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Chapter 3

An Extension of The Bala-Carter Theory

Introduction

In this chapter we present a new classification of nilpotent orbits of real reductive Lie algebras under the action of their adjoint group. Maintaining the above notations, we prove that the orbits $K_{\mathbb{C}}.e$ are in one-to-one correspondence with the triples of the form $(l, q_l, \mathfrak{w})$, where e is a non zero nilpotent in $\mathfrak{p}_{\mathbb{C}}$, l is a minimal (θ, σ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e, q_l is a θ stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}$ is a certain $L \cap K_{\mathbb{C}}$ prehomogeneous subspace of $q_l \cap \mathfrak{p}_{\mathbb{C}}$ containing e. Thus, we obtain a classification for real nilpotents G-orbits in $\mathfrak{g}$ via the Kostant-Sekiguchi correspondence. This classification generalizes the one given by P. Bala and R. Carter in 1976, for complex semisimple Lie algebras [**C**].

1. The Bala-Carter correspondence

The Bala-Carter Classification can be expressed as follows:

Theorem 3.1.1 (Bala, Carter). *There is a one-to-one correspondence between nilpotent orbits of* $\mathfrak{g}_{\mathbb{C}}$ *and* $G_{\mathbb{C}}$*-conjugacy classes of pairs* (m, p_m)*, where* m *is a Levi subalgebra of* $\mathfrak{g}_{\mathbb{C}}$ *and* p_m *is a distinguished parabolic subalgebra of the semisimple algebra* $[m, m]$.

Proof. Bala &Carter **[B-C1]** (Also see Carter **[Ca]**, Collingwood & MacGovern **[C-Mc]**).

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What really makes the correspondence possible is the fact that the orbit of a distinguished nilpotent element X in $\mathfrak{g}_{\mathbb{C}}$ is uniquely determined by the Jacobson-Morozov p attached to X or any of its representatives. It is the Richardson orbit attached to p. This fails for arbitrary orbits.

2. Extension of the Bala-Carter correspondence

Let $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$. Then, $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$. Let κ be the Killing form on $\mathfrak{g}_{\mathbb{C}}$ and $\tau = \theta \circ \sigma$, the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_u$. For X and Y in $\mathfrak{g}_{\mathbb{C}}$ define $\kappa'(X, Y) = -\kappa(X, \tau(Y))$. It is known that κ' is a hermitian form on $\mathfrak{g}_{\mathbb{C}}$. Let $X = A + iB$ and $Y = C + iD$ with A, B, C, and D in $\mathfrak{g}_u$. Then

1. If $X \neq 0$ then
$$\kappa'(X, X) = -\kappa(A + iB, A - iB) = -\kappa(A, A) - \kappa(B, B) > 0$$
because $\kappa|_{\mathfrak{g}_u}$ is negative definite.

2. If λ and μ are complex numbers then

$$\kappa'(\lambda X, \mu Y) = -\kappa(\lambda X, \tau(\mu Y)) = \lambda\bar{\mu}(-\kappa(X, \tau(Y)) = \lambda\overline{\mu}\kappa'(X, Y).$$

3. $\kappa'(X, Y) = -\kappa(A, C) - \kappa(B, D) + i(\kappa(A, D) - \kappa(B, C)) = \overline{\kappa'(Y, X)}$.

The next lemma is also known.

Lemma 3.2.1. *Let* $\mathfrak{q} = \bigoplus_{j\geq 0} \mathfrak{g}_{\mathbb{C}}^{(j)}$ *be the Jacobson-Morozov parabolic subalgebra associated with the triple* (x, e, f) *where* $x \in i\mathfrak{g}_u$. *Then for* $j \neq k$, $\mathfrak{g}_{\mathbb{C}}^{(j)}$ *is orthognal to* $\mathfrak{g}_{\mathbb{C}}^{(k)}$ *relative* κ'.

Proof. We can assume $k > 0$. Observe that $\tau(x) = -x$, since $x \in i\mathfrak{g}_u$. Let $y \in \mathfrak{g}_{\mathbb{C}}^{(k)}$, $z \in \mathfrak{g}_{\mathbb{C}}^{(j)}$ and $k \neq j$. Then

$$[x, \tau(z)] = \tau([\tau(x), z]) = \tau([-x, z]) = -j\tau(z).$$

Hence $\tau(z) \in \mathfrak{g}_{\mathbb{C}}^{(-j)}$. Using associativity of the Killing form κ, we have:

$$\kappa'(y, z) = -\kappa(y, \tau(z)) = -\frac{1}{k}\kappa([x, y], \tau(z))$$

$$= \frac{1}{k}\kappa(y, [x, \tau(z)]) = -\frac{j}{k}\kappa(y, \tau(z)) = \frac{j}{k}\kappa'(y, z).$$

Hence

$$\kappa'(y, z) = 0.$$

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We will need the following lemma.

Lemma 3.2.2. *For any normal triple (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$, $[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}$.*

Proof. From the $\mathfrak{sl}(2, \mathbb{C})$ representation theory we know that

$$[\mathfrak{p}_{\mathbb{C}}^x, e] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

Therefore, since $e \in \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ we have

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] \supseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

On the other hand, clearly,

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] \subseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

The lemma follows.

