z_s and Z_n of Z also lies in $\mathfrak{s}^{\mathfrak{t}}$, whence $Z_s \in \mathfrak{t}$ by maximality. Under the adjoint action $\mathfrak{t}$ acts trivially on $\mathfrak{s}^{\mathfrak{t}}$, it follows that Z acts on $\mathfrak{s}^{\mathfrak{t}}$ by the nilpotent endomorphism Z_n. Hence $\mathfrak{s}^{\mathfrak{t}}$ is nilpotent (Engel's Theorem). The fact that $Z_s \in \mathfrak{t}$ shows that all semisimple elements of $\mathfrak{s}^{\mathfrak{t}}$ lie in $\mathfrak{t}$.

To show that $\mathfrak{s}^{\mathfrak{t}}$ is self-normalizing, observe that $\mathfrak{t}$ acts on the normalizer $\mathfrak{n}$ of $\mathfrak{s}^{\mathfrak{t}}$ in $\mathfrak{s}$ via the adjoint representation. Hence this normalizer is the sum of its $\mathfrak{t}$-weight spaces. But the 0-weight space is exactly $\mathfrak{s}^{\mathfrak{t}}$, and the independence of weight spaces prevents any other weight from occurring in $\mathfrak{n}$.

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Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two maximal toral subalgebras of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}_1}$ is $K_{\mathbb{C}}^e$-conjugate to $\mathfrak{s}^{\mathfrak{t}_2}$ by a map that must send $\mathfrak{t}_1$ to $\mathfrak{t}_2$ because $K_{\mathbb{C}}^e$ preserves semisimplicity and nilpotence in $\mathfrak{k}_{\mathbb{C}}$. Hence any two maximal toral subalgebras of $\mathfrak{k}_{\mathbb{C}}^e$ are conjugate under $K_{\mathbb{C}}^e$.

Lemma 1.1.2. *If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$, then $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$.*

Proof. We know that $\mathfrak{k}_{\mathbb{C}}^e = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \oplus u_e$ where u_e is an ad_x invariant nilpotent ideal of $\mathfrak{k}_{\mathbb{C}}^e$. Let $\mathfrak{t}'$ be a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$ such that $\mathfrak{t} \subsetneq \mathfrak{t}'$. Any element z of $\mathfrak{t}' \setminus \mathfrak{t}$ can be written in the form of

$$z = z_1 + z_2 \text{ with } z_1 \in \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \text{ and } z_2 \in u_e.$$

But

$$[\mathfrak{t}, z] = [\mathfrak{t}, z_1] \oplus [\mathfrak{t}, z_2] = 0.$$

Hence

$$[\mathfrak{t}, z_1] = 0 \text{ and } [\mathfrak{t}, z_2] = 0,$$

Since $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ normalizes itself and $\mathfrak{g}_{\mathbb{C}}$.

Clearly z_1 is in $\mathfrak{t}$, for $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$. It follows that $z - z_1$ is in $\mathfrak{t}'$, that is, $z - z_1$ is semisimple. Since $z - z_1 = z_2$ and z_2 is nilpotent, $z_2 = 0$. Hence z must be in $\mathfrak{t}$.

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Proposition 1.1.3. *If l is a minimal (σ, θ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing a nilpotent element e of $\mathfrak{p}_{\mathbb{C}}$ then $l = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$, where $\mathfrak{t}$ is a maximal toral subalagebra of $\mathfrak{k}_{\mathbb{C}}^{e}$.*

Proof. Let (H, E, F) be a KS-triple in $\mathfrak{g}$. By definition $l = \mathfrak{g}_{\mathbb{C}}^{z}$ for some z in $i\mathfrak{k}$. Since $iz \in \mathfrak{k}$ we can find a maximal torus $\mathfrak{t}_0$ of $\mathfrak{k}^{(H,E,F)}$ containing iz. Therefore $\mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ containing z. From lemma 1.1.2 it is also a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^{e}$. Hence $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}} \subseteq l$.

