Allez

Chapter 2

Noticed Nilpotent Orbits in Symmetric Spaces

Introduction

The *noticed* nilpotent orbits of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$ play a fundamental role in this work. We will give a very simple characterization of *noticed* nilpotent elements of $\mathfrak{p}_{\mathbb{C}}$. They are similar to the *distinguished* nilpotents used by Bala and Carter [**B-C1**]. We will also prove several dimension theorems regarding θ-stable parabolic subalgebras containing a *noticed* nilpotent.

1. Noticed nilpotent elements and θ-stable parabolic subalgebras

Definition. A nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ (or its $K_{\mathbb{C}}$-orbit) is *noticed* if the only

(θ, σ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e (or eqivalently meeting $K_{\mathbb{C}}.e$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

A Levi subalgebra l contains e if and only if $[l, l]$ does. Thus if e is noticed in l it is actually noticed in the semi-simple subalgebra $[l, l]$ for if l is (θ, σ)-stable then so is $[l, l]$, and any nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ is noticed in any minimal (θ, σ)-stable Levi subalgebra l containing it.

Let (x, e, f) be a normal triple. Then from proposition 1.1.3, e is noticed if and only if $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = \{0\}$. Recall the $\mathbb{Z}_2$-gradation:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(i)} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$$

Using this gradation it is easy to give a criterion for a nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ to be *noticed*

Lemma 2.1.1. *Retain the above notations. Then a nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ is noticed if and only if dim $\mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. Dragomir Djoković proved the following result in section 12 of [**D**]:

Let (x, e, f) be a normal triple of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{l}(e)$ be a Levi factor of $\mathfrak{k}_{\mathbb{C}}^{e}$. For any integer i let:

$$N(0, i) = \dim \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{k}_{\mathbb{C}},$$

$$N(1, i) = \dim \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Then $\dim \mathfrak{l}(e) = N(0, 0) - N(1, 2)$.

Note that $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ is a Levi factor of $\mathfrak{k}_{\mathbb{C}}^{e}$. Hence,

$$\dim \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = \dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} \text{ - } \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$$

But e is *noticed* if and only if $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = 0$. The lemma follows.

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The next proposition gives an upper bound for the dimension of $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$ when e is noticed.

Proposition 2.1.2. *If e is noticed then dim $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} \leq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. By Richardson [**R1**] (See also Vinberg [**Vi**]) $L \cap K_{\mathbb{C}}$ has a dense orbit on $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$. Hence

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \leq \dim l \cap \mathfrak{k}_{\mathbb{C}}.$$

But from lemma 1.2.2 we have

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Since e is noticed,

$$\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Hence,

$$\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} \leq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}.$$

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Later, we shall give a classification of the conjugacy classes of the nilpotent orbits of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$. It is a generalization of the Bala-Carter classification for $G_{\mathbb{C}}$-nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ [**B-C1**]. Bala and Carter use the notion of *distinguished* parabolic subalgebra in order to parametrize the nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$. Such parabolic algebras always contain a *distinguished* nilpotent element in $\mathfrak{g}_{\mathbb{C}}$.

Definition. A nilpotent element X in $\mathfrak{g}_{\mathbb{C}}$ (or its $G_{\mathbb{C}}$-orbit) is *distinguished* if the only Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing X (or eqivalently meeting $G_{\mathbb{C}}.X$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

Clearly any distinguished nilpotent belonging to $\mathfrak{p}_{\mathbb{C}}$ is automatically noticed. Furthermore Bala and Carter give the following criterion for a nilpotent $X \in \mathfrak{g}_{\mathbb{C}}$ to be distinguished:

Lemma 2.1.3. *Retain the above notations. Then X is distinguished if and only if $dim\ \mathfrak{g}_{\mathbb{C}}^{(0)} = dim\ \mathfrak{g}_{\mathbb{C}}^{(2)}$.*

Proof. See Bala & Carter [**B-C1**] or Collingwood & MacGovern [**C-Mc**].

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Observe the similarity beteween the criterion for X to be distinguished and that for e to be noticed.