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Let $\mathfrak{q} = l \oplus u$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{m}$ be the orthogonal complement of $[u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]]$ relative to κ' inside $u \cap \mathfrak{p}_{\mathbb{C}}$. Define $\mathfrak{w}$ to be an $L \cap K_{\mathbb{C}}$ module in $\mathfrak{m}$. Finally, let $\hat{\mathfrak{w}} = \mathfrak{w} \oplus [l \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{w}]$. Clearly, $\hat{\mathfrak{w}}$ is θ-stable.

Definition. Define $\mathfrak{L}$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ such that:

1. $\mathfrak{w}$ has a dense $L \cap K_{\mathbb{C}}$ orbit: $(L \cap K_{\mathbb{C}}) \cdot \hat{e}$

2. $\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \mathfrak{w}$.
3. $L{\cdot}\hat{e}$ is dense in $\hat{\mathfrak{w}}$.
4. $\hat{\mathfrak{w}} \perp [u,[u,u]]$
5. $[u,\hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$
6. $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$

Property 3 implies that $\hat{\mathfrak{w}}$ is an L-module for

$$\overline{L.\hat{e}} = \hat{\mathfrak{w}}.$$

Hence $\hat{\mathfrak{w}}$ is L stable.

Let $\mathfrak{S}$ be the set of noticed KS-triples (x,e,f) of $\mathfrak{g}_{\mathbb{C}}$.

We have a map $\mathfrak{F}$ from $\mathfrak{S}$ to $\mathfrak{L}$ which associates a triple (x,e,f) of $\mathfrak{S}$ to an element $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}_x, \mathfrak{w})$ of $\mathfrak{L}$ where $\mathfrak{q}_x$ is the θ-stable Jacobson-Morosov parabolic subalgebra of (x,e,f).

Let

$$\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}.$$

From Kostant and Rallis [**K-R**] we know that $L.e$ (respectively $L \cap K_{\mathbb{C}}$.e) is dense in $\mathfrak{g}_{\mathbb{C}}^{(2)}$ (respectively $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$). Also

$$\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}},$$

because e is noticed.

By definition $l \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^x$.

Since

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}},$$

by lemma 3.2.2, we must have

$$\hat{\mathfrak{w}} = \mathfrak{g}_{\mathbb{C}}^{(2)}.$$

Hence conditions 1, 2, 3 in the above definition are verified. Furthermore, we know that:

$$u = \bigoplus_{j \in \mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}.$$

it follows that

$$\mathfrak{g}_{\mathbb{C}}^{(2)} \perp [u, [u, u]],$$

by lemma 3.2.1

To see that condition 5 holds it is enough to observe that

$$[\mathfrak{g}_{\mathbb{C}}^{(i)}, \mathfrak{g}_{\mathbb{C}}^{(j)}] \subseteq \mathfrak{g}_{\mathbb{C}}^{(i+j)}.$$

This observation, the definition of u and lemma 3.2.1 imply that

$$[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}.$$

From the representation theory of $\mathfrak{sl}_2$ we have

$$[\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}] = u \cap \mathfrak{p}_{\mathbb{C}}.$$

Hence

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}].$$

Therefore $\mathfrak{F}$ is well defined.

From a theorem of Kostant & Rallis [**K-R**], there is a bijection between the non-zero nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ and the $K_{\mathbb{C}}$-conjugacy classes of normal triples. Two normal noticed triples (x, e, f) and (x', e', f') are $K_{\mathbb{C}}$ conjugate if and only if their corresponding triples $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}', \mathfrak{w}')$ are $K_{\mathbb{C}}$ conjugate. Hence, $\mathfrak{F}$ induces a one to one map from $K_{\mathbb{C}}$-conjugacy classes of $\mathfrak{S}$ and the $K_{\mathbb{C}}$-conjugacy classes of the triples of $\mathfrak{L}$. The next theorem tells us that such a map is also surjective.

If $q = l \oplus u$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, then there exists $z \in i\mathfrak{k}$ such that

u = sum of eigenspaces of ad_z for positive eigenvalues

l = eigenspace of ad_z for eigenvalue 0

$\bar{u}$ = sum of eigenspaces of ad_z for negative eigenvalues.[**K-V**]

Furthermore l is (θ, σ)-stable. Also u and $\bar{u}$ are both θ-stable. Finally

$$\mathfrak{g}_{\mathbb{C}} = \bar{u} \oplus l \oplus u,$$

$\tau(u) = \bar{u}$, and from lemma 3.2.1 the spaces l, u and $\bar{u}$ are mutually orthogonal relative to κ'.

Theorem 3.2.3. *For any triple* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ *of* $\mathfrak{L}$ *there exists a normal triple* (x, e, f)

in $\mathfrak{S}$ *such that* $\mathfrak{q}$ *is the Jacobson-Morosov parabolic subalgebra for* (x, e, f) *and* $\mathfrak{w}$ $= \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

Proof. Let $\tau(\mathfrak{w}) = \bar{\mathfrak{w}}$. First, we prove that

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l \cap \mathfrak{k}_{\mathbb{C}}.$$

Clearly $[\mathfrak{w}, \bar{\mathfrak{w}}] \subset \mathfrak{k}_{\mathbb{C}}$ since both $\mathfrak{w}$ and $\bar{\mathfrak{w}}$ are subsets of $\mathfrak{p}_{\mathbb{C}}$. Therefore it is enough to show that

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l.$$

But

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \perp u,$$

because for all $X \in \mathfrak{w}$, $Y \in \bar{\mathfrak{w}}$ and $U \in u$ we have:

$$\kappa'(U, [X, Y]) = -\kappa(U, \tau([X, Y])) = -\kappa(U, [\tau(X), \tau(Y)])$$

$$= \kappa(U, [\tau(Y), \tau(X)])) = -\kappa([\tau(Y), U], \tau(X)) = \kappa'([\tau(Y), U], X) = 0,$$

since κ is invariant, $\tau(Y) \in \mathfrak{w}$ and $[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$.