We shall now prove that $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ is a (σ, θ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$. Then $\mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{t}_0$, where $\mathfrak{t}_0$ is a maximal torus of $\mathfrak{k}^{(H,E,F)}$. Furthermore $\sigma(\mathfrak{t}) = \mathfrak{t}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ be the root system relative to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{t} \subset \mathfrak{h}$.

The set $\Phi = \{\alpha \in \Delta | \alpha(\mathfrak{t}) = 0\}$ is a root subsystem of Δ and any positive root system for Φ can be extended to a positive root system for Δ.

To prove that claim we need to find z' in $\mathfrak{h}$ such that $\alpha(z') = 0$ for all α in Φ and $\delta(z') \neq 0$ for all δ in $\Delta \setminus \Phi$. Denote by P_δ the hyperplane perpendicular to any root

δ in $\Delta \setminus \Phi$. Then $\mathfrak{t} \not\subseteq \bigcup_{\delta} P_{\delta}$. Otherwise we would have $\mathfrak{t} = \mathfrak{t} \cap (\bigcup_{\delta} P_{\delta}) = \bigcup_{\delta} \mathfrak{t} \cap P_{\delta}$, which would mean that $\mathfrak{t} \subset P_{\delta}$ for some fixed $\delta \in \Delta \setminus \Phi$, for $\mathfrak{t}$ cannot be the union of finitely many of its subspaces. This is a contradiction. Therefore we can choose z' in $i\mathfrak{t}_0$, with the desired properties, and we obtain $\Delta^+ = \Phi^+ \cup \{\delta \in \Delta \setminus \Phi | \delta(z') > 0\}$. By construction we have $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}} = \mathfrak{g}_{\mathbb{C}}^{z'} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, where the $\mathfrak{g}_{\mathbb{C}}^{\alpha}$'s are root spaces . From the previous lemma we know that $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^{e}$. Therefore $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ is a (σ, θ)-stable Levi-subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e. By minimality $l \subseteq \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$.

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We proceed to prove the following theorem.

Theorem 1.1.4. *Any two minimal (σ, θ)-stable Levi subalgebras of $\mathfrak{g}_{\mathbb{C}}$ containing a nilpotent element e of $\mathfrak{p}_{\mathbb{C}}$ are $K_{\mathbb{C}}^{e}$ conjugate.*

Proof. Let l_1 and l_2 be two minimal (σ, θ)-stable minimal Levi subalgebras of $\mathfrak{g}_{\mathbb{C}}$ containing e. Then $l_1 = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}_1}$ and $l_2 = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}_2}$, where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two maximal toral subalgebras of $\mathfrak{k}_{\mathbb{C}}^{e}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are $K_{\mathbb{C}}^{e}$-conjugate, so are l_1 and l_2.

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2. Some denseness theorems

Let (x, e, f) to be a KS-triple with $x \in i\mathfrak{k}$. From the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}_{\mathbb{C}}$ has the following eigenspace decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(j)} \text{ where } \mathfrak{g}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{g}_{\mathbb{C}} | [x,z] = jz\}.$$

The subalgebra $\mathfrak{q} = \bigoplus_{j\in\mathbb{N}} \mathfrak{g}_{\mathbb{C}}^{(j)}$ is a parabolic sub algebra of $\mathfrak{g}_{\mathbb{C}}$ with a Levi part $l = \mathfrak{g}_{\mathbb{C}}^{(0)}$ and nilradical $u = \bigoplus_{j\in\mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}$.

Call $\mathfrak{q}$ the Jacobson-Morosov parabolic subalgebra of e relative to the triple (x,e,f). Our choice of the triple (x,e,f) forces $\mathfrak{q}$ to be θ-stable in Vogan's sense.

Retain the above notations. Let Q and L, be the connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{q}$ and l respectively. Define $L \cap K_{\mathbb{C}}$ to be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $l \cap \mathfrak{k}_{\mathbb{C}}$. We shall prove some facts about some eigenspaces of x in $\mathfrak{p}_{\mathbb{C}}$.