Recall that a non zero nilpotent X of $\mathfrak{g}_{\mathbb{C}}$ is even if and only if $\mathfrak{g}_{\mathbb{C}}^{(i)} = 0$ whenever i is odd. Bala and Carter also proved that:

Theorem 2.1.4. *Any distinguished nilpotent $X \in \mathfrak{g}_{\mathbb{C}}$ is even.*

Proof. See Bala & Carter [**B-C1**] or Collingwood & MacGovern [**C-Mc**].

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The above eveness property is not shared by noticed nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$. We shall give some examples. But first we give the Bala-Carter characterization of *distinguished parabolic* subalgebras.

Let H, X, Y be a standard triple of $\mathfrak{g}_{\mathbb{C}}$ and let p the Jacobson-Morozov parabolic subalgebra of X with Levi decomposition $p = m \oplus v$. Then we have:

Theorem 2.1.5. *X is distinguished if and only if dim m = dim $\frac{v}{[v,v]}$.*

Proof. See Bala & Carter [**B-C1**] or Collingwood & MacGovern [**C-Mc**].

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Therefore it makes sense to define an arbitrary parabolic subalgebra $p = m \oplus v$ to be *distinguished* if dim m = dim $\frac{v}{[v,v]}$.

If e is an even noticed nilpotent element of $\mathfrak{p}_{\mathbb{C}}$ then we can prove a theorem similar to theorem 2.1.5.

Theorem 2.1.6. *Let e be an even nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. Let $\mathfrak{q}$ be a θ-stable Jacobson-Morosov parabolic subalgebra of e relative to a triple (x, e, f) defined as*

above. Then $\mathfrak{q} = l \oplus u$ *and* $dim\ l \cap \mathfrak{k}_{\mathbb{C}} = dim\ \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$ *if and only if* e *is noticed.*

Proof. We may assume that $\mathfrak{q}$ is defined as above. Let $u' = \bigoplus_{i \geq 4} \mathfrak{g}_{\mathbb{C}}^{(i)}$. Then

$$\dim\ l \cap \mathfrak{k}_{\mathbb{C}} = \dim\ \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}},$$

while

$$\dim\ \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} = \dim\ u \cap \mathfrak{p}_{\mathbb{C}} \text{ - } \dim\ u' \cap \mathfrak{p}_{\mathbb{C}}.$$

By definition we have

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subset u' \cap \mathfrak{p}_{\mathbb{C}}.$$

By the representation theory of $\mathfrak{sl}_2$,

$$\text{if } z \in \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}} \text{ and } i \geq 4 \text{ then } z = [e, z'] \text{ for some } z' \text{ in } \mathfrak{g}_{\mathbb{C}}^{(i-2)} \cap \mathfrak{k}_{\mathbb{C}} \subset u \cap \mathfrak{k}_{\mathbb{C}}.$$

Hence

$$u' \cap \mathfrak{p}_{\mathbb{C}} \subset [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}].$$

The conclusion follows at once.

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For exceptional Lie algebras one obtains the following:

Proposition 2.1.7. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be an exceptional simple complex Lie algebra. Let* $\mathfrak{q} = l \oplus u$ *be a* θ*-stable Jacobson-Morosov parabolic subalgebra of* e *relative to a normal*

triple (x, e, f) *defined as above. If* e *is noticed then dim* $l \cap \mathfrak{k}_{\mathbb{C}}$ = *dim* $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Proof. For our purpose we use the tables obtained by D. Djoković **[D]** in order to isolate the orbits of interest. In view of the previous theorem, we only need to consider non even orbits for which $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)} = 0$. It turns out there are only 5 of them. (See Table 1)

Since $\mathfrak{q}$ is the Jacobson-Morosov parabolic subalgebra of e, we have:

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]} \cong \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \oplus \frac{\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]}$$

as $L \cap K_{\mathbb{C}}$-modules.

Following Djoković we find that the only relevent cases come from the real forms $E_{7(7)}$, $E_{8(8)}$ and $E_{6(6)}$.