Furthermore

$$[\bar{u}, \bar{\mathfrak{w}}] \perp \bar{\mathfrak{w}},$$

because for all B and $C \in \bar{\mathfrak{w}}$ and $A \in \bar{u}$ we have:

$$\kappa'([A, B], C) = -\kappa(\tau(C), \tau[\tau(A), \tau(B)]) = \kappa'(\tau(C), [\tau(A), \tau(B)]) = 0,$$

since $\tau(B), \tau(C) \in \mathfrak{w}$ and $\tau(A) \in u$ and $[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$.

Moreover for all $X \in \mathfrak{w}$, $Y \in \bar{\mathfrak{w}}$ and $U \in \bar{u}$ we have:

$$\kappa'(U,[X,Y]) = -\kappa(U,\tau([X,Y])) = -\kappa(U,[\tau(X),\tau(Y)])$$

$$= -\kappa([U,\tau(X)],\tau(Y)) = \kappa'([U,\tau(X)],Y) = 0,$$

since κ is invariant, $\tau(X) \in \bar{\mathfrak{w}}$ and $[\bar{u}, \bar{\mathfrak{w}}] \perp \bar{\mathfrak{w}}$. Hence $[\mathfrak{w}, \bar{\mathfrak{w}}] \perp \bar{u}$.

It follows that $[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l$, for l, u and $\bar{u}$ are mutually orthogonal relative to κ'. Therefore we must have:

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l \cap \mathfrak{k}_{\mathbb{C}}.$$

Since $L \cap K_{\mathbb{C}}$ has a dense orbit in $\mathfrak{w}$ there exists $Z_{\mathfrak{w}}$ in $\mathfrak{w}$ such that

$$[l \cap \mathfrak{k}_{\mathbb{C}}, Z_{\mathfrak{w}}] = \mathfrak{w},$$

We now claim that $\mathfrak{g}_{\mathbb{C}}^{Z_{\mathfrak{w}}} \cap \bar{\mathfrak{w}} = 0$. Indeed, let y be a non zero element of $\bar{\mathfrak{w}}$ such that $[y, Z_{\mathfrak{w}}] = 0$. Then

$$\kappa(\mathfrak{w}, y) = \kappa([Z_{\mathfrak{w}}, l \cap \mathfrak{k}_{\mathbb{C}}], y) = -\kappa(l \cap \mathfrak{k}_{\mathbb{C}}, [Z_{\mathfrak{w}}, y]) = 0,$$

which contradicts the fact that $\mathfrak{w}$ and $\bar{\mathfrak{w}}$ are paired non degenerately by the Killing for κ of $\mathfrak{g}_{\mathbb{C}}$; but

$$\mathfrak{g}_{\mathbb{C}}^{Z_{\mathfrak{w}}} \text{ is } \kappa\text{-orthogonal to } [\mathfrak{g}_{\mathbb{C}}, Z_{\mathfrak{w}}] \supset [l \cap \mathfrak{k}_{\mathbb{C}}, Z_{\mathfrak{w}}] = \mathfrak{w}.$$

Hence,

$$\dim\,[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = \dim\,\bar{\mathfrak{w}} = \dim\,\mathfrak{w} = \dim\,l \cap \mathfrak{k}_{\mathbb{C}}$$

whence we have

$$[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}},$$

since $Z_{\mathfrak{w}} \in \mathfrak{w}$.

Fix a θ-stable positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. To construct $\mathfrak{h}_{\mathbb{C}}$ one starts with a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$ and adjoins a subspace $\mathfrak{a}$ of $\mathfrak{p}$ that is maximal with respect to the properties of being abelian and commuting with $\mathfrak{k}$ (see Knapp and Vogan [**K,V**]). Then $\mathfrak{h}_{\mathbb{C}}$ is the complexification of

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}.$$

Furthermore since $\mathfrak{q}$ is a θ-stable parabolic subalgebra the subspaces u and $[u, u]$ have the properties mentioned in proposition 1.2.4. If α is a weight of $\mathfrak{g}_{\mathbb{C}}$ then $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ denotes the corresponding weight space and $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ is called a weight vector.

Choose $x \in \mathfrak{k}_{\mathbb{C}}$ such that:

for any simple root β of $\mathfrak{g}_{\mathbb{C}}$,

$$\beta(x) = 0 \qquad \text{if } \mathfrak{g}_{\mathbb{C}}^{\beta} \subseteq l,$$

$$\beta(x) = 2 \qquad \text{if } \mathfrak{g}_{\mathbb{C}}^{\beta} \subseteq \hat{\mathfrak{w}},$$

$$\beta(x) = 1 \qquad \text{otherwise.}$$

The next step is to prove that $\mathfrak{w}$ is the 2 eigenspace of ad_x on $\mathfrak{p}_{\mathbb{C}}$. Observe that , since l, $\hat{\mathfrak{w}}$ and the set of simple roots of $\mathfrak{g}_{\mathbb{C}}$ whose roots spaces are not in $\hat{\mathfrak{w}}$ are θ-stable, we have

$$\beta(x) = \theta\beta(x) \text{ , for all simple roots } \beta \text{ of } \mathfrak{g}_{\mathbb{C}}.$$

Furthermore two different simple roots of $\mathfrak{g}_{\mathbb{C}}$ cannot restrict to the same weight of $\mathfrak{p}_{\mathbb{C}}$. (See **[K-V]**, page 257) In other words $\mathfrak{t}$-weights in $u \cap \mathfrak{p}_{\mathbb{C}}$ occur with multiplicity 1.