Kostant and Rallis ([**[K-R], Lemma 4**]) proved that $L \cap K_{\mathbb{C}}\cdot e$ is dense in $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ which is therefore a prehomogeneous space in Sato's sense.

Let $\mathfrak{q}$ be the Jacobson-Morozov parabolic subalgebra of e relative to the normal triple (x,e,f). Then

Proposition 1.2.1. *$Q \cap K_{\mathbb{C}}.e$ is dense in $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. Moreover if e is even, that is $\mathfrak{g}_{\mathbb{C}}^{(i)} = 0$ for i odd, then $\overline{Q \cap K_{\mathbb{C}}\cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. It is a result of Carter [**Ca,Proposition 5.7.3**] that the orbit of Q on $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)}$ containing e is a dense open subset of $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)}$. It follows that $[\mathfrak{q}, e] = \bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)}$ which implies that

$$[\mathfrak{q}, e] = [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, e] \oplus [\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}, e]$$

Since each $\mathfrak{g}_{\mathbb{C}}^{(i)}$ is θ-stable,

$$\bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} = \bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$$

The fact that $e \in \mathfrak{p}_{\mathbb{C}}$ and the previous direct sum decomposition force

$$[\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, e] = \bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Hence, the map $ad_e : \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} \to \bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$ is surjective.

It is clear that $\bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$ is a $Q \cap K_{\mathbb{C}}$-module under the adjoint action. Also

$$e \in \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} \subseteq \bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Thus the map $z \to Ad_z(e)$ is a morphism from $Q \cap K_{\mathbb{C}}$ to $\bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. Its differential is the map:

$$-ad_e : \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} \to \bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

This is surjective. Thus the given morphism is dominant and separable. Since the image of such a map is open in its closure, the $Q \cap K_{\mathbb{C}}$-orbit of e is a non-empty open dense subset of $\bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$.

And if e is even then $\bigoplus_{i\geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}} = u \cap \mathfrak{p}_{\mathbb{C}}$. Hence $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$.

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lemma 1.2.2. *With the above notation dim* $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$ = *dim* $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$.

Proof. By the $\mathfrak{sl}_2$-theory the $\mathfrak{g}_{\mathbb{C}}^{(-1)} \cap \mathfrak{k}_{\mathbb{C}}$ and the $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$ modules are paired. Since

$$\dim \mathfrak{g}_{\mathbb{C}}^{(-1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$$

we conclude that

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$$

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The author wishes to thank Cary Rader for suggestions regarding the proof of the following theorem.

Theorem 1.2.3. $L \cap K_{\mathbb{C}}$ *has finitely many orbits on each eigenspace* $\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$.

Proof. Let $S = \{1, \theta\}$ and define $H = L \times_\phi S$ to be the semidirect product of L and S where ϕ is a homomorphism from S to $Aut(L)$ such that for $s \in S$ and $y \in L$, $\phi(s)(y) = s(y)$. If we identify L and S with $L \times 1$ and $1 \times S$ respectively then they are closed subgroups of H. If (y_1, s_1) and (y_2, s_2) are two elements of H then the group multiplication law on H is defined so that:

$$(y_1, s_1) \times_\phi (y_2, s_2) = (y_1 \phi(s_1) y_2, s_1 s_2).$$

Observe that S is diagonalizable. Let $(y,s) \in H$ and $(y_1,1) \in L \times 1$ then

$$(y,s) \times (y_1,1) \times (y,s)^{-1} = (y,s) \times (y_1,1) \times (\phi(s^{-1})y^{-1}, s^{-1}) \in L \times 1.$$