Now, let $\mathfrak{h}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. Then :

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{s}, \text{ where } \mathfrak{t} \text{ is a Cartan subalgebra of } \mathfrak{k} \text{ and } \mathfrak{s} \subseteq \mathfrak{p}.$$

$E_{7(7)}$ and $E_{8(8)}$ are of inner type that is a fundamental Cartan sublagebra for these two algebras is also Cartan subalgebra of their compact part $\mathfrak{k}$. It turns out that for $E_{(7,7)}$ and $E_{(8,8)}$, $\mathfrak{k}_{\mathbb{C}}$ is of type to A_7 and D_8 respectively. Therefore we can evaluate the roots on the neutral element x relative to the orbit under consideration. Hence, we obtain the dimensions of the spaces of interest easily since a root is either compact or non compact.

The case $E_{6(6)}$ is more involved because we are forced to deal with complex roots.

Starting with a fundamental Cartan subalgebra we choose a strongly orthogonal set of imaginary roots φ_1, φ_2, φ_3 and φ_4. Using linear combinations of those roots we can create a subalgebra isomorphic to C_4 with positive simple roots β_1, β_2, β_3 and β_4.The φ_i's will be the compact roots. We know that:

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \ compact \ imaginary} \mathbb{C}X_\alpha \bigoplus \sum_{\alpha \ complex} \mathbb{C}(X_\alpha + \theta(X_\alpha))$$

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} \oplus \sum_{\alpha \ non \ compact \ imaginary} \mathbb{C}X_\alpha \bigoplus \sum_{\alpha \ complex} \mathbb{C}(X_\alpha - \theta(X_\alpha))$$

Each root relative to the fundamental Cartan subalgebra is a linear combination of the β_i's. Therefore we can evaluate the roots on the desired neutral element x and proceed as above.

Computations using the software Lie on a Macintosh IIci reveal that:

$$dim\mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = dim\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$$

$$dim\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = dim[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]$$

as indicated in the following table.

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Table 1[1]

Algebra	orbits	$l \cap \mathfrak{k}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$	$[,]$
$E_{7(7)}$	1111111	7	7	7	7	7
$E_{8(8)}$	11111111	8	8	8	8	8
$E_{8(8)}$	11111121	8	7	7	8	7
$E_{6(6)}$	1111	4	4	4	4	4
$E_{6(6)}$	1112	4	3	3	4	3

The following theorem gives one more type of noticed element $e \in \mathfrak{p}_{\mathbb{C}}$ such that the θ-stable parabolic subalgebra relative to a normal triple (x, e, f) has the above property.

Theorem 2.1.8. *Let e be a noticed nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. Let $\mathfrak{q} = l \oplus u$ be a θ-stable Jacobson-Morosov parabolic subalgebra of e relative to a triple (x, e, f) defined as above. Assume that dim $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$ = dim $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$. Then*

dim $l \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Proof. We have

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]} \cong \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \oplus \frac{\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]} \tag{2.1.8}$$

as $L \cap K_{\mathbb{C}}$-modules. Hence

[1] $[,]$ stands for $[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]$.

$$\dim \frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}},u\cap\mathfrak{p}_{\mathbb{C}}]} = \dim\, \mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{p}_{\mathbb{C}} + \dim\, \mathfrak{g}_{\mathbb{C}}^{(2)}\cap\mathfrak{p}_{\mathbb{C}} \text{ - } \dim\, [\mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{p}_{\mathbb{C}}].$$

But

$$[\mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{p}_{\mathbb{C}}] \subseteq \mathfrak{g}_{\mathbb{C}}^{(2)}\cap\mathfrak{p}_{\mathbb{C}}.$$

Therefore

$$\dim\, [\mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{p}_{\mathbb{C}}] \leq \dim\, \mathfrak{g}_{\mathbb{C}}^{(2)}\cap\mathfrak{p}_{\mathbb{C}} = \dim\, \mathfrak{g}_{\mathbb{C}}^{(1)}\cap\mathfrak{p}_{\mathbb{C}}.$$

We see then that

$$\dim \frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}},u\cap\mathfrak{p}_{\mathbb{C}}]} \geq \dim\, \mathfrak{g}_{\mathbb{C}}^{(2)}\cap\mathfrak{p}_{\mathbb{C}}\ .$$