Define $\mathfrak{y}$ and $\hat{\mathfrak{y}}$ such that:

$$u \cap \mathfrak{p}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \oplus \mathfrak{y}$$

and

$$u = [u, u] \oplus \hat{\mathfrak{y}}.$$

Clearly $\mathfrak{y} = \hat{\mathfrak{y}} \cap \mathfrak{p}_{\mathbb{C}}$

Let ν be a weight of $\mathfrak{p}_{\mathbb{C}}$ such that $\mathfrak{g}^{\nu}_{\mathbb{C}} \subseteq \hat{\mathfrak{w}}$. Then we only need to consider two cases.

case 1. ν is the restriction of a simple root α and $\mathfrak{g}^{\alpha}_{\mathbb{C}}$ lies in $\hat{\mathfrak{y}}$. Then $\alpha(x) = 2$ since $\hat{\mathfrak{y}} \subseteq \hat{\mathfrak{w}}$. Hence $\nu(x) = 2$.

case 2. ν is a weight of $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$. Let X_{ν} be a non-zero vector in $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ such that:

$$[x, X_\nu] = \nu(x) X_\nu.$$

Then

$$X_\nu \in [u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{y}] \oplus [u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]].$$

Since $\hat{\mathfrak{w}} \perp [u, [u, u]]$ X_ν must be in $[u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{y}]$. But

$$u \cap \mathfrak{k}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{k}_{\mathbb{C}}] \oplus \mathfrak{v}.$$

Therefore

$$X_\nu \in [\mathfrak{v}, \mathfrak{y}],$$

for $\hat{\mathfrak{w}} \perp [u, [u, u]]$.

Now any weight α such that $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ lies in $\mathfrak{v}$ or $\mathfrak{y}$ must be the restriction of a simple root of $\mathfrak{g}_{\mathbb{C}}$, otherwise X_ν would be in $[u, [u, u]]$.

Moreover, since

$$[u, \hat{\mathfrak{w}}] \cap \hat{\mathfrak{w}} = \{0\} ,$$

we must have

$$\alpha(x) = 1.$$

Hence

$$\nu(x) = 2.$$

Therefore $\mathfrak{w} \subseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

Next we show that $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{w}$.

If ν is a weight of $\mathfrak{m}$ but not of $\mathfrak{w}$ then either $X_\nu \notin [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ or $X_\nu \in [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ for

$$\mathfrak{m} \perp [u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]].$$

In the first case we have $\nu(x) = 1$.

If $X_\nu \in [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$ then either $X_\nu \in [l \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$ or $X_\nu \in [u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$. Since $\mathfrak{w}$ is an $L \cap K_{\mathbb{C}}$-module, we must have

$$X_\nu \in [u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}].$$

It follows that $\nu(x) \geq 3$
Therefore if $\nu = \beta|_{\mathfrak{t}}$ then X_β must be in $\hat{\mathfrak{w}}$. Hence

$$\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} \text{ and } \hat{\mathfrak{w}} = \mathfrak{g}_{\mathbb{C}}^{(2)}.$$

Choose $e = Z_{\mathfrak{w}}$. Note that the fact that $[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}}$ makes it possible to find f in $\bar{\mathfrak{w}}$ such that $[e, f] = x$, but then since $\mathfrak{w}$ is the 2-eigenspace of x in $\mathfrak{p}_{\mathbb{C}}$, $\bar{\mathfrak{w}}$ is the -2 eigenspace. Therefore $[x, f] = -2f$. The triple (x, e, f) is normal. The nilpotent e is noticed since dim $l \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\mathfrak{w}$.

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The following theorem completes the classification.

Let l be a (θ, σ)-stable Levi Subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Define the set of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ to have the same properties as the triples of $\mathfrak{L}$, replacing $\mathfrak{g}_{\mathbb{C}}$ by l. Here $\mathfrak{q}_l$ is a θ-stable parabolic subalgebra of $[l, l]$. Then we have:

Theorem 3.2.4. *There is a one-to-one correspondence between nilpotent $K_{\mathbb{C}}$-orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ in $\mathfrak{L}$.*

Proof. We noted before that a Levi subalgebra l contains a nilpotent element $e \in \mathfrak{p}_{\mathbb{C}}$ if and only if $[l, l]$ does. Two Levi subalgebras are $K_{\mathbb{C}}$-conjugate if and only if their derived subalgebras are. Each nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ can be put in a normal triple (x, e, f) inside of the minimal Levi subalgebra l containing e. By theorem 1.1.4 two minimal Levi subalgebras containing e are conjugate under $K_{\mathbb{C}}^e$. Hence the theorem follows from theorem 3.2.3

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3. Example

Let $\mathfrak{g}$ be $\mathfrak{sl}(3, \mathbb{R})$, the set of 3×3 real matrices of trace 0, Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(3, \mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of 3×3 complex symmetric matrices. The Cartan involution θ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Denote by $\bar{Y}$, the complex conjugate of a matrix $Y \in \mathfrak{g}_{\mathbb{C}}$.