Hence L is a normal subgroup of H. Moreover $\mathfrak{g}_{\mathbb{C}}$ is stable under the adjoint representation of H. Let $L^S = \{y \in L | sys^{-1} = y \text{ for every } s \in S\}$. It is obvious that $L^S = L \cap K_{\mathbb{C}}$ and that the (-1) weight space of S in $\mathfrak{g}_{\mathbb{C}}^{(i)}$ is $\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. The fact that L has a finite number of orbits on $\mathfrak{g}_{\mathbb{C}}^{(i)}$ **[R2 ,THEOREM E]** implies that

$$\mathfrak{g}_{\mathbb{C}}(i) \cap \mathfrak{p}_{\mathbb{C}} \subseteq \bigcup_{j=1}^{n} L.z_j \text{ for } z_j \in \mathfrak{g}_{\mathbb{C}}^{(i)}.$$

Our theorem now follows from a result of R.W. Richardson **[R2 ,THEOREM A]** that if $z \in \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$, then $L \cap K_{\mathbb{C}}$ has only a finite number of orbits on the intersection $(L.z) \cap (\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}})$.

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We shall now prove a theorem of great importance to us. We shall show that $L \cap K_{\mathbb{C}}$ has a finite number of orbits on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$. But first, we give a Lie algebra version of the proof of a theorem of R.W. Richardson . See (**[Röh,Theorem 1.4]**).

Some definitions are needed. Let Φ be a root system of $\mathfrak{g}_{\mathbb{C}}$ and Δ the set of simple roots determined by our choice of a positive root system Φ^+. It is a well known fact that any parabolic subalgebra is $G_{\mathbb{C}}$-conjugate to a parabolic of the form $\mathfrak{p} = l \oplus u$, where

$$l = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Lambda \rangle} \mathfrak{g}_{\mathbb{C}}^{\alpha}$$

and

$$u = \sum_{\alpha \in \Phi^+ \setminus \langle \Lambda \rangle} \mathfrak{g}_{\mathbb{C}}^{\alpha}.$$

Here, $\Lambda \subseteq \Delta$, $\langle \Lambda \rangle$ denotes the subroot system of Φ generated by Λ and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ relative to Δ. See **[C-Mc]**.

Proposition 1.2.4. *Retaining the above notation*

1. u and $[u,u]$ are the direct sum of their 1-dimensional root spaces.

2. A root α of u is a root of $[u,u]$ if and only if it is the sum of two roots of u. Such roots are called decomposable.

3. A root in u is indecomposable if and only if it is the sum of one simple root not in Λ and various simple roots in Λ.

Proof. See Collingwood McGovern **[C-Mc] Proposition 8.2.7**.

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Let $\mathfrak{q} = l + u$ be the Jacobson-Morozov parabolic subalgebra relative to the standard triple (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$. Let L be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l. We obtain:

Theorem 1.2.5 (Richardson). *L has only a finite number of orbits on $\frac{u}{[u,u]}$. In*

particular there exists a unique dense orbit and so dim $l \geq dim\ \frac{u}{[u,u]}$.

Proof. First we show that $\frac{u}{[u,u]}$ is a weight space for some non zero weight λ of a torus s of $\mathfrak{g}_{\mathbb{C}}$, and $l = \mathfrak{g}^s_{\mathbb{C}}$.

Retaining the above definitions of Φ^+ and Δ it is easy to see that l is determined by the subset Λ of Δ such that

$$\Lambda = \{\alpha \in \Delta | \alpha(x) = 0\}.$$

Let ϕ be an indecomposable root. Then we can find $\gamma \in \Delta \setminus \Lambda$ such that

$$\phi = \gamma + \sum_{\alpha \in \Lambda} n_\alpha \alpha \ (n_\alpha\text{'s are integers}).$$

Under the Killing form κ of $\mathfrak{g}_{\mathbb{C}}$ we can identify any simple root β with the fundamental weight w_β which , in turn, can be identified with $H_{w_\beta} \in \mathfrak{h}$ via $w_\beta(Y) = \kappa(Y, H_{w_\beta})$ for any $Y \in \mathfrak{h}$.