However we have proved in theorem 1.2.6 that

$$\dim \frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}},u\cap\mathfrak{p}_{\mathbb{C}}]} \leq \dim\, l\cap\mathfrak{k}_{\mathbb{C}}.$$

It follows from theorem 2.1.1 that

$$\dim\, l\cap\mathfrak{k}_{\mathbb{C}} = \dim \frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}},u\cap\mathfrak{p}_{\mathbb{C}}]} \quad \text{if } e \text{ is noticed.}$$

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2. An important counter-example

It is not true in general that if e is noticed then

$$\dim\, l\cap\mathfrak{k}_{\mathbb{C}} = \dim \frac{u\cap\mathfrak{p}_{\mathbb{C}}}{[u\cap\mathfrak{k}_{\mathbb{C}},u\cap\mathfrak{p}_{\mathbb{C}}]}.$$

We shall give the following example.

Let $\mathfrak{g}$ be $\mathfrak{sl}(7,\mathbb{R})$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(7,\mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(7,\mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of 7×7 complex symmetric matrices. The Cartan involution θ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Let

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$E = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Choose $F = E^T$. Then $\theta(H) = -H$, $\theta(E) = -F$. Hence (H, E, F) is a KS-triple, and $x = i(E - F)$ is in $\mathfrak{k}_{\mathbb{C}}$. In fact under the Kostant-Sekiguchi map (H, E, F) corresponds to a normal triple (x, e, f), with $e = \frac{1}{2}(E + F + iH)$ and $f = \frac{1}{2}(E + F - iH)$. Next we compute the following eigenspaces of x:

$\mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}}$ is the set of matrices of the form $A = M - M^T$

where

$$M = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2}(a+b) & 0 & a & n\sqrt{3} & s\sqrt{3} & 0 \\ 0 & 0 & b & 0 & -s & n & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2}(a+b) & n & s & 0 \\ 0 & 0 & 0 & 0 & -s\sqrt{3} & n\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$a, b, n, s, t \in \mathbb{C}$.

$\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ is the set of matrices of the form $A = N + D + N^T$

where

$$D = \begin{pmatrix} \frac{a\sqrt{3}+3ib}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-a\sqrt{3}+ib}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a\sqrt{3}-ib}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-a\sqrt{3}-3ib}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & it & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -it & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & \frac{-ia+b\sqrt{3}}{2} & a & -ia\sqrt{3} & \frac{2is-n}{\sqrt{3}} & \frac{s+2in}{\sqrt{3}} & 0 \\ 0 & 0 & b & -a & s & n & 0 \\ 0 & 0 & 0 & \frac{-ia+b\sqrt{3}}{2} & -n & s & 0 \\ 0 & 0 & 0 & 0 & \frac{s+2in}{\sqrt{3}} & \frac{-2is+n}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$a, b, n, s, t \in \mathbb{C}$.

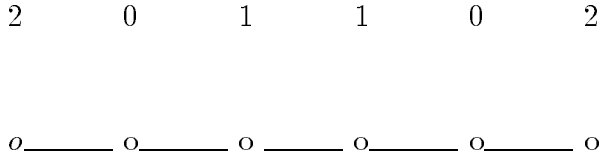
Clearly dim $\mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} = 5$. Hence the triple (x, e, f) is noticed.

From the theory of the classification of the real nilpotent orbits of $\mathfrak{sl}_n$[**C,Mc**] and by Sekiguchi [**Se**]

$$G_{\mathbb{C}}.e = G_{\mathbb{C}}.E$$

,

and the triple (x, e, f) can be associated to the partition $[4, 2, 1]$ of 7 and the corresponding weighted Dynkin diagram is



One sees that dim $\mathfrak{g}_{\mathbb{C}}^{(1)} = 4$. Consequently dim $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} = 2$. A calculation shows that (K_1, K_2) and (P_1, P_2) below are bases for $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$ respectively. Here

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & & 0 & 0 & -i\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & i & -1 & i\sqrt{3} & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -i & -1 & 0 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & & 0 & 0 & -i\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & -i & 1 & -i\sqrt{3} & 0 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i & 1 & 0 \end{pmatrix}$$

Hence dim $[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}] = 3$