The set of orthogonal matrices ($K_{\mathbb{C}}$) preserves the set of symmetric matrices ($\mathfrak{p}_{\mathbb{C}}$) under conjugation. The nilpotent orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ are parametrized by the partitions of 3. Therefore, there are exactly two non zero nilpotent classes since the zero nilpotent class corresponds to the partition $[1,1,1]$. A computation shows that the following matrices

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}$$

generate the only θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$. Let

$$\mathfrak{b} = \mathbb{C}H_1 \oplus \mathbb{C}E_3 \oplus \mathbb{C}H_2 \oplus \mathbb{C}E_1 \oplus \mathbb{C}E_2.$$

Of course $\mathfrak{b}$ is conjugate to the set of upper triangular matrices of $\mathfrak{sl}(3,\mathbb{C})$.

We see that $l \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_1$ and $\mathfrak{m} = \mathbb{C}E_1 \oplus \mathbb{C}E_2$. Let $\mathfrak{w}_1 = \mathbb{C}E_1$ and $\mathfrak{w}_2 = \mathbb{C}E_2$. Clearly, $L \cap K_{\mathbb{C}}$ has a dense orbit on $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively for $[H_1, E_1] = 2E_1$ and $[H_1, E_2] = E_2$. Also

$$\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \mathfrak{w}_1 = \dim\, \mathfrak{w}_2 = 1.$$

For each $\mathfrak{w}_i$ one verifies easily that the triple ,$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_i)$, satisfies all the requirements specified above.

Thus we obtain the following correspondence:

$$(H_1, E_1, \bar{E}_1) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_1)$$

$$(2H_1, E_2, \bar{E}_2) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_2)$$

4. Two important special cases

In some special cases the θ-stable parabolic subalgebra in the triple $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ is characterized by a dimension criterion similar to that of the *distinguished* parabolic subalgebras that play an essential role in the Bala-Carter theory. One of these cases is very important, because it classifies all the noticed orbits of the classical real Lie algebras of type B, C and D and those of the exceptional algebras of type $E_{6(2)}$, $F_{4(4)}$, $G_{2(2)}$. We shall say more about this in the next chapter.

Define $\overline{\mathfrak{L}}$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\mathfrak{L}$ such that:

1. $\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.
2. $u \cap \mathfrak{p}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \oplus \mathfrak{w}$.

Let $\overline{\mathfrak{S}}$ be the set of KS-triples (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$ such that e is even and noticed then we have a map from $\overline{\mathfrak{S}}$ to $\overline{\mathfrak{L}}$ which associates a triple (x, e, f) of $\overline{\mathfrak{S}}$ to an element $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$) of $\overline{\mathfrak{L}}$ where $\mathfrak{q}$ is the θ-stable Jacobson Morozov parabolic subalgebra relative to (x, e, f) and $\mathfrak{w} = \mathfrak{g}^2_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$. From the previous results (see theorem 2.1.6)

it is clear that such a map is well defined and induces a one to one map from $K_{\mathbb{C}}$-orbits of $\overline{\mathfrak{S}}$ and the $K_{\mathbb{C}}$-conjugacy classes of the triples of $\overline{\mathfrak{L}}$, which is also surjective by the following theorem.

Theorem 3.4.1. *For any triple* $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ *of* $\overline{\mathfrak{L}}$ *there exist a normal triple* (x, e, f) *in* $\overline{\mathfrak{S}}$ *such that* $\mathfrak{q}$ *is the* θ*-stable parabolic subalgebra relative to* (x, e, f) *and* $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

Proof. As in theorem 3.2.3 we can find $Z_{\mathfrak{w}} \in \mathfrak{w}$ such that

$$[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}}.$$

Choose a θ-stable positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that l is defined by a θ-stable subset Λ of Δ^+.

We consider two cases:

1. Assume that $\mathfrak{g}_{\mathbb{C}}$ is of inner type. We can choose $\mathfrak{h} \subset \mathfrak{k}_{\mathbb{C}}$. Now take $x \in \mathfrak{h}$ such that for any simple root α of $\mathfrak{g}_{\mathbb{C}}$, $\alpha(x) = 0$ if $\alpha \in \Lambda$ and $\alpha(x) = 2$ if $\alpha \in \Delta \setminus \Lambda$. Choose $e = Z_{\mathfrak{w}}$.

We use Proposition 1.2.4 to classify the roots whose corresponding root spaces lie in $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ as decomposable that is the sum of two roots of u and those whose corresponding root spaces lie in $\mathfrak{w}$ as indecomposable which means that they are the sum of one simple root not in Λ and various simple roots in Λ. Therefore

$$[x, w] = 2w \text{ for any } w \in \mathfrak{w}.$$

Moreover $\mathfrak{w}$ is the intersection of the 2-eigenspace of ad_x and $\mathfrak{p}_{\mathbb{C}}$ for any non zero

element of $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ is an eigenvector of ad_x with eigenvalue at least 4. In other words if $y \in [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ is an eigenvector of ad_x then $y = [y_1, y_2]$ with $y_1 \in [u \cap \mathfrak{k}_{\mathbb{C}}]$ and $y_2 \in [u \cap \mathfrak{p}_{\mathbb{C}}]$.By the Jacobi identity we have

$$[x, y] = [x, [y_1, y_2]] = -[y_1[y_2, x]] - [y_2[x, y_1]] = c[y_1, y_2] = cy \text{ where } c \geq 4$$

Note that the fact that $[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}}$ makes it possible to find f in $\bar{\mathfrak{w}}$ such that $[e, f] = x$ and $[x, f] = -2f$. The triple (x, e, f) is normal. The nilpotent e is even by construction and it is also noticed since dim $l \cap \mathfrak{k}_{\mathbb{C}} =$ dim $\mathfrak{w}$.