Define

$$\hat{H} = \sum_{\beta \in \Delta \setminus \Lambda} \frac{2H_{w_\beta}}{\kappa(\beta, \beta)}.$$

Then

$$\phi(\hat{H}) = \gamma(\hat{H}) + \sum_{\beta \in \Delta \setminus \Lambda, \alpha \in \Lambda} \frac{2n_\alpha \alpha(H_{w_\beta})}{\kappa(\beta, \beta)},$$

$$\phi(\hat{H}) = \gamma(\hat{H}) + \sum_{\beta \in \Delta \setminus \Lambda, \alpha \in \Lambda} \frac{2n_\alpha \kappa(w_\beta, \alpha)}{\kappa(\beta, \beta)} = \gamma(\hat{H}),$$

for $\kappa(w_\beta, \alpha) = 0$.

Hence

$$\phi(\hat{H}) = \frac{2\gamma(H_{w_\gamma})}{\kappa(\gamma,\gamma)} = \frac{2\kappa(w_\gamma,\gamma)}{\kappa(\gamma,\gamma)} = 1.$$

But $\frac{u}{[u,u]}$ is isomorphic as an L-module to V where V is the span of the root spaces of the indecomposable roots. Thus:

$$u = V \oplus [u, u]$$

Thus V is the 1-eigenspace of $\hat{H}$, for any element in [u,u] considered as an eigenvector of $\hat{H}$ will have eigenvalue strictly larger than 1.

Moreover the fact that $l = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Lambda \rangle} \mathfrak{g}^\alpha_{\mathbb{C}}$ implies that $[l, \hat{H}] = 0$, that is $l \subseteq \mathfrak{g}^{\hat{H}}_{\mathbb{C}}$. In fact $l = \mathfrak{g}^{\hat{H}}_{\mathbb{C}}$, because no non zero element of u commutes with $\hat{H}$, since

$$[\hat{H}, X_\beta] = X_\beta \text{ for any } \beta \in \Delta \setminus \Lambda \text{ and } X_\beta \in \mathfrak{g}^\beta_{\mathbb{C}}.$$

The theorem follows from a result of Richardson.[**R2 ,THEOREM E**]

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Now, let (x, e, f) be a normal triple and $\mathfrak{q} = l \oplus u$ its Jacobson-Morozov parabolic.

The next theorem is very important. It shows that $L \cap K_{\mathbb{C}}$ has a dense orbit on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Theorem 1.2.6 *$L \cap K_{\mathbb{C}}$ has only a finite number of orbits on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.*

In particular there exists a unique dense $(L \cap K_{\mathbb{C}})$-orbit and so dim $l \cap \mathfrak{k}_{\mathbb{C}} \geq$ dim $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Proof. Define S and H as in theorem 1.2.3 . Then $L \cap K_{\mathbb{C}} = L^S$. First we observe that $\frac{u \cap \mathfrak{p}_{\mathbb{C}} + [u,u]}{[u,u]}$ is included in $\left(\frac{u}{[u,u]}\right)_{-1}$, the -1 weight of S in $\frac{u}{[u,u]}$. It is a general fact (see **[Br, Theorem 5.20]**) that

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}} + [u,u]}{[u,u]} \cong \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u,u] \cap (u \cap \mathfrak{p}_{\mathbb{C}})}$$

.

Since

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u,u] \cap (u \cap \mathfrak{p}_{\mathbb{C}})} = \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]},$$

we conclude that

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]} \subseteq \left(\frac{u}{[u,u]}\right)_{-1}.$$

From the previous theorem $\frac{u}{[u,u]}$ is included in a finite union of L orbits. Each such orbit intersected with $\left(\frac{u}{[u,u]}\right)_{-1}$ is a finite union of $L \cap K_{\mathbb{C}}$ orbit by Richardson.**[R2, Theorem A]**. The desired result follows.

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Theorems 1.2.3 and 1.2.6 tell us that $\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$ and $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$ are prehomogeneous spaces for $L \cap K_{\mathbb{C}}$ in Sato's sense.

The importance of the preceding theorem will become clear in the process of classifying the $K_{\mathbb{C}}$-nilpotent orbits on $\mathfrak{p}_{\mathbb{C}}$. Next we characterize *noticed* nilpotents $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$.