2. If $\mathfrak{g}_{\mathbb{C}}$ is not of inner type then retaining the previous notations we have from case(1)

$$[x, w] = [x_{\mathfrak{k}_{\mathbb{C}}}, w] + [x_{\mathfrak{p}_{\mathbb{C}}}, w] = 2w,$$

where $x = x_{\mathfrak{k}_{\mathbb{C}}} + x_{\mathfrak{p}_{\mathbb{C}}}$ with $x_{\mathfrak{k}_{\mathbb{C}}} \in \mathfrak{k}_{\mathbb{C}}$, $x_{\mathfrak{p}_{\mathbb{C}}} \in \mathfrak{p}_{\mathbb{C}}$ and $w \in \mathfrak{w}$.

Hence

$$[x_{\mathfrak{p}_{\mathbb{C}}}, w] = 0.$$

This proves that $\mathfrak{w}$ is inthe 2-eigenspace of $ad_{x_{\mathfrak{k}_{\mathbb{C}}}}$ in $\mathfrak{p}_{\mathbb{C}}$. Recall that a basis of $u \cap \mathfrak{k}_{\mathbb{C}}$ is made of root vectors of the form X_α and $X_\phi + \theta X_\phi$ where α is a compact imaginary root and ϕ a complex one. Similary a basis of $u \cap \mathfrak{p}_{\mathbb{C}}$ is made of root vectors of the form X_β and $X_\phi - \theta X_\phi$ where β is a non compact imaginary root and ϕ is complex. If α is an indecomposable root then

$$\alpha = \alpha_1 + \sum_i m_i \beta_i \text{ where } \alpha_1 \in \Delta \setminus \Lambda,\ m_i \geq 0 \text{ and } \beta_i \in \Lambda.$$

The root $\theta(\alpha)$ must be indecomposable also. Otherwise we would have

$$\theta(\alpha) = \alpha'_1 + \cdots + \alpha'_k + \sum_j m_j\beta_j \text{ where } \alpha'_i s \in \Delta \setminus \Lambda,\ m_j \geq 0 \text{ and } \beta_j \in \Lambda.$$

But

$$\theta(\theta(\alpha)) = \theta(\alpha'_1) + \cdots + \theta(\alpha'_k) + \sum_j m_j\theta(\beta_j) = \alpha.$$

Since Λ is θ-stable all the $\theta(\alpha_i) \in \Lambda$ This would mean that α is decomposable. This is a contradiction.

Moreover for any y in the 2-eigenspace of ad_x in $\mathfrak{p}_{\mathbb{C}}$ we have

$$[x, y] = [x_{\mathfrak{k}_{\mathbb{C}}}, y] + [x_{\mathfrak{p}_{\mathbb{C}}}, y] = 2y. \text{ Thus } [x_{\mathfrak{p}_{\mathbb{C}}}, y] = 0.$$

The facts that $\phi_{|\mathfrak{k}_{\mathbb{C}}} = \frac{(\phi+\theta(\phi))}{2}$ and $\theta(u) = u$ imply that any eigenvector of ad_x in $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ has eigenvalue at least 4. Hence $\mathfrak{w}$ is the intersection of the 2-eigenspace of $ad_{x_{\mathfrak{k}_{\mathbb{C}}}}$ and $\mathfrak{p}_{\mathbb{C}}$. Next we show that $\mathfrak{q}$ is a parabolic subalgebra relative to a triple containing $x_{\mathfrak{k}_{\mathbb{C}}}$ as a neutral element. Assume that α is a root such that $\alpha(x) \geq 0$ then

if α is an imaginary root then $\alpha(x_{\mathfrak{k}_{\mathbb{C}}}) = \alpha(x) \geq 0$.

If α is a complex root then $\theta(\alpha) \geq 0$, because $\mathfrak{q}$ is θ-stable.But

$$\alpha(x_{\mathfrak{k}_{\mathbb{C}}}) = \frac{(\alpha+\theta(\alpha))}{2}(x) \geq 0.$$

So $l \subset l' = \mathfrak{g}_{\mathbb{C}}^{x_{\mathfrak{k}_{\mathbb{C}}}}$ and $u \subset u'$ where u' is the sum of positive eigenspaces of $ad_{x_{\mathfrak{k}_{\mathbb{C}}}}$ on $\mathfrak{g}_{\mathbb{C}}$. Since

$$\dim \mathfrak{g}_{\mathbb{C}} = \dim l + 2\dim u = \dim l' + 2\dim u',$$

we must have

$$l = l' \text{ and } u = u'$$

Choose $e = Z_{\mathfrak{w}}$ using the same argument used at the beginning of the proof of the previous theorem and find f in $\bar{\mathfrak{w}}$ such that $[e, f] = x_{\mathfrak{k}_{\mathbb{C}}}$ and $[x_{\mathfrak{k}_{\mathbb{C}}}, f] = -2f$. The triple $(x_{\mathfrak{k}_{\mathbb{C}}}, e, f)$ is normal . The nilpotent e is even by construction and it is also noticed since dim $l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{w}$.

■

We may summarize the foregoing results as follows.

Theorem 3.4.2. *There is a one-to-one correspondence between even nilpotent $K_{\mathbb{C}}$-orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$ conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ where l is a (σ, θ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and*

0. *$\mathfrak{q}_l = m_l \oplus u_l$ is a θ-stable parabolic subalgebra of $[l, l]$.*
1. *$\mathfrak{w}_l \subseteq u_l \cap \mathfrak{p}_{\mathbb{C}}$.*
2. *$\mathfrak{w}_l$ is $M_l \cap K_{\mathbb{C}}$ stable. (M_l connected Lie group of $G_{\mathbb{C}}$ with Lie algebra m_l)*
3. *$\mathfrak{w}_l$ has a dense $M_l \cap K_{\mathbb{C}}$ orbit.*
4. *dim $m_l \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\mathfrak{w}_l$.*
5. *dim $m_l \cap \mathfrak{k}_{\mathbb{C}}$ = dim $\frac{u_l \cap \mathfrak{p}_{\mathbb{C}}}{[u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}]}$.*
6. *$u_l \cap \mathfrak{p}_{\mathbb{C}} = [u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}] \oplus \mathfrak{w}_l$.*

Proof. Similar to the proof of theorem 3.3.4.

■

Example 3.4.3.

We illustrate this theorem by computing the number of $K_{\mathbb{C}}$ nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$ for the split real form of the exceptional complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of type G_2. This algebra is of inner type. That is $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ have the same rank. Let α_1 and α_2 be the usual simple roots of $\mathfrak{g}_{\mathbb{C}}$. Assume that α_1 is short and α_2 is long. Then it is well known and easy to check that the set of positive roots can be partitioned into compact and non compact roots roots as follows :

$$\text{compact roots : } \alpha_1,\, 3\alpha_1 + 2\alpha_2$$

$$\text{non compact roots : } \alpha_2,\, \alpha_1 + \alpha_2,\, 2\alpha_1 + \alpha_2,\, 3\alpha_1 + \alpha_2.$$

Hence $\mathfrak{k}_{\mathbb{C}}$ is the Cartan subalgebra together with the direct sum of the root spaces corresponding to the compact roots, and $\mathfrak{p}_{\mathbb{C}}$ is the direct sum of the root spaces corresponding to the non compact roots. Every Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is $\theta-$stable. There are only 3 different conjugacy classes of Borel sublagebras under $K_{\mathbb{C}}$. To obtain them one uses the reflection S_{α_2} to conjugate (α_1, α_2) to ($\alpha_1 + \alpha_2$, -α_2) which is conjugated to (-α_1 - α_2, $3\alpha_1 + 2\alpha_2$) by $S_{\alpha_1+\alpha_2}$.

The $\theta-$stable parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ that are part of an element of $\overline{\mathfrak{L}}$ are:

$b = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2}$

$\oplus\mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{\alpha_1} + \mathbb{C} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$

where $l_b \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2}$

and $\mathfrak{w}_b = \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2}$

$p = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{-\alpha_1}$

$\oplus\mathbb{C}X_{\alpha_2} \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$

where $l_p \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{-\alpha_1}$

and $\mathfrak{w}_p = \mathbb{C}X_{\alpha_2} \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2}$

$q = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-\alpha_1-\alpha_2}$

$\oplus\mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$

where $l_q \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2}$

and $\mathfrak{w}_q = \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2}$

A careful introspection of the data and some computations, using a specific Chevalley basis for $\mathfrak{g}_{\mathbb{C}}$, show that we can set up the following correspondence:

$(H_{\alpha_1} + 4H_{3\alpha_1+2\alpha_2}, X_{-\alpha_2} + X_{\alpha_1+\alpha_2}, -2X_{\alpha_2} + 4X_{-\alpha_1-\alpha_2}) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, b, \mathfrak{w}_b)$

$(2H_{3\alpha_1+2\alpha_2}, X_{\alpha_2} + X_{2\alpha_1+\alpha_2}, X_{-\alpha_2} + X_{-2\alpha_1-\alpha_2}) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, p, \mathfrak{w}_p)$

$$(H_{\alpha_1} + H_{3\alpha_1+2\alpha_2}, X_{-\alpha_2} + X_{2\alpha_1+\alpha_2}, -2X_{\alpha_2} + 4X_{-2\alpha_1-\alpha_2}) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, q, \mathfrak{w}_q)$$

$$(H_{2\alpha_1+\alpha_2}, X_{2\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}) \longleftrightarrow (l_1, q_{l_1}, \mathfrak{w}_{l_1})$$

$$(H_{3\alpha_1+\alpha_2}, X_{3\alpha_1+\alpha_2}, X_{-3\alpha_1-\alpha_2}) \longleftrightarrow (l_2, q_{l_2}, \mathfrak{w}_{l_2})$$

One shows directly that:

$$l_1 = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-2\alpha_1-\alpha_2}$$

$$q_{l_1} = \mathbb{C}H_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2}$$

$$\mathfrak{w}_{l_1} = \mathbb{C}X_{2\alpha_1+\alpha_2}$$

$$l_2 = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-3\alpha_1-\alpha_2}$$

$$q_{l_2} = \mathbb{C}H_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2}$$

$$\mathfrak{w}_{l_2} = \mathbb{C}X_{3\alpha_1+\alpha_2}$$

This example shows how the conjugacy classes of the normal triples parametrizing the $K_{\mathbb{C}}-$ nilpotent orbits on $\mathfrak{p}_{\mathbb{C}}$ are in one to one correspondence with the triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ described above.

Under certain conditions that will be stated later we shall prove a result similar to the previous theorem in the case where $e \in \mathfrak{p}_{\mathbb{C}}$ is non even.

Define $\mathfrak{L}'$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\mathfrak{L}$ such that:

1. $\mathfrak{w} \subseteq [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$, $\mathfrak{w} \perp [u[u,u]]$ relative to κ'.
2. $\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Let $\mathfrak{S}'$ be the set of normal triples (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ such that

1. e is not even.
2. e is noticed.
3. $\dim\, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} = \dim\, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

From (2.1.8) we know that

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]} \cong \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \oplus \frac{\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]}$$

as $L \cap K_{\mathbb{C}}$-modules. Consequently if e is noticed then

$$\dim\, \mathfrak{g}_{\mathbb{C}}^{2} \cap \mathfrak{p}_{\mathbb{C}} = \dim\, [\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}].$$

Hence

$$\mathfrak{g}_{\mathbb{C}}^{2} \cap \mathfrak{p}_{\mathbb{C}} = [\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}],$$

and

$$\mathfrak{g}_{\mathbb{C}}^{2} \cap \mathfrak{p}_{\mathbb{C}} \perp [u, [u,u]] \text{ relative to } \kappa'.$$

We have a map from $\mathfrak{S}'$ to $\mathfrak{L}'$ which associates a triple (x, e, f) of $\mathfrak{S}'$ to an element $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ of $\mathfrak{L}'$ where $\mathfrak{q}$ is the θ-stable Jacobson-Morozov parabolic subalgebra rel-

ative to (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}$. From theorem 2.1.8 it is clear that such a map is well defined. Next we prove that it is actually surjective.

Theorem 3.4.4. *For any triple $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\mathfrak{L}'$ there exist a normal triple (x, e, f) in $\mathfrak{S}'$ such that $\mathfrak{q}$ is the θ-stable parabolic subalgebra relative to (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. Choose $\mathfrak{w}'$ such that $u \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{w}' \oplus [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$. Hence dim $\mathfrak{w}'$ = dim $l \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{w}'$ is $L \cap K_{\mathbb{C}}$ stable. It follows that the triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}')$ is an element of $\overline{\mathfrak{L}}$ and from theorem 3.4.2 it is possible to associate it to a triple (x', e', f') of $\overline{\mathfrak{S}}$ such that $\mathfrak{w}'$ is the intersection of the 2-eigenspace of x' and $\mathfrak{p}_{\mathbb{C}}$.

From lemma 3.2.1, $\mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}}$, the 4 eigenspace of x' in $\mathfrak{p}_{\mathbb{C}}$ is orthogonal to [u,[u,u]] relative to κ'. It is also $L \cap K_{\mathbb{C}}$-stable. Furthermore $\mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}} \subseteq [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ as we have seen in the proof oftheorem 3.4.1. By definition $\mathfrak{w} \cap \mathfrak{w}' = \{0\}$. Hence $\mathfrak{w} \subset \mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}}$. But the fact that $L \cap K_{\mathbb{C}}$ has a dense orbit on $\mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}}$ implies that

$$\dim \mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}} \leq \dim l \cap \mathfrak{k}_{\mathbb{C}}$$

.

Therefore we conclude that

$$\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(4)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Let $x = \frac{x'}{2}$ and $\bar{\mathfrak{w}} = \mathfrak{g}_{\mathbb{C}}^{(-4)} \cap \mathfrak{p}_{\mathbb{C}}$. Then using an argument similar to the one in the proof of theorem 3.2.3, we can show the existence of a non zero nilpotent e in $\mathfrak{w}$ such that

$$[e, \bar{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}}.$$

That makes it possible to find f in $\bar{\mathfrak{w}}$ such that the triple $(x, e, f) \in \mathfrak{S}'$.

■

A triple (x, e, f) in $\mathfrak{S}'$ will be refered to as a half-even triple. The orbit $K_{\mathbb{C}} \cdot e$ in $\mathfrak{p}_{\mathbb{C}}$ is also called a half-even orbit.

Finally, as in the even case, we have:

Theorem 3.4.5. *There is a one-to-one correspondence between half-even nilpotent $K_{\mathbb{C}}$-orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ where l is a (σ, θ)-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and*

0. *$\mathfrak{q}_l = m_l \oplus u_l$ is a θ-stable parabolic subalgebra of $[l, l]$.*
1. *$\mathfrak{w}_l \subseteq [u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}]$.*
2. *$\mathfrak{w}_l$ is $M_l \cap K_{\mathbb{C}}$ stable. (M_l connected Lie group of $G_{\mathbb{C}}$ with Lie algebra m_l)*
3. *$\mathfrak{w}_l$ has a dense $M_l \cap K_{\mathbb{C}}$ orbit.*
4. *$dim\ m_l \cap \mathfrak{k}_{\mathbb{C}} = dim\ \mathfrak{w}_l$.*
5. *$dim\ m_l \cap \mathfrak{k}_{\mathbb{C}} = dim\ \frac{u_l \cap \mathfrak{p}_{\mathbb{C}}}{[u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}]}$.*
6. *$\mathfrak{w}_l \perp [u_l, [u_l, u_l]]$*

Proof. Similar to the proof of theorem 3.3.4.

■

We note that a generalization of proposition 1.2.1 to non-even nilpotent elements

of $\mathfrak{p}_{\mathbb{C}}$ would make the space $\mathfrak{w}$ in theorem 3.2.3 more accessible. We are currently investigating the possibility of obtaining such a